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ON THE DIMENSION OF THE LINEAR SYSTEMS CUT OUT BY HYPERSURFACES OF GIVEN DEGREE ON THE NORMALIZATION OF STRANGE PROJECTIVE CURVES

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Abstract: Here we give inequalities which measure the failure of projective normality in each degree for a linear system on a smooth curve whose image is a strange projective curve.

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1. Strange Curves

Let $C \subset \mathbf{P}^n$, $n \geq 2$, be an integral non-degenerate curve. We recall that C is said to be strange if there is $P \in \mathbf{P}^n$ (called the strange point of C) such that every tangent line to C at one of its smooth points contains P, i.e. if and only if the rational map from C into $\mathbf{P}^{n-1}$ induced by the linear projection $\pi_P : \mathbf{P}^n \backslash \{P\} \to \mathbf{P}^{n-1}$ is not separable. Thus non-degenerate strange curves do not exist in characteristic zero. Here we work over an algebraically closed field $\mathbf{K}$ with $p :=$

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char($\mathbf{K}$) > 0. By a Theorem of Luis the only strange smooth non-degenerate curves are the smooth plane conic and in this case we have $p = 2$ ([5], Proposition 3). For every prime $p > 0$ there is a recipe to construct all the strange curves (see [3] for $n = 2$ and [1] for $n \geq 3$). In particular for every prime p and every $n \geq 2$ there are huge families (with arbitrarly large dimension) of strange non-degenerate curves with arbitrarly large degree. For any integral curve $Y \subset \mathbf{P}^m$ and any integer $t > 0$ let $\rho_{Y,t,m} : H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(t)) \to H^0(Y, \mathcal{O}_Y(t))$ be the restriction map. Set $M(Y;t,m) := \mathrm{Im}(\rho_{Y,t,m})$ amd $m(Y;t,m) := \dim(M(Y,t,m))$. For any integral projective curve and any dominant morphism $h : X \to Y$ set $M(X,h;t,m) := h^*(M(Y;t,m) \subseteq H^0(X, L^{\otimes t}))$, where $L := h^*(\mathcal{O}_Y(1))$. Set $m(X,h;t,m) := \dim(M(X,h;t,m))$. Hence $m(X,h;t,m) = m(Y;t,m)$.

Remark 1. Let $C \subset \mathbf{P}^n$ be an integral non-degenerate strange curve and P its strange point. Set $d := \deg(C)$. Let m be the multiplicity of C at P, k the separable degree of the rational map π_P induced by the linear projection from P and q its inseparable degree. Thus $m \geq 0$, $m = 0$ if and only if $P \notin C$, $k > 0$, $q > 0$ and q is a power of p. We have $d = m + kq(\deg(T))$.

Here we prove the following result concerning the failure of projective normality in degree t for the linear system associated to a strange curve C on the normalization $h : X \to C$ of C.

Theorem 1. *Let $C \subset \mathbf{P}^n$ be a non-degenerate strange curve, $h : X \to C$ the normalization map, $P \in \mathbf{P}^n$ the strange point of C and $L := h^*(\mathcal{O}_C(1))$. Let m be the multiplicity of C at P, k the separable degree of the rational map π_P induced by the linear projection from P and q its inseparable degree. Let $T \subset \mathbf{P}^{n-1}$ be the image of C through the projection π_P. Hence $d := \deg(C) = \deg(L) = m + kq(\deg(T))$. For every integer $t > 0$ we have $m(X,h;t,n) \leq \binom{n+t-1}{n-1} + m(T;t,n-1) \leq \binom{n+t-1}{n-1} + t((d-m)/kq) + 1$ and $h^0(X, L^{\otimes t}) \geq q(m(T;t,n-1)) + 1 - q$.*

Proof. Let $u : D \to T$ be the normalization and $f : X \to D$ the induced morphism. Notice that $\deg(f \circ u^*(\mathcal{O}_T(1)))) = d - m$ and that there is a degree m effective divisor B on X supported by $h^{-1}(P)$ and such that $M(X, f \circ u; 1, n-1)$ is a subsystem of $H^0(X, L(-B))$. By [4], Lemma 9, f factors trough the degree q Frobenius map. Hence we have $f^*(M(D,u;t,n-1)) \subseteq M(X,h;t,n)$. Thus there is an inclusion of $M(D,u;t,n-1)^{\otimes q}$ in $H^0(X, L^{\otimes t})$. Iterating q times Hopf Lemma we obtain the last inequality. The term $\binom{n+t-1}{n-1}$ in the first two inequalities is just $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) - h^0(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(t))$. We have $m(T;t,n-1) \leq t((d-m)/kq) + 1$ by Riemann inequality, because $\deg(T) = (d-m)/kq$

(Remark 1). □

We proved the case $t = 1$ of Theorem 1 in [2] and obtained that the linear system on X associated to $h : X \to \mathbf{P}^n$ is not complete, unless $p = n = 2$ and C is a smooth plane conic.

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