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TREES IN PROJECTIVE SPACES: NORMAL BUNDLES AND EXTREMAL HYPERPLANE SECTIONS

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Abstract: Let $X \subset \mathbf{P}^n$ be a tree, i.e. a connected union of finitely many lines with only ordinary nodes as singularities and $p_a(X) = 0$. Here we study the normal bundle of X and the possible extremal behaviour of a transversal hyperplane section of X in terms of the combinatorial structure of the lines of X.

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1. Introduction

Let X be a reduced projective curve. We will say that X is *an abstract tree* if X is connected, with only ordinary nodes as singularities and $p_a(X) = 0$. Hence every irreducible component of X is smooth and rational. Let X be an abstract tree and d the number of its irreducible components. We will say that X has degree d. X is smooth if and only if $d = 1$. If $d \geq 2$ the connectedness of X implies the existence of an ordering $X_1, \dots, X_d$ of the irreducible components of X such that all subcurves $X_1 \cup \cdots \cup X_i$, $2 \leq i \leq d$, are connected. Any

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such ordering will be called *a compatible ordering* of X. Hence for every integer $i \in \{2, \dots, d\}$ there is an integer $j(i) \in \{1, \dots, i-1\}$ such that $X_i \cup X_{j(i)} \neq \emptyset$. Since each point of X is contained in at most two irreducible components of X, the integer $j(i)$ is uniquely determined. Hence, setting $\tau(i) = j(i)$ for $2 \leq i \leq d$ we define a function $\tau : \{2, \dots, d\} \to \{1, \dots, d-1\}$ such that $\tau(i) < i$ for all i. The function τ may depend from the choice of the compatible ordeing. The function τ will be called a type of X. The set $\mathcal{A}(d, \tau)$ of all degree d abstract trees with a compatible ordering whose associated function is τ is parametrized by an irreducible variety. If $\mathcal{A}(d, \tau) \cap T(d, \tau') \neq \emptyset$, then $\mathcal{A}(d, \tau) = \mathcal{A}(d, \tau')$ and hence the ambiguity coming from the choice of a compatible ordering will not create any trouble. Let $Y \subset \mathbf{P}^n$ be a reduced projective curve. We will say that Y is a *tree* if each irreducible component of Y is a line and Y is an abstract tree. Let $\mathcal{B}(n, d, \tau)$ be the set of all degree d trees of $\mathbf{P}^n$ with τ as associated function for some compatible ordering of the set of lines of Y. $\mathcal{B}(n, d, \tau)$ is parametrized by a smooth and irreducible quasi-projective variety. Fix any $Y \in B(n, d, \tau)$ and let N_Y or $N_{Y, \mathbf{P}^n}$ its normal bundle. Since Y has only ordinary nodes as singularities, N_Y is a rank $n-1$ vector bundle on Y. Let T be an irreducible component of Y and $b(T)$ the number of irreducible components of Y different from T and intersecting T. We have $\deg(N_Y|T) = n - 1 + b(T)$ and $N_Y|T$ is obtained from $N_T \cong \mathcal{O}_T(1)^{\oplus(n-1)}$ making $b(T)$ positive elementary transformations (see [1]). Since T is smooth and rational, each vector bundle on T is isomorphic to a direct sum of line bundles. Hence there are integers $a_1 \geq \cdots \geq a_{n-1}$ such that $a_{n-1} \geq 1$, $a_1 + \cdots + a_{n-1} = n - 1 + b(T)$ and $N_Y|T \cong \mathcal{O}_T(a_1) \oplus \cdots \oplus \mathcal{O}_T(a_{n-1})$. A *phantom tree in* $\mathbf{P}^n$ is a triple (X, L, V), where X is an abstract tree, $L \in \mathrm{Pic}(X)$, $\deg(L|T) = 1$ for every irreducible component T of X, $V \subseteq H^0(X, L)$ spans L and the morphism $\phi_{L,V} : X \to \mathbf{P}(V^*)$ induced by V is generically injective and everywhere an immersion, i.e. its differential is everywhere invertible. For any integer $n \geq \dim(V) - 1$ we will say that the phantom tree (X, L, V) is a phantom tree of $\mathbf{P}^n$ spanning a $(\dim(V) - 1)$ linear subspace of $\mathbf{P}^n$. For all d and τ let $\mathcal{F}(n, d, \tau)$ be the set of all degree d phantom trees in $\mathbf{P}^n$ with type τ. Fix any embedding $j : \mathbf{P}(V^*) \subseteq \mathbf{P}^n$ as a linear subspace and set $\phi : j \circ \phi_{L,V}$. Since ϕ is a local embedding, the normal sheaf N_ϕ to ϕ is a rank $n-1$ vector bundle. Let T be an irreducible component of Y and $b(T)$ the number of irreducible components of Y different from T and intersecting T. We have $\deg(N_Y|T) = n - 1 + b(T)$ and $N_Y|T$ is obtained from $N_T \cong \mathcal{O}_T(1)^{\oplus(n-1)}$ making $b(T)$ positive elementary transformations and there are integers $a_1 \geq \cdots \geq a_{n-1}$ such that $a_{n-1} \geq 1$, $a_1 + \cdots + a_{n-1} = n - 1 + b(T)$ and $N_Y|T \cong \mathcal{O}_T(a_1) \oplus \cdots \oplus \mathcal{O}_T(a_{n-1})$. If $X = X_1 \cup \cdots \cup X_d$ for some compatible ordering and τ an associated type the integer $b(T)$ may be computed in the following way. If $d = 1$, then $b(T) = 0$. Assume $d \geq 2$; we have $b_(X_d) = 1$, $b(X_1) = \text{card}(\{j \in \{2, \dots, d\} : \tau(j) = 1\})$, for every i with $2 \leq 2 < d$ $b(X_i) = 1 + \text{card}(\{j \in \{i+1, \dots, d\} : \tau(j) = i\})$. The integers $(b(X_1), \dots, b(X_d))$ or $b(\tau(1)), \dots, b(\tau(d))$ will be called the normal sequence of τ.

In Section 2 we will prove the following existence result.

Theorem 1. *Fix integers $n \geq 2$ and $d \geq 1$ and a type τ. Let $(b_1, \dots, b_d)$ be the normal sequence of τ. Fix integers $a_{i,j}$, $1 \leq i \leq d$, $1 \leq j \leq n-1$, such that $a_{i,1} \geq \cdots \geq a_{i,n-1} \geq 1$ and $a_{i,1} + \cdots + a_{i,n-1} = n - 1 + b_i$ for all i with $1 \leq i \leq d$. Then there exists $(X, V, L) \in \mathcal{F}(n, d, \tau)$ such that, calling ϕ the associated unramified birational morphism and $X = X_1 \cup \cdots \cup X_d$ an admissible ordering with τ as type, we have $N_\phi | X_i \cong \mathcal{O}_{X_i}(a_{i,1}) \oplus \cdots \oplus \mathcal{O}_{X_i}(a_{i,n-1})$ for all i with $1 \leq i \leq d$.*

Theorem 1 cannot be extended to the case of trees in $\mathbf{P}^n$ (see Proposition 1 and Example 1). We may at least prove that the general splitting type may be realized by embedded trees (see Proposition 2). The study of the restricted tangent bundle $T\mathbf{P}^n|X$, $X \in \mathcal{B}(n, d, \tau)$, is very easy (see Proposition 9). In Section 3 and Section 4 we will classify types of trees in $\mathbf{P}^3$ with some extremal behaviour from the point of view of the postulation or the postulation of a transversal hyperplane section.

2. Preliminaries and Proof of Theorem 1

Let $X = X_1 \cup \cdots \cup X_d$ be a degree d tree of type τ. A line X_i of X will be called $final$ if either $d = 1$ or X_i intersects exactly another line, i.e. $b_i = 1$: Notice that X_d is a final line. X is called *a bamboo* if either $d = 1$ or $X_i \cup X_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Hence X is a bamboo if and only if either $d = 1$ or X has exactly two final lines. For any compatible ordering of any degree $d \geq 2$ bamboo the associated type τ is unique; indeed, $\tau(i) = i - 1$ for every $i \in \{2, \dots, d\}$. Any bamboo of degree $d \geq 2$ has exactly two compatible orderings of its irreducible components.

Remark 1. Set $T := \mathbf{P}^1$ and fix $P \in T$. Let $E \cong \mathcal{O}_T(c_1) \oplus \cdots \oplus \mathcal{O}_T(c_r)$ and $G \cong \mathcal{O}_T(c_1) \oplus \cdots \oplus \mathcal{O}_T(c_r)$ be two rank r vector bundles on T. E is obtained from G making a positive elementary transformation if and only if there is an integer i such that $1 \leq i \leq r$, $c_j = d_j$ if $j \neq i$ and $c_i = d_i + 1$. Furthermore, if this condition is satisfied then E may be obtained from G making a positive elementary transformation supported by P.

Remark 2. Let T be a final line of a phantom tree of $\mathbf{P}^n$. Since $b(T) = 1$, we have $N_\phi|T \cong \mathcal{O}_T(2) \oplus \mathcal{O}_T(1) \oplus \cdots \oplus \mathcal{O}_T(1)$.

Remark 3. Let $M \subset \mathbf{P}^n$ be a codimension $t > 0$ linear subspace. Then $N_{M,\mathbf{P}^n} \cong \mathcal{O}_M(1)^{\oplus t}$ and for every locally complete intersection curve $C \subseteq M$ we have $N_{C,\mathbf{P}^n} \cong N_{C,M} \oplus \mathcal{O}_C(1)^{\oplus t}$. Let (X, L, V) a phantom tree for $\mathbf{P}^n$ and ϕ the associated immersion. Assume $t := n - \dim(V) + 1 > 0$. Then N_ϕ has $L^{\oplus t}$ as a direct factor.

Proof of Theorem 1. Since the cases $d = 1$ and $d = 2$ are trivial (use respectively a line and a reducible plane conic), we may assume $d \geq 3$ and use induction on the integer d. Set $\eta := \tau|\{2, \dots, d-1\}$. By the inductive assumption there is $(Y, W, R) \in \mathcal{F}(n, d-1, \eta)$ satisfying Theorem 1, say $Y = Y_1 \cup \cdots \cup Y_{d-1}$. We have $b(\eta(i)) = b(\tau(i))$ if $1 \leq i \leq d-1$ and $i \neq \tau(d)$. We have $b(\eta(\tau(d))) = b(\tau(\tau(d))) - 1$. By the inductive assumption there is $Y \in \mathcal{F}(n, d-1, \eta)$ satisfying the thesis of Theorem 1 with respect to the integers $a'_{i,j}$, $1 \leq i \leq d-1$, $1 \leq j \leq n-1$ with $a'_{i,j} = a_{i,j}$ if either $i \neq \tau(d)$ or $j \neq 1$, $a'_{\tau(d),1} = a_{\tau(d),1} - 1$; here we use the convention that if $a_{\tau(d),1} = a_{\tau(d),2}$, then we reorder the integers $a'_{\tau(d),1}, \dots, a'_{\tau(d),n-1}$ in non-decreasing order. By Remark 2 any phantom tree $Y_1 \cup \cdots \cup Y_{d-1}$ with an immersion ψ with $N_\phi|Y_i$ with the prescribed splitting type $a'_{i,1} \geq \cdots \geq a'_{i,n-1}$. Fix $P \in D := Y_{\tau(d)}$ which is a smooth point of Y and set $X = Y \cup X_d$ with $X_d \cap Y = \{P\}$ and X_d intersecting quasi-transversally Y at P. Choose as ϕ an extension of ψ with β as associated direction. Every normal vector in $T\mathbf{P}^n$ to the line $\psi(D)$ at $\psi(D)$ is induced by a plane A containing $\psi(P)$ and, conversely, each such plane gives a unique such normal vector. Thus every positive elementary transformation of $N_\phi|D$ is associated to a plane B through $\phi(D)$, i.e. to a tangent direction β to $\psi(D)$ at $\phi(P)$, i.e. to the unique line R with $\psi(P) \in R \in B$. Let $u : X_d \to R$ be an isomorphism with $u(P) = \psi(P)$. Since $R \neq T$, we see that the map $\phi : X \to \mathbf{P}^n$ defined by $\phi|Y = \psi$ and $\phi|X_d = u$ is unramified and generically injective. By construction $N_\phi|Y_{\tau(d)}$ is obtained from $N_\psi|Y_{\tau(d)}$ making a positive elementary transformation supported by P and associated to the given normal vector. By Remark 1 $N_\phi|X_{\tau(d)}$ is as required for the statement Theorem 1. By Remark 2 $N_\phi|X_d \cong \mathcal{O}_T(2) \oplus \mathcal{O}_T(1) \oplus \cdots \oplus \mathcal{O}_T(1)$, as required for the statement Theorem 1. □

Now we will show that Theorem 1 is not true for embedded trees, a counterexample being bamboos of any degree $d \geq 3$.

Proposition 1. *Fix integers $n \geq 3$ and $d \geq 2$ and a degree d bamboo $X \subset \mathbf{P}^n$. Let $X = X_1 \cup \cdots \cup X_d$ be a compatible ordering of the lines of X. Then:*

(a) $N_X|X_i \cong \mathcal{O}_{X_i}(2) \oplus \mathcal{O}_{X_i}(1)^{\oplus(n-2)}$ *if* $i = 1$ *or* $i = d$*;*

(b) $N_X|X_i \cong \mathcal{O}_{X_i}(2)^{\oplus 2} \oplus \mathcal{O}_{X_i}(1)^{\oplus(n-3)}$ *if* $1 < i < d$*.*

Proof. Part (a) follows from Remark 2. Assume $1 < i < d$ and set $Y := X_{i-1} \cup X_i \cup X_{i+1}$. Thus Y is a degree 3 bamboo and hence it spans a 3-dimensional linear subspace M of $\mathbf{P}^n$. We have $N_{Y,\mathbf{P}^n} \cong N_{Y,M} \oplus \mathcal{O}_Y(1)^{\oplus(n-3)}$ by Remark 3. We have $N_X|X_i \cong N_{Y,\mathbf{P}^n}$ because X and Y coincide in a neighborhood of X_i. Hence we reduce part (b) to the case $n = 3$ and $d = 3$ and hence $i = 2$. In this case X is contained in a smooth quadric surface S as a divisor of type $(2, 1)$, the line X_2 being of type $(0, 1)$ on S. Hence $N_{X,S}|X_2 \cong \mathcal{O}_{X_2}(2)$ and $N_{S,\mathbf{P}^3} \cong \mathcal{O}_S(2)$. Restricting to X_2 the normal bundle exact sequence

$$0 \to N_{X,S} \to N_{X,\mathbf{P}^3} \to N_{S,\mathbf{P}^3}|X \to 0\,, \tag{1}$$

we obtain that $N_{X,\mathbf{P}^3}|X_2$ is an extension of $\mathcal{O}_{X_2}(2)$ by itself. Since $X_2 \cong \mathbf{P}^1$, this extension splits. □

Example 1. Fix integers $n \geq 3$ and $d \geq 3$. Let $X = X_1 \cup \cdots \cup X_d$ be a compatible ordering of a bamboo. Fix an index i with $1 < i < d$ and set $T := X_i$. There are two isomorphism classes of vector bundles on T obtained from $\mathcal{O}_T(1)^{\oplus(n-1)}$ making two positive elementary transformations: $\mathcal{O}_T(2)^{\oplus 2} \oplus \mathcal{O}_T(1)^{\oplus(n-3)}$ and $\mathcal{O}_T(3) \oplus \mathcal{O}_T(1)^{\oplus(n-2)}$. By Theorem 1 both isomorphism classes arise as restrictions to T of the normal bundle of a phantom bamboo in $\mathbf{P}^n$. By Proposition 1 the second isomorphism class does not arise as $N_X|T$ for some embedding of T in $\mathbf{P}^n$.

Remark 4. Let $X \subset \mathbf{P}^n$, $n \geq 3$, be a degree d bamboo and $X = X_1 \cup \cdots \cup X_d$ a compatible ordering of its lines. It is easy to check by induction on d that $\mathrm{Pic}(X) \cong \mathbf{Z}^{\oplus d}$ and that each line bundle on X is uniquely determined by the degree of its restriction to each X_i. Every vector bundle on X is a direct sum of line bundles (see [3], Proposition 3.1).

Proposition 2. *Fix integers* $n \geq 2$ *and* $d \geq 1$ *and a type* τ*. Let* $(b_1, \ldots, b_d)$ *be the normal sequence of* τ*. Fix integers* $a_{i,j}$*,* $1 \leq i \leq d$*,* $1 \leq j \leq n-1$*, such that* $a_{i,1} \geq \cdots \geq a_{i,n-1} \geq 1$*,* $a_{i,n-1} \geq a_{i,1} - 1$ *and* $a_{i,1} + \cdots + a_{i,n-1} = n - 1 + b_i$ *for all* i *with* $1 \leq i \leq d$*. Then there exists* $X = X_1 \cup \cdots \cup X_d \in \mathcal{B}(n, d, \tau)$ *such that* $N_X|X_i \cong \mathcal{O}_{X_i}(a_{i,1}) \oplus \cdots \oplus \mathcal{O}_{X_i}(a_{i,n-1})$ *for all* i *with* $1 \leq i \leq d$*.*

Proof. Just follow the proof of Theorem 1, taking as R a general line through $\phi(P)$. Thus $R \cap \phi(D) = \{P\}$ and the injectivity of ψ (obtained by the inductive assumption) implies the injectivity of ϕ. □

Proposition 3. *Let $Y \subset \mathbf{P}^n$ be a reduced and connected curve and N_Y its normal sheaf. The following conditions are equivalent:*

(a) Y is contained in a hyperplane of $\mathbf{P}^n$;

(b) N_Y has $\mathcal{O}_Y(1))$ as a direct factor;

(c) there is a morphism of sheaves $f : N_Y \to \mathcal{O}_Y(1)$ which is non-zero at the general point of each irreducible component of Y;

(d) there is a morphism $f : N_Y \to \mathcal{O}_Y(1)$ such that Coker(f) *has finite support.*

Proof. Since $\mathcal{O}_Y(1)$ is a line bundle, (c) and (d) are equivalent. Obviously (b) implies (c). If Y is contained in a hyperplane H, then $N_Y \cong N_{Y,H} \oplus \mathcal{O}_Y(1)$ and hence (b) holds. Assume (d). Consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)} \to T\mathbf{P}^n \to 0\,. \tag{2}$$

Since Y is reduced, it has only finitely many singular points and hence the natural map $u : T\mathbf{P}^n \to N_Y$ is surjective outside finitely many points. Restricting (2) to Y we obtain a map $f : \mathcal{O}_Y(1)^{\oplus(n+1)} \to \mathcal{O}_Y(1)$ which is surjective outside finitely many points. Since Y is reduced and connected, f is induced by $n+1$ constants $a_0, \dots, a_n$, not all zero. Choose homogeneous coordinates $z_0, \dots, z_n$ on $\mathbf{P}^n$. The map $\gamma : \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)}$ in (2) is defined by $\gamma(x) = (xz_0, \dots, xz_n)$. Since Y is connected and $u \circ (\gamma|Y) \equiv 0$, Y is contained in the hyperplane $\{a_0z_0 + \cdots + a_nz_n = 0\}$. Hence (d) implies (a). □

3. Postulation

Let $T \subset \mathbf{P}^n$ be a closed subscheme. We will say that T has *maximal rank* if for all integers $t \geq 0$ the restriction map $\rho_{X,n,t} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(T, \mathcal{O}_T(t))$ has maximal rank, i.e. it is injective or surjective. If $\dim(T) \leq 1$ and there is an integer k such that $h^1(T, \mathcal{O}_T(k-1)) = 0$ and $H^1(\mathbf{P}^n, \mathcal{I}_T(k)) = 0$ then $H^1(\mathbf{P}^n, \mathcal{I}_T(t)) = 0$ for all $t > k$ (Castelnuovo-Mumford Lemma). Hence if we also assume $h^0(\mathbf{P}^n, \mathcal{I}_T(k-1)) = 0$, then the scheme T has maximal rank.

Proposition 4. *Let $X \subset \mathbf{P}^3$ be a degree 5 tree. Then:*

(a) for every integer $t \geq 3$ the restriction map $\rho_{X,3,t}$ is surjective;

(b) *X has maximal rank if of only if X is not contained in a smooth quadric surface;*

(c) *X is contained in a smooth quadric surface if and only if $h^0(\mathbf{P}^3, \mathcal{I}_X(2)) = 0$;*

(d) *if X is a bamboo, then it has maximal rank;*

(e) *define $\tau_1 : \{2,3,4\} \to \{1,2,3\}$ by $\tau_1(i) = 1$ for all i; if X is not a bamboo, then $X \in \mathcal{B}(3,5,\tau_1)$; there are $Y, Z \in \mathcal{B}(3,5,\tau_1)$ such that Y has maximal rank, while Z has not maximal rank.*

Proof. Let $X_1 \cup \cdots \cup X_5$ a good ordering of the lines of X and H the plane spanned by $X_1 \cup X_2$. Since X is a tree, H contains no line X_i with $i > 2$. Hence $X_3 \cup X_4 \cup X_5$ is the residual scheme of X with respect to the Cartier divisor H of $\mathbf{P}^3$. Since X is connected, $X_3 \cap H = X_3 \cap (X_1 \cup X_2)$. Hence the scheme $X \cap H$ is the union of $X_1 \cup X_2$ and a zero-dimensional scheme of lenght at most two. Thus $\rho_{X \cap H,2,3}$ is surjective. It is easy to check that $\rho_{X_3 \cup X_4 \cup X_5,3,3}$ is surjective. Hence $\rho_{X,3,3}$ is surjective by Horace Lemma. By Castelnuovo-Mumford Lemma $\rho_{X,3,x}$ is surjective for every $x > 3$, proving part (a). We have $h^0(X, \mathcal{O}_X(2)) = 11$ (Riemann-Roch) and $h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) = 10$. Hence $\rho_{X,3,2}$ is not surjective. Hence if $\rho_{X,3,2}$ is not injective, then X has not maximal rank. If $\rho_{X,3,2}$ is injective, X has maximal rank by part (a). Any plane contains at most two lines of a tree. Every line of an irreducible quadric cone contains the vertex of the cone and hence any quadric cone contains at most two lines of a tree. Hence if a quadric surface S contains a degree 5 tree, then S is smooth and T is a divisor of type $(4,1)$ or of type $(1,4)$ on T. No such divisor is a bamboo, but there are divisors of type $(4,1)$ and divisors of type $(1,4)$ which are trees. Any type for degree 5 abstract trees is either the type of a bamboo or the type τ_1 or the type τ_2 defined by $\tau_2(2) = 1$, $\tau_2(3) = \tau_2(4) = \tau_2(5) = 2$. Since $\mathcal{A}(3,5,\tau_1) = \mathcal{A}(3,5,\tau_2)$, we have $\mathcal{B}(3,5,\tau_1) = \mathcal{B}(3,5,\tau_2)$. Hence all the assertions of the Proposition are true. □

The proof of part (a) of Proposition 4 gives the following result (hint: use induction on the integer n and Horace Lemma with respect to a hyperplane).

Proposition 5. *Let $X \subset \mathbf{P}^n$ be a non-degenerate degree $n+2$ tree. For every integer $t \geq 3$ the restriction map $\rho_{X,3,t}$ is surjective.*

Lemma 1. *Let $X \subset \mathbf{P}^n$, $n \geq 4$, be a degree $n+2$ tree. If there is a linear subspace M of $\mathbf{P}^n$ such that $\dim(M) = 3$ and M contains at least 5 lines of X, then X has not maximal rank.*

Proof. Call T (resp. Y) the union of the lines of X contained (resp. not contained) in M. For every integer x we have the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_X(x) \to \mathcal{O}_Y(x) \oplus \mathcal{O}_T(x) \to \mathcal{O}_{Y\cap T}(x) \to 0\,. \tag{3}$$

Since $p_a(X) = 0$, every connected component T has arithmetic genus zero. Call c the number of the connected components of T. Since $p_a(X) = 0$, we easily see from (3) that the restriction map $H^0(X, \mathcal{O}_X(2)) \to H^0(T, \mathcal{O}_T(2))$ is surjective. Hence if $\rho_{X,n,2}$ is surjective, then $\rho_{T,3,2}$ is surjective. Since $h^0(M, \mathcal{O}_M(2)) = 10 < 2(\deg(T)) + c = h^0(T, \mathcal{O}_T(2))$ (Riemann-Roch), $\rho_{T,3,2}$ is not surjective and hence $\rho_{X,n,2}$ is not surjective. Since $n \geq 4$, we have $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2)) = (n+2)(n+1)/2 > 2(n+2)+1 = h^0(X, \mathcal{O}_X(2))$. Thus X has not maximal rank. □

It seems to be very long and not interesting to list all non-degenerate degree $n+2$ trees $X \subset \mathbf{P}^n$, $n \geq 4$, which have not maximal rank. We only state the following example.

Example 2. Let $X \subset \mathbf{P}^4$ be a degree 6 bamboo such that no 5 lines of it are contained in a hyperplane. Choose a compatible ordering $X_1, \dots, X_6$ of the lines of X and call H the hyperplane spanned by $X_1 \cup X_2 \cup X_3$. First assume $X_4 \subset H$. Call A (resp. B) the plane spanned by $X_1 \cup X_2$ (resp. $X_3 \cup X_4$). The scheme-theoretic intersection $X \cap H$ is the disjoint union of $X_1 \cup X_2 \cup X_3 \cup X_4$ and the point $X_6 \cap H$, while $X_5 \cup X_6$ is the residual scheme of X with respect to H. Hence applying Horace Lemma with respect to H and Riemann-Roch to the scheme $X \cap H$ we see that $\rho_{X,4,2}$ is surjective (i.e. by Proposition 5, X has maximal rank) if and only if $X_6 \cap H \notin A \cup B$. Now assume that X_4 is not contained in H but X_5 is contained in H. Since X is a bamboo and no 5 lines of X are contained in H, then $X \cap H = X_1 \cup X_2 \cup X_3 \cup X_5$. Since $X_1 \cup X_2 \cup X_3 \cup X_5$ is not contained in a quadric surface of H, using Horace Lemma we obtain that X has maximal rank. Similarly, if H contains X_6 but neither X_4 nor X_5, then X has maximal rank. Now assume that H contains neither X_4 nor X_5 nor X_6: Since X is a bamboo, the scheme $X \cap H$ is the disjoint union of $X_1 \cup X_2 \cup X_3$ and the lenght two scheme $H \cap (X_5 \cup X_6)$. Using the exact sequence (3) with $T := X_1 \cup X_2 \cup X_3$ we obtain the surjectivity of the restriction map $H^0(X, \mathcal{O}_X(2)) \to H^0(X \cap H, \mathcal{O}_{X\cap H}(2))$. Hence using Horace Lemma we see that $\rho_{X,4,2}$ is surjective (i.e. X has maximal rank) if and only if $X \cap H$ is contained in a unique quadric surface. This is the case for most (but not for all) $X_5 \cup X_6$.

4. Hyperplane Sections

Proposition 6. *Fix an integer $d \geq 5$, say $d = 3a + b$ with $0 \leq b \leq 2$. Let $X \subset \mathbf{P}^3$ be a degree d bamboo and D a line not contained in X. Then* lenght$(X \cap D) \leq 2a + b$ *(scheme-theoretic intersection). Conversely, for every integer x with $4 \leq x \leq 2a + b$ there is a line $L \subset$ and a degree d bamboo such that L intersects Y transversally at exactly x points; if $x \geq 5$ and $a \geq 3$, we may even find (Y, L) so that for every line $R \neq L$, $R \subseteq Y$, we have* length$(R \cap Y) < x$.

Proof. Fix a compatible ordering $X_1, \ldots, X_d$ of the lines of X. For every $1 \leq i < d$ the lines X_i and X_{i+1} are coplanar by nthe very definition of bamboo. If length$(D \cap (X_i \cup X_{i+1}) \geq 2$, then D is contained in the plane $\langle X_i \cup X_{i+1} \rangle$ spanned by $X_i \cup X_{i+1}$ and length$(D \cap (X_i \cup X_{i+1}) = 2$. Furthermore, if $i \leq d-2$ we have $D \cap X_{i+2} = \emptyset$. Hence lenght$(X \cap D \leq 2a + b$. Starting from $i = 2$ the same trick (L intersecting X_1, X_2, X_4, X_5, but not $X_3 \cup X_5 \cup \ldots$ and so on) show the existence of the pair (Y, L). The last assertion can be checked because if L intersects 3 disjoint lines A, A', A'' of Y, then it lies on the unique smooth quadric surface containing $A \cup A' \cup A''$, while if L intersects two lines A, B with $A \neq B$ and $A \cap B \neq \emptyset$, then $L \subset \langle A \cup B \rangle$. □

Remark 5. Fix integers $d \geq 5$ and $t > 2d/3$ and a zero-dimensional scheme $Z \subset \mathbf{P}^2$ such that $h^1(\mathbf{P}^2, \mathcal{I}_Z(t)) \neq 0$. By [2], Remarks (i)(1) at p. 116, there is a line $D \subset \mathbf{P}^2$ such that length$(D \cap Z) \geq t + 1$.

Fix an integer $d \geq 4$ and define $\tau_0 : \{2, \ldots, d\} \to \{1, \ldots, d-1\}$, $\tau_1 : \{2, \ldots, d\} \to \{1, \ldots, d-1\}$ and $\tau_2 : \{2, \ldots, d\} \to \{1, \ldots, d-1\}$ by the relations $\tau_0(i) = i - 1$ for every i, $\tau_1(i) = 1$ for every i, $\tau_2(2) = 1$ and $\tau_2(j) = 2$ for every j with $3 \leq j \leq d$. Hence τ_0 is the type of a degree d bamboo, while $\mathcal{A}(d, \tau_1) = \mathcal{A}(d, \tau_2)$ (exchange X_1 and X_2).

Remark 6. Let τ be a type of degree $d \geq 2$ abstract trees and $X \in \mathcal{A}(d, \tau)$ with associated compatible ordering $X_1, \ldots, X_d$ of the irreducible components of X. Let $\beta(X)$ be the maximal number of mutually disjoint irreducible components of X. Set $\beta(\tau) := \beta(X)$. The integer $\beta(\tau)$ is well-defined because it depends only on τ, not on the choice of X. If $\mathcal{A}(d, \tau) = \mathcal{A}(d, \eta)$, then $\beta(\tau) = \beta(\eta)$. We have $\beta(\tau_0) = [(d + 1)/2]$ and $\beta(\tau_1) = \beta(\tau_2) = d - 1$. Use that for any compatible ordering $X_1, \ldots, X_d$ of the irreducible components of X the curves X_1 and $X_1 \cup X_2$ are connected, to check te following assertions:

(a) $\mathcal{A}(d, \tau) = \mathcal{A}(d, \tau_1)$ if and only if either $\tau = \tau_1$ or $\tau = \tau_2$;

(b) $\beta(\tau) \leq d - 1$;

(c) $\beta(\tau) = d - 1$ if and only if either $\tau = \tau_1$ or $\tau = \tau_2$.

Proposition 7. *Let $X \in \mathcal{B}(3, d, \tau)$, $d \geq 5$, be a tree such that there are infinitely many lines $D_i \subset \mathbf{P}^3$, $i \in I$, with each D_i not an irreducible component of X. Then $\mathcal{B}(3, d, \tau) = \mathcal{B}(3, d, \tau_1)$, X is contained in a smooth quadric surface S as a divisor of type $(d - 1, 1)$ or $(1, d - 1)$ and $D_i \subset S$ for every $i \in I$.*

Proof. First assume that there is at least 3 mutually disjoint lines among the lines D_i, $i \in I$, say D, D' and D''. There is a unique quadric surface S containing $D \cup D' \cup D''$ and S is smooth. By Bezout Theorem every irreducible component of X is contained in S. Since X is connected and $p_a(X) = 0$, X is a divisor of type $(1, d - 1)$ or a divisor of type $(d - 1, 1)$ on S and we conclude in this case. If no such 3 mutually disjoint lines do not exist, then there is an infinite subset J oof I such that $D_i \cap D_j \neq \emptyset$ for all $i, j \in J$. It is well-known and easy to check that this implies that one of the following cases occur:

(a) there is a plane M such that $D_i \subset M$ for every $i \in J$;

(b) there is $P \in \mathbf{P}^3$ such that $P \in D_i$ for every $i \in J$.

Assume that we are in case (a). Then $X \subset M$ by Bezout Theorem, contradicting the inequality $d > 2$. Assume that we are in case (b). In a surface with vertex P different from a plane any two lines, A, B, meets only at P and any line meeting bothe A and B either contains P or it is in the plane $\langle A \cup B \rangle$. Since $d \geq 4$ and X has only ordinary nodes as singularities, even case (b) is impossible, proving the exitence of the 3 mutually disjoint lines D, D', D''. $\square$

In the same way we obtain the following result.

Proposition 8. *Let $X \in \mathcal{B}(3, d, \tau)$, $d \geq 5$, such that there are 3 disjoint lines $D_i \subset \mathbf{P}^3$, $1 \leq i \leq 3$, with each D_i not an irreducible component of X. Then $\mathcal{B}(3, d, \tau) = \mathcal{B}(3, d, \tau_1)$, X is contained in a smooth quadric surface S as a divisor of type $(d-1, 1)$ or $(1, d-1)$ and S is the only quadric surface containing $D_1 \cup D_2 \cup D_3$.*

Remark 7. Fix $X \in \mathcal{B}(3, d, \tau)$, $d \geq 5$. Any line $D \subset \mathbf{P}^3$ not in X is contained in a pencil of planes and the general such plane does not contain any line of X. If D intersects quasi-transversally X, i.e. if D does not contains any singular point of X, then the general plane in this pencil is transversal to X. For any smooth quadric surface S there is a two-dimensional irreducible family of planes of $\mathbf{P}^3$ containing one of the lines of type $(1, 0)$ of S. By Remark 5 the following conditions are equivalent:

(a) X is contained in a quadric surface;

(b) X is contained in a smooth quadric surface;

(c) there is a two-dimensional irreducible family Ψ of planes of $\mathbf{P}^3$ such that a general $M \in \Psi$ does not contains any line of X and $h^0(M, \mathcal{I}_{X\cap M,M}(d-3)) \neq 0$;

(d) there is a two-dimensional irreducible family Ψ of planes of $\mathbf{P}^3$ such that a general $M \in \Psi$ is transversal to X and $h^0(M, \mathcal{I}_{X\cap M,M}(d-3)) \neq 0$.

Furthermore, if any of the previous equivalent conditions is satisfied, then $\mathcal{B}(3,d,\tau) = \mathcal{B}(3,d,\tau_1)$.

5. Restricted Tangent Bundle

Fix integers $n \geq 2$ and $d \geq 1$. Let (X, L, V) be a degree d phantom tree of $\mathbf{P}^n$ and $\phi : X \to \mathbf{P}^n$ the associated unramified morphism. The rank n vector bundle $\phi^*(T\mathbf{P}^n)$ will be called the *restricted tangent bundle* of ϕ or or (X, L, V) or, just, of X. For every irreducible component D of X the vector bundle $\phi^*(T\mathbf{P}^n)|D \cong (\phi|D)^*(T\mathbf{P}^n)$ has splitting type $(1, 0, \dots, 0)$. Hence $(\phi|D)^*(T\mathbf{P}^n)$ has a unique degree 2 line subbundle. Assume $d \geq 2$ and fix irreducible components D, R of X such that $D \neq R$ and $D \cap R \neq \emptyset$. We will say that $(\phi|D)^*(T\mathbf{P}^n)$ and $(\phi|R)^*(T\mathbf{P}^n)$ *glue together* if the degree two line subbundle of $(\phi|D)^*(T\mathbf{P}^n)$ and $(\phi|R)^*(T\mathbf{P}^n)$ induce a degree 4 line subbundle of $D \cup R$, i.e. if and only if $h^0(D \cup R, \phi^*(T\mathbf{P}^n(-2))|D \cup R) \neq 0$.

Proposition 9. *Fix integers $n \geq 2$ and $d \geq 2$. Let (X, L, V) be a degree d phantom tree of $\mathbf{P}^n$ and $\phi : X \to \mathbf{P}^n$ the associated unramified morphism. Then:*

(a) for all irreducible components D, R of X such that $D \neq R$ and $D \cap R \neq \emptyset$ the vector bundles $(\phi|D)^*(T\mathbf{P}^n)$ and $(\phi|R)^*(T\mathbf{P}^n)$ does not glue together;

(b) $h^0(X, \phi^*(T\mathbf{P}^n(-2))) = 0$ and $\phi^*(T\mathbf{P}^n(-1))$ is spanned by its global sections.

Proof. Let $A, B \subset \mathbf{P}^n$ be lines such that $A \neq B$ and $A \cap B \neq \emptyset$. Set $M := \langle A \cup B \rangle$. We have $T\mathbf{P}^n|A \cup B \cong TM|A \cup B \oplus \mathcal{O}_{A\cup B}(1)$. Hence to prove part (a) it is sufficient to prove that the two degree 2 line subbundles of $TM|A$ and $TM|B$ does not glue together to a degree 4 line subbundle of

$TM|A \cup B$, i.e, $h^0(A \cup B, TM(-2)|A \cup B) = 0$ Set $\{P\} := A \cap B$. Assume $h^0(A \cup B, TM(-2)|A \cup B) \neq 0$ and take $\sigma_B \in H^0(A \cup B, TM(-2)|A \cup B))$ with $\sigma_B \neq 0$. Thus σ_B is unique up to a non-zero multiplicative constant and it induces a uniquely determined degree zero line subbundle L_B of $TM(-2)|A \cup B$. Now we fix A and $\{P\}$ and vary B among all lines $\neq A$ and passing through P. All reducible conics are projectively equivalent and hence for any such line B' we obtain a line subbundle L'_B. These rank one subbundles glue together to a degree zero line subbundle of $TM(-2)$. Since $h^0(M, TM(-2)) = 0$ (e.g. by the Euler Sequence (2)), we obtained a contradiction and prove part (a). By part (a) we obtain $h^0(X, \phi^*(T\mathbf{P}^n(-2))) = 0$. The vector bundle $\phi^*(T\mathbf{P}^n(-1))$ is spanned by its global sections because $T\mathbf{P}^n(-1)$ is spanned by its global sections. □

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