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AN APPLICATION OF THE MINKOWSKI INEQUALITY

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Abstract: We use the Minkowski product inequality to improve classical upper bounds for the Euler totient function.

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1. Introduction

Let $x_1, x_2, \ldots, x_k$ be positive real numbers. Then the following inequality is referred to as Minkowski inequality (see [1], Chapter 2):

$$\prod_{i=1}^{k}(1+x_i) \geq (1+\sqrt[k]{x_1 \cdot \cdots \cdot x_k})^k, \quad \text{where} \quad x_1, \ldots, x_k \geq 0. \tag{$*$}$$

One should note that inequality $(*)$ becomes equality if and only if $x_1 = \cdots = x_k$

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or $k = 1$; this fact follows immediately from the proof of $(*)$ depended on the well-known connection between the geometric and arithmetic means.

Minkowski inequality has interesting interpretations both in the theory of Hermitian matrices ([2], Theorem 7.8.8) and geometric inequalities ([1], Chapter 2).

In this paper we use inequality $(*)$ to improve classical upper bounds for the Euler totient function $\varphi(n)$, where n is a composite positive integer.

It is known (see [3], Theorem 5, p. 248) that for n composite we have

$$\varphi(n) < n - \sqrt{n}\,. \tag{S}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Moreover, let $s(n) := p_1 p_2 \ldots p_r$ be the square-free part of n, and put $n_0 = n/s(n)$. With these notations we prove the following theorem.

Theorem. *If n is a composite positive integer, then for every natural number $k = 1, 2, \ldots, \omega(n)$, where $\omega(n) = r$ is the number of all distinct prime divisors of n, we have*

$$\varphi(n) < (\sqrt[r]{n} - \sqrt[r]{n_0})^r \leq (\sqrt[k]{n} - \sqrt[k]{n_0})^k. \tag{$**$}$$

2. Corollaries

Since $n_0 \geq 1$, for the case $k = 2$ in Theorem we immediately obtain

Corollary 1. *If n is a composite positive integer with $\omega(n) \geq 2$, then we have*

$$\varphi(n) < n - 2\sqrt{n} + 1. \tag{$***$}$$

The above inequality $(***)$ improves inequality (S) because, for every positive integer n with $\omega(n) \geq 2$, we have $n - 2\sqrt{n} + 1 < n - \sqrt{n}$.

Using inequality $(**)$ we can obtain a lower bound for the square-free part $s(n)$ of n. For this purpose consider the function $\epsilon_n : [1, \infty) \to \mathbf{R}_+$ defined by the rule

$$1/\epsilon_n(x) = x \cdot (1 - (\frac{\varphi(n)}{n})^{1/x})\ .$$

One can easily check that $\epsilon_n(x)$ decreases, with $x \to \infty$, to $\log(n/\varphi(n))$. Hence, by $(**)$, we get

Corollary 2. *If n is a composite positive integer, then*

$$s(n) > (\epsilon_n(r) \cdot r)^r \geq (\frac{1}{\log(n/\varphi(n))} \cdot r)^r,$$

where $r = \omega(n)$.

3. The Proof of the Theorem

From the Minkowski inequality $(*)$ it follows that, for pairwise distinct real numbers $x_1, \dots, x_r \in (0,1)$, we have

$$\prod_{i=1}^{r}(1-x_i) < \big(1 + (\prod_{i=1}^{r} \frac{x_i}{1-x_i})^{1/r}\big)^{-r} \ . \tag{3.1}$$

Let

$$X = (\prod_{i=1}^{r}(1-x_i))^{1/r}, \quad \text{and} \quad Y = \prod_{i=1}^{r} x_i. \tag{3.2}$$

Then from (3.1) and (3.2) it follows that $X < ((1 + (Y/X))^{-1}$, that is,

$$X < 1 - Y. \tag{3.3}$$

From (3.2) and and (3.3) we obtain

$$\prod_{i=1}^{r}(1-x_i) < (1 - (\prod_{i=1}^{r} x_i)^{1/r})^r \ . \tag{3.4}$$

Since for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ we have

$$\varphi(n)/n = \prod_{i=1}^{r}(1 - \frac{1}{p_i}), \tag{3.5}$$

putting in (3.4) $x_i = 1/p_i$, where $i = 1, \dots, r$, we obtain, by (3.5), that

$$\varphi(n)/n < \big(1 - \frac{1}{\sqrt[r]{s(n)}}\big)^r, \tag{3.6}$$

and this is an equivalent form of the sharp inequality in $(**)$.

Now, for a given $a \in (0,1)$, consider the function $t \mapsto (1 - a^{1/t})^t$ defined on $[1, \infty)$. It is easy to check this function is decreasing, and applying this to the right part of inequality (3.6), we obtain the weak inequality in $(**)$.

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