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AN ALTERNATING GROUP METHOD OF PARALLEL COMPUTING FOR THE HEAT EQUATIONS

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Abstract: In this paper, an alternating group method for solving diffusion equations is presented. It is shown that the method is unconditional stable and has the obvious property of parallelism. The numerical experiments presented here show that the method can be used directly on parallel computers.

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Key Words: diffusion equation, alternating group method, stability, parallel computing

1. Introduction

Consider the following initial-boundary value problem of the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in \Omega \times [0,T]\,, \tag{1a}$$

$$u(x,0) = f(x), \quad x \in \Omega\,, \tag{1b}$$

$$u(0,t) = g_1(t), u(1,t) = g_2(t), \quad (x,t) \in \partial\Omega \times [0,T]\,, \tag{1c}$$

where $\Omega = [0,1]$ and $\partial\Omega$ is the bound of Ω.

There have been many numerical methods for this problem, such as classical explicit scheme, Grarnk-Nicolson scheme [8], finite element method [9], and so on. These methods can be summed up to two classes, one is explicit scheme,

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it need not to solve equations and has good parallel property, and is feasible to computing on the parallel computers. However this scheme is of conditional stable and it confines fixed-time increment quite strongly. The other is implicit scheme that is of unconditional stability, but it need to solve equation, so it is not convenient to be applied on the parallel computers. Therefore, how to construct a numerical method that is feasible to parallel computers becomes a concerned problem. On 1980's, Evans raised an Alternating Group Explicit method (AGE) [6], [5] that is feasible parallel computers, Zhang Baolin developed this method and raided an Alternating Section Explicit-implicit method(ASE-I) [12], [3]. In this paper, we present an alternating group method, using non-symmetric Saul'yev scheme which is feasible to parallel computing. It is shown that the method is unconditionally stable. and has the obvious property of parallelism. Moreover the numerical experiments presented here show that the method can be used directly on parallel computers and has higher degree of accuracy.

2. Alternating Group Method

Let us first split the domain $(0,1)\times(0,T)$, and suppose that Δx and Δt are the space step and the time step respectively. $\Delta x = \frac{1}{J} = h,\ \ x_i = ih,\ \ t_n = n\Delta t$, $i = 0, 1, \cdots, J,\ \ n = 0, 1, \cdots, [\frac{T}{\Delta t}]$, where J is a positive integer, the mesh point $(x_i, t_n)$is denoted by (i, n).

Define

$$\delta_x u_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \quad \delta_{\bar{x}} u_i^n = \frac{u_i^n - u_{i-1}^n}{h}, \quad \delta_t u_i^n = \frac{u_i^{n+1} - u_i^n}{h}.$$

In order to construct alternating group, we give the following second class Saul'yev non symmetric difference scheme (see [10]) of approximation equation (1) at $(x_i, t_{n+\frac{1}{2}})$:

$$\delta_t u_i^n = \frac{1}{2}(\delta_x(\delta_{\bar{x}} u)_i^n + h^{-1}(\delta_x u_i^{n+1} - \delta_{\bar{x}} u_i^n)), \tag{2}$$

$$\delta_t u_i^n = \frac{1}{2}(\delta_x(\delta_{\bar{x}} u)_i^n + h^{-1}(\delta_x u_i^{n} - \delta_{\bar{x}} u_i^{n+1})), \tag{3}$$

$$\delta_t u_i^n = \frac{1}{2}(\delta_x(\delta_{\bar{x}} u)_i^{n+1} + h^{-1}(\delta_x u_i^{n+1} - \delta_{\bar{x}} u_i^n)), \tag{4}$$

$$\delta_t u_i^n = \frac{1}{2}(\delta_x(\delta_{\bar{x}} u)_i^{n+1} + h^{-1}(\delta_x u_i^{n} - \delta_{\bar{x}} u_i^{n+1})). \tag{5}$$

(2)-(5)can be written as

$$(1+r)u_i^{n+1} - ru_{i+1}^{n+1} = 2ru_{i-1}^n + (1-3r)u_i^n + ru_{i+1}^n, \tag{6}$$

$$-ru_{i-1}^{n+1} + (1+r)u_i^{n+1} = ru_{i-1}^n + (1-3r)u_i^n + 2ru_{i+1}^n, \tag{7}$$

$$-ru_{i-1}^{n+1} + (1+3r)u_i^{n+1} - 2ru_{i+1}^{n+1} = ru_{i-1}^n + (1-r)u_i^n, \tag{8}$$

$$-2ru_{i-1}^{n+1} + (1+3r)u_i^{n+1} - ru_{i+1}^{n+1} = (1-r)u_i^n + ru_{i+1}^n, \tag{9}$$

where $r = \frac{\triangle t}{2h^2}$

Before discussing alternating group method, we give three group modules used in this paper. Because what we use are four-point GM group consisted of four interior points and two-point GL group near to the left boundary and two-point GR group near to the right boundary. Four-point GM group consists of four interior points, (x_i, t_{n+1}), (x_{i+1}, t_{n+1}), (x_{i+2}, t_{n+1}), (x_{i+3}, t_{n+1}), the difference scheme of the approximation equation (1) at $(x_i, t_{n+\frac{1}{2}})$, $(x_{i+1}, t_{n+\frac{1}{2}})$, $(x_{i+2}, t_{n+\frac{1}{2}})$,
$(x_{i+3}, t_{n+\frac{1}{2}})$, are given by (6), (8), (9), (7):

$$(1+r)u_i^{n+1} - ru_{i+1}^{n+1} = 2ru_{i-1}^n + (1-3r)u_i^n + ru_{i+1}^n, \tag{10}$$

$$-ru_i^{n+1} + (1+3r)u_{i+1}^{n+1} - 2ru_{i+2}^{n+1} = ru_i^n + (1-r)u_{i+1}^n, \tag{11}$$

$$-2ru_{i+1}^{n+1} + (1+3r)u_{i+2}^{n+1} - ru_{i+3}^{n+1} = (1-r)u_{i+2}^n + ru_{i+3}^n, \tag{12}$$

$$-ru_{i+2}^{n+1} + (1+r)u_{i+3}^{n+1} = ru_{i+2}^n + (1-3r)u_{i+3}^n + 2ru_{i+4}^n. \tag{13}$$

In order to use different group alternatively at neighboring layers and avoid the one-point-group influencing the global accuracy, we use two-point-group at the places near the left and right boundary. The GR two-point consist of two interior points $(x_{m-2}, t_{n+\frac{1}{2}})$, $(x_{m-1}, t_{n+\frac{1}{2}})$, near the right boundary. Applying the scheme (6), (8) respectively:

$$(1+r)u_{m-2}^{n+1} - ru_{m-1}^{n+1} = 2ru_{m-3}^n + (1-3r)u_{m-2}^n + ru_{m-1}^n, \tag{14}$$

$$-ru_{m-2}^{n+1} + (1+3r)u_{m-1}^{n+1} = ru_{m-2}^n + (1-r)u_{m-1}^n + 2ru_m^{n+1}. \tag{15}$$

The GL two-point consist of two interior points (x_1, t_{n+1}), (x_2, t_{n+1}) near the left boundary. Applying the scheme (7), (9), respectively:

$$-ru_1^{n+1} + (1+r)u_2^{n+1} = ru_1^n + (1-3r)u_2^n + 2ru_3^{n+1}, \tag{16}$$

$$(1+3r)u_1^{n+1} - ru_2^{n+1} = 2ru_0^{n+1} + (1-r)u_1^n + ru_2^n. \tag{17}$$

If we write : $\bar{u}^n = (u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n)^T$, $\quad b_i = (2ru_{i-1}^n, 0, 0, 2ru_{i+4}^n)^T$, then $A\bar{u}^{n+1} = B\bar{u}^n + b_i$, where

$$A = \begin{bmatrix} 1+r & -r & 0 & 0 \\ -r & 1+3r & -2r & 0 \\ 0 & -2r & 1+3r & -r \\ 0 & 0 & -r & 1+r \end{bmatrix},$$

$$B = \begin{bmatrix} 1-3r & -r & 0 & 0 \\ -r & 1-r & -2r & 0 \\ 0 & -2r & 1-r & r \\ 0 & 0 & r & 1-3r \end{bmatrix}.$$

Assume n is even number and the numerical solution on the n-th layer u_i^n is given. We want to get the numerical solutions on the $(n+1)$-th and $(n+2)$-th layers denoted by $u_i^{n+1}$and u_i^{n+2}, respectively. We shall construct two kinds of four points alternating group schemes as follows, where the division-point number is odd number and the interior-point $m-1$ is even number.

(1) $m-1 = 4k+2$, where k is a positive integer.

Group the interior points on the $(n+1)$-th laver into $k+1$ group, from the left to the right, they are continuous k-1 GM four-point groups from first group to k-th groups, each group consists of four interior points. Applying the remaining two points of difference scheme (10)-(13), each constitutes a GR two-point group denoted by(14)-(15). The grouping method on the $(n+2)$-th level: group the interior points on the $(n+2)$-th level into $k+1$ groups, from the left to the right, the first group is GL two-point set, whose scheme is denoted by (16)-(17), each of other k GM four-point sets consist of four interior points, denoted by the different scheme (10)-(13). Using the scheme of two layers above alternatively, we get the alternating group method of four points. It can be written as the following matrix form

$$(I + rG_1)u^{n+1} = (I - rG_2)u^n + b_1\,, \tag{18a}$$

$$(I + rG_2)u^{n+2} = (I - rG_1)u^{n+1} + b_2\,, \tag{18b}$$

where

$$u^n = (u_1^n, u_2^n, ..., u_{N-1}^n)^T, b_1 = (2ru_0^n, 0, ..., 0, 2ru_m^{n+1})^T,$$

$$b_2 = (2ru_0^{n+2}, 0, ..., 0, 2ru_m^{n+1})^T,$$

$$G_1 = \mathrm{diag}\,(A_1, A_2, ..., A_k, A_1^*), G_2 = \mathrm{diag}\,(A_2^*, A_1, ..., A_{k-1}, A_k),$$

$$A_1^* = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_2^* = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix},$$

$$A_j = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \ j = 1, 2, ..., k-1$$

(2) $m-1 = 4k$, where k is a positive integer, assume n is even number. The grouping method on the $(n+1)$-th level is as follows: group the interior points on the $(n+1)$-th level into $k+1$ groups, where it is close to the left and right boundary denoted by (14)-(15) and (16)-(17), respectively. Each group consists of four points from the second group to the k-th group. Apply the difference scheme (10)-(13) to it.

The grouping method on the $(n+2)$-th level: group the interior points on the $(n+2)$-th into k groups, each group consists of four interior points. Apply the difference scheme (10)-(13) to it.

Using the scheme of two layers above alternately, we get the alternating group method of four points. It can be written as the following matrix form:

$$(I + r\bar{G}_1)u^{n+1} = (I - r\bar{G}_2)u^n + b\,, \tag{19a}$$

$$(I + r\bar{G}_2)u^{n+2} = (I - r\bar{G}_1)u^{n+1} + b\,, \tag{19b}$$

where

$$b = (2ru_0^{n+1}, 0, ..., 2ru_m^{n+1})^T,$$

$$\bar{G}_1 = \mathrm{diag}\,(A_2^*, A_1, ..., A_{k-1}, A_1^*), \bar{G}_2 = \mathrm{diag}\,(A_1, A_1, ..., A_{k-1}, A_k).$$

By the construction of the matrices $G_1, G_2, \bar{G}_1, \bar{G}_2$, it is seen that the subsystem constructed on the $(n+1)$-th and $(n+2)$-th levels of (18) and (19) can be computed independently. So they are feasible to parallel computing.

3. Stability

Lemma. (Kellogg, see [7]) *Assume that $r > 0$. If G is a nonnegative matrix, that is, $G + G^T$ is nonnegative definite, then $\|(I + rG)^{-1}\|_2 \le 1$, $\|(I - rG)(I + rG)^{-1}\|_2 \le 1$.*

Proof. It is easily to prove that $G_1 + G_1^T$ and $G_2 + G_2^T$ are nonnegative definite matrixes, because the matrix-block of matrix G_1 and G_2:

$$A_j + A_j^T = 2 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

are nonnegative definite, and similarly $A_1^*+(A_1^*)^T$, $A_2^*+(A_2^*)^T$ are nonnegative. □

Let n be even. For the stability of (18), we only need consider the case with the homogeneous boundary condition. By (18), we have

$$u^{n+2} = Gu^n,$$

where

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I + rG_2).$$

Let $\bar{G} = (I + rG_2)G(I + rG_2)^{-1}$. By the above Lemma,

$$\|(I - rG_i)(I + rG_i)^{-1}\|_2 \leq 1, \quad i = 1, 2,$$

we have

$$\rho(G) = \rho(\bar{G}) \leq \|\bar{G}\|_2 \leq 1.$$

Therefore (18) is unconditionally stable. Hence we have the following result.

Theorem. *For all $r > 0$, the alternating group method (18) and (19) for solving the convection-diffusion equation is of unconditional stability.*

4. Numerical Experiment

Consider the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in \Omega \times [0,T],$$

$$u(x,0) = 4x(1-x), \quad x \in [0,1],$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in [0,T].$$

The accurate solution to it is ([9])

$$u(x,t) = \frac{32}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^3} \sin((2k+1)\pi x) e^{-(2k+1)\pi^2 t}. \tag{20}$$

Let us apply the difference scheme (18) to the above example, taking $m = 11$, and choosing $r = 0.5$ and $t = 0.5, 0.6$, respectively. Through the numerical experiments presented in the Table l and Table 2, we conclude that our method has better stability and higher degree of accuracy than that of the AGE method.

Absolute error (Eu):

$$e_j^n = \left| u_j^n - U(x_j, t^n) \right|.$$

Relative error (Ru):

$$E_{nj} = \frac{|e_j^n|}{|u(x_j, t_n)|} \times 100.$$

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x_j		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Eq(18)	Eu	0E-5	0E-5	3E-5	3E-5	3E-5	4E-5	2E-5	1E-5	1E-5
	Ru	11E-5	15E-5	60E-5	33E-5	24E-5	36E-5	23E-5	9E-5	6E-5
AGE[2]	Eu	101E-5	69E-5	193E-5	89E-5	62E-5	99E-5	75E-5	41E-5	49E-4
	Ru	96E-4	826E-4	564E-5	78E-4	73E-4	67E-4	81E-4	36E-4	84E-4
Exact solution		119E-5	226E-4	311E-4	366E-4	384E-4	366E-4	311E-4	226E-4	119E-4

Table 1: Comparison of the absolute error and percentage error of the numerical solution for problem (1), (20) ($t = 0.1$, $m = 11$, $\Delta t = 0.01$).

x_j		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Eq(18)	Eu	3E-4	7E-5	13E-5	15E-5	17E-5	19E-5	17E-5	14E-5	11E-5
	Ru	21E-7	54E-7	229E-7	197E-7	187E-7	204E-7	7E-6	66E-7	33E-7
AGE[2]	Eu	92E-5	123E-5	381E-5	278E-5	252E-5	289E-5	117E-5	151E-5	143E-5
	Ru	265E-4	339E-4	3196E-5	3325E-4	331E-4	327E-4	337E-4	305E-4	339E-4
Exact solution		23E-5	43E-4	60E-4	70E-4	70E-4	71E-4	60E-4	43E-4	23E-4

Table 2: Comparison of the absolute error and percentage error of the numerical solution for problem (1), (20) ($t = 0.5$, $m = 11$, $\Delta t = 0.02$).

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