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HOLOMORPHIC VECTOR BUNDLES WITH INFINITE-DIMENSIONAL FIBERS ON COMPLEX PROJECTIVE VARIETIES

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Abstract: Here we consider holomorphic vector bundles with infinite-dimensional fibers on complex integral projective varieties from the point of view of their semistable finite rank subsheaves.

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1. Introduction

Let X be an integral n-dimensional complex projective variety and F a torsion free coherent analytic sheaf on X. A subsheaf $G \neq 0$ of F is said to be saturated in F if either $G = F$ or the coherent sheaf F/G has no torsion. The sheaf F is said to be reflexive if the natural map $F \to \mathrm{Hom}(\mathrm{Hom}(F, \mathcal{O}_X), \mathcal{O}_X)$ is bijective (see [1]). If F is locally free, then it is reflexive. If F is reflexive, then it is locally free outside $\mathrm{Sing}(X) \cup T$, where T is a closed analytic subset of X with codimension at least three in X (see [1], Corollary 1.4). If F is

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reflexive and G is saturated in F, then G is reflexive. Fix an ample $H \in \mathrm{Pic}(X)$. Fix a reflexive coherent sheaf F on X. $\det(F)$ may be defined as a line bundle at each point on which F is locally free. First assume that $\mathrm{Sing}(X)$ has codimension at least two in X. Hence there is a unique Weil divisor $\det(F)$ of X which extends the divisor $\det(F)$ previously defined on a big open subset of X. Set $\mu(F) := \det(F) \cdot H^{n-1}/\mathrm{rank}(F)$ (or $\mu_H(F)$ if there is any danger of misunderstandings); here $\det(F) \cdot H^{n-1}$ denotes the intersection product of $n-1$ Cartier divisors with a Weil divisor. If X is an integral curve, set $\deg(F) := \mathrm{rank}(F)(p_a(X) - 1) + \chi(F)$ and $\mu(F) := \deg(F)/\mathrm{rank}(F)$. The rational number $\mu(F)$ is called the slope of F. Notice that if $n = 1$ $\mu(F)$ does not depend from the choice of H. Recall that F is said to be H-stable (resp. H-semistable) if for all subsheaves, A, of F with lower rank we have $\mu(A) < \mu(F)$ (resp. $\mu(A) \leq \mu(F)$); this is the notion of stability or semistability in the sense of Mumford and Takemoto.

Theorem 1. *Let X be an integral complex projective variety, H an ample line bundle and E a holomorphic vector bundle on X with a locally convex topological vector space as a fiber. Assume that E satisfies the following condition:*

(A) There is an integer t, a locally convex topological vector space V and an inclusion $j : E \to (H^{\otimes t})^V$.

Then one of the following cases occurs:

(a) There is no holomorphic vector bundle with finite rank F on X such that there is an injective map $u : F \to E$.

(b) The set $\mathfrak{S}$ of all coherent sheaves A on X such that there is an inclusion $j : A \to E$ is not empty.

We are in case (a) if and only if for every integer a the vector bundle $E \otimes H^{\otimes a}$ has no non-zero section. We are in case (a) if and only if $\mathfrak{S}$ contains no line bundle. Assume case (b), i.e. assume $\mathfrak{S} \neq \emptyset$ and set $\tau := \sup_{F \in \mathfrak{S}} \mu(F)$. Then either τ is a maximum or it is not a maximum. If τ is a maximum and the set of all integers $\mathrm{rank}(A)$ for $A \in \mathfrak{S}$ with $\mu(A) = \tau$ is bounded, then there is a saturated $D \in \mathfrak{S}$ such that $\mu(D) = \tau$ and every $A \in \mathfrak{S}$ with $\mu(A) = \tau$ is contained in D. If τ is a maximum and the set of all integers $\mathrm{rank}(A)$ for $A \in \mathfrak{S}$ with $\mu(A) = \tau$ is not bounded, then there is a sequence of saturated and stable sheaves $A_n \in \mathfrak{S}$, $n \geq 1$, such that $\mu(A_i) = \tau$ for every i and for every $n \geq 1$ E contains a saturated subsheaf isomorphic to $\sum_{I=1}^{n} A_i$. If τ is not a maximum, then there is a sequence of coherent saturated subsheaves $F_n \subset E$, $n \geq 1$, such that $\mu(F_{n+1}) > \mu(F_n)$ and $F_n \subset F_{n+1}$ for every $n \geq 1$.

Take X, H and E as in Theorem 1. If E satisfies the condition (A) stated there, we will say that E is right H-bounded. Easy examples shows that all cases may occur for any integral projective curve X. Furthermore, we may exasily construct (e.g. taking a direct sum) example in which τ is a maximum, the maximal rank subsheaf D exists, is locally free and E/D is locally free, but all cases may occur for E/D. The only think that it is always true is that E/D is right bounded (with an obvious extension of this definition if E/D is not locally free).

2. The Proof

Lemma 1. *Let X be an integral complex projective variety, H an ample line bundle on X, E a holomorphic vector bundle on X with a locally convex topological vector space as a fiber and A, B coherent subsheaves of E. Assume $\mu(A) = \mu(B)$ and that there is no coherent subsheaf D of E such that $\mu(D) > \mu(A)$. Then A and B are saturated in E and there is a reflexive subsheaf M of E such that $\mu(M) = \mu(A)$ and M contains subsheaves A', B' with $A' \cong A$ and $B' \cong B$. Any such A is H-semistable. Furthermore, for all inclusions $i : A \to E$ and $j : B \to E$ the subsheaf $i(A) + j(B)$ is a coherent and saturated subsheaf of E with slope $\mu(A)$. Either $i(A) \cap j(B) = \{0\}$, and hence D contains a subsheaf isomorphic to $A \oplus B$ or $\mu(i(A) \cap j(B)) = \tau$; if A is H-stable, then the latter case occur if and only if $i(A) = j(B)$.*

Proof. If A is not saturated in E, then there is a coherent subsheaf M of E such that $A \subset M$, $\text{rank}(A) = \text{rank}(M)$ and the support of M/A contains a non-empty hypersurface T. Hence $\mu(M) \geq T \cdot H^{n-1}/\text{rank}(M)$ contradicting the maximality of $\mu(A)$ among the slopes of all elements of $\mathfrak{S}$. The proofs of all other statements are very similar and we omit them. □

Proof of Theorem 1. If $E \otimes H^{\otimes a}$ has a global section, then there is an inclusion $H^{\otimes -a} \to E$ and hence $\mathfrak{S}$ contains the line bundle $H^{\otimes -a}$. Conversely, assume $\mathfrak{S} \neq \emptyset$ and fix an inclusion $j : F \to E$ with $F \neq 0$ and F with finite rank. Since H is ample, there is an t_0 such that $h^0(X, F(t)) \neq 0$ for every $t \geq t_0$ and in particular $H^{\otimes -t_0} \in \mathfrak{S}$. Now assume that we are in case (b) and that τ is a maximum. Set $\mathfrak{G} := \{A \in \mathfrak{S} : \mu(A) = T\}$. By Lemma 1 every $A \in \mathfrak{G}$ is saturated. First assume the existence of $D \in \mathfrak{G}$ such that $\text{rank}(D)$ is maximal among all $B \in \mathfrak{G}$. By Lemma 1 D is saturated and contains all other members of $\mathfrak{G}$. Now assume there is no such D. If $A \in \mathfrak{S}$ and $\mu(A) = \tau$, then there is a stable subsheaf M of A with $\mu(A) := \tau$: just take the smallest rank non-zero subsheaf of A with τ as its slope. Let $\mathfrak{B}$ be the set of all $A \in \mathfrak{S}$ such that $\mu(A) = \tau$ and A is stable. The non-existence of such a sheaf D implies that $\mathfrak{B}$ is infinite. Take any infinite subset $A_n \in \mathfrak{B}$ and use the last assertion of Lemma 1 to obtain that $A_1 + \cdots + A_n \cong \oplus_{i=1}^n A_n$ for every $n \geq 1$. Now assume that τ is not a maximum. Take any sequence $G_n \in \mathfrak{S}$, $n \geq 1$, such that $\sup_{n\geq 1} \mu(G_n) = \tau$ and take as F_n the saturation of $G_1 + \cdots + G_n$. □

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References

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