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ON TANGENT BUNDLES OF CERTAIN HOMOGENEOUS SPACES

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Abstract: We define spaces X_n and Y_n as follows. First embed $\mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$ by

$$A + \sqrt{-1}B \in \mathrm{U}(2) \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{SO}(4).$$

For $n \geq 4$, we set $X_n = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{SU}(2)}$ and $Y_n = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{U}(2)}$. We first determine $c(Y_n)$, the total Chern class of the tangent bundle of Y_n. The result gives $w(X_n)$ and $p(X_n)$, the total Stiefel-Whitney class and the total Pontrjagin class of the tangent bundle of X_n, respectively. We also calculate $w(X_n)$ from the Wu formula. The result about $p(X_n)$ implies that X_n is parallelizable if and only if $n = 4$ or 5.

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1. Introduction

For a compact connected Lie group G, denote by $M(k, G)$ the framed moduli space of G-instantons over S^4 with instanton number k. There is an inclusion $M(k, G) \hookrightarrow \Omega^3_k G$. The algebra $H_*(\Omega^3_0 G; \mathbf{Z}/p)$ (where p is a prime) is known for all G, but the homology $H_*(M(k, G); \mathbf{Z}/p)$ is still difficult. In [4], a method of

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constructing new homology classes in $H_*(M(k,G);\mathbf{Z}/p)$ was introduced. The idea for this is as follows: First let $C = C_G(\mathrm{SU}(2))$ be the centralizer of a certain $\mathrm{SU}(2)$ in G. Then we have $M(1,G) \simeq G/C$. Next we have a loop sum Pontrjagin product $* : M(k,G) \times M(l,G) \to M(k+l,G)$ and the homology operations $Q_i : H_q(M(k,G);\mathbf{Z}/p) \to H_{pq+i(p-1)}(M(pk,G);\mathbf{Z}/p)$ $(i=1,2)$ which are compatible with those defined in $\Omega^3 G$ (see [4]). Applying the homology operations on elements of $H_*(M(1,G);\mathbf{Z}/p)$ iteratively and constructing the loop sum of these elements, one obtains new elements in $H_*(M(k,G);\mathbf{Z}/p)$.

Thus to study the topology of G/C has a certain significance. Note that in order to obtain $M(1,G)$, one need to choose an appropriate embedding $\mathrm{SU}(2) \hookrightarrow G$. When $G = \mathrm{SU}(n)$ or $\mathrm{Sp}(n)$, G/C is given as follows (see [4]): When $G = \mathrm{SU}(n)$, G/C is homeomorphic to the unit tangent bundle of $\mathbf{C}P^{n-1}$; and when $G = \mathrm{Sp}(n)$, $G/C \cong \mathbf{R}P^{4n-1}$.

In [5], the case for $G = \mathrm{SO}(n)$ was considered. In this case $M(k,G)$ consists of real instantons and G/C is given by the following X_n: First embed $\mathrm{SU}(2) \hookrightarrow \mathrm{SO}(4)$ by $A + \sqrt{-1}B \in \mathrm{SU}(2) \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{SO}(4)$. For $n \geq 4$, we set

$$X_n = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{SU}(2)}.$$

The space X_n is a compact orientable manifold of dimension $4n-13$. One has the following examples: $X_4 \cong \mathbf{R}P^3$; $X_5 \cong \mathbf{R}P^7$; and X_6 is homeomorphic to the unit tangent bundle of $\mathbf{C}P^3$ (we can prove these examples using the isomorphisms $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ and $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$). The purpose of this paper is to determine $w(X_n)$ and $p(X_n)$, the total Stiefel-Whitney class and the total Pontrjagin class of the tangent bundle of X_n, respectively. For that purpose, we define a space Y_n as follows.

$$Y_n = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{U}(2)},$$

where the embedding $\mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$ is defined to be the extension of the embedding $\mathrm{SU}(2) \hookrightarrow \mathrm{SO}(4)$. There is a fiber bundle $S^1 \to X_n \to Y_n$. Since $\mathrm{SO}(n-4) \times \mathrm{U}(2)$ is the centralizer of a certain S^1 in $\mathrm{SO}(n)$, Y_n is a homogeneous Kähler manifold, Y_n has no torsion and its Betti numbers vanish in odd dimensions (see [1, 3]). For $n = 1$ or 2, let $c_i \in H^{2i}(Y_n;\mathbf{Z})$ be the Chern class of the vector bundle associated to the principal bundle $\mathrm{U}(2) \to \mathrm{SO}(n)/\mathrm{SO}(n-4) \to Y_n$. Now, $c(Y_n)$, the total Chern class of the tangent bundle of Y_n, is given by the following theorem.

Theorem A.

$$c(Y_n) = \frac{(1+c_1+c_2)^n}{(1+c_1)(1-c_1^2+4c_2)(1+2c_1+4c_2)} \quad \textit{in } H^*(Y_n;\mathbf{Z}).$$

From Theorem A, we obtain the following main theorem.

Theorem B. (i) $w(X_n) = (1+c_2)^n$ *in* $H^*(X_n;\mathbf{Z}/2)$.
(ii) $p(X_n) = \dfrac{(1-c_2)^{2n}}{(1-4c_2)^4}$ *in* $H^*(X_n;\mathbf{Z})$.

Remark. The cohomology ring $H^*(X_n;\mathbf{Z}/p)$ was determined in [5] (see also Section 2 for $p=2$). In particular, the element c_2 has the following properties:
(1) We set $t = \left[\frac{n}{4}\right]$. Then we have $c_2^{t-1} \neq 0$ but $c_2^t = 0$ in $H^*(X_n;\mathbf{Z}/2)$.
(2) We set $m = \left[\frac{n}{2}\right]$. Then we have $c_2^{m-2} \neq 0$ but $c_2^{m-1} = 0$ in $H^*(X_n;\mathbf{Q})$.

We can prove Theorem B (i) alternatively by applying the Wu formula to the following theorem.

Theorem C. *Let* $v_k \in H^k(X_n;\mathbf{Z}/2)$ *be the Wu class see* ([6]). *Then*

$$v_k = \begin{cases} \binom{n-1-i}{i} c_2^i & k = 4i\,, \\ 0 & k \not\equiv 0 \pmod 4. \end{cases}$$

Theorem B (i) implies the following corollary.

Corollary D. *The class* $w(X_n)$ *is equal to* 1 *if and only if* n *is one of the following cases*: (a) $n = 2^f$ $(f \geq 2)$; (b) $n = 3\cdot 2^f$ $(f \geq 1)$; *or* (c) $n = 5, 7, 10$.

Theorem B (ii) implies the following corollary.

Corollary E. (i) $p_1(X_n) = (-2n+16)c_2$ *and* $p_2(X_n) = (2n^2-33n+160)c_2^2$.
(ii) *The manifold* X_n *is parallelizable if and only if* $n = 4$ *or* 5.

Remark. The classifying map $f_n : X_n \to \mathrm{BSU}\,(2)$ is a homotopy equivalence up to a certain dimension which tends to ∞ as $n \to \infty$. Hence, it would be interesting to compare Theorem B (ii) with the fact that $p(\mathbf{H}P^m) = \frac{(1+u)^{2m+2}}{1+4u}$, where $u \in H^4(\mathbf{H}P^m;\mathbf{Z})$ is a generator (see [6, p. 248]).

2. Proofs

Proof of Theorem A. We prove Theorem A using Theorem 10.8 in [2]. First we consider the case $n = 2m$. Then we have the following result about roots:

Positive roots of SO $(2m)$: $\pm x_i + x_j$ $(1 \leq i < j \leq m)$

Positive roots of SO $(2m-4) \times$ U (2): $\pm x_i + x_j$ $(1 \leq i < j \leq m-2)$, $-x_{m-1} + x_m$

Complementary roots:

$$\pm x_i + x_{m-1}, \pm x_i + x_m, x_{m-1} + x_m, \tag{1}$$

where $1 \leq i \leq m-2$. From Theorem 10.8 in [2], we have

$$\begin{aligned} c(Y_{2m}) = \Bigg[\prod_{i=1}^{m-2} & (1 + x_i + x_{m-1})(1 - x_i + x_{m-1}) \\ & \times (1 + x_i + x_m)(1 - x_i + x_m)\Bigg](1 + x_{m-1} + x_m). \end{aligned} \tag{2}$$

Since the symmetric polynomials in $x_1^2, \dots, x_m^2$ are zero in $H^*(Y_{2m}; \mathbf{Z})$, we have

$$\prod_{i=1}^{m} (z + x_i)(z - x_i) = z^{2m}, \tag{3}$$

where z is an indeterminate. We replace in (3) the indeterminate z by $1 + x_{m-1}$ or $1 + x_m$, and substitute the results for (2). Then we obtain a description of $c(Y_{2m})$ in terms of x_{m-1} and x_m. Finally, setting $c_1 = x_{m-1} + x_m$ and $c_2 = x_{m-1}x_m$, we obtain Theorem A for $n = 2m$.

For the case $n = 2m+1$, the roots of SO $(2m+1)$ complementary to SO $(2m-3) \times$ U (2) are the roots in (1), x_{m-1} and x_m. Hence

$$\begin{aligned} c(Y_{2m+1}) &= c(Y_{2m})(1 + x_{m-1})(1 + x_m) \\ &= c(Y_{2m})(1 + c_1 + c_2). \end{aligned}$$

Thus we obtain Theorem A for $n = 2m+1$. This completes the proof of Theorem A. □

Proof of Theorem C. From the result of [5], $H^*(X_n; \mathbf{Z}/2)$ has the form

$$H^*(X_n; \mathbf{Z}/2) \cong \mathbf{Z}/2[c_2]/(c_2^t) \otimes \Delta(e_1, e_2, e_3),$$

where $t = \left[\frac{n}{4}\right]$ and $\Delta(e_1, e_2, e_3)$ is a module with a simple system of generators $\{e_1, e_2, e_3\}$. Moreover, all squaring operations are determined. In particular, $Sq^a c_2 = 0$ for $a = 1, 2$ or 3. For simplicity, we write the result only for $n = 4t+3$.

(i) $\deg e_1 = 4t$, $\deg e_2 = 4t+1$ and $\deg e_3 = 4t+2$.

(ii) $Sq^1 e_r = \begin{cases} e_3 & r = 2\,, \\ 0 & r = 1, 3. \end{cases}$

(iii) $Sq^2 e_r = \begin{cases} c_2 e_1 & r = 3\,, \\ 0 & r = 1, 2. \end{cases}$

(iv) $Sq^{4j} e_r = \binom{t+1}{j} c_2^j e_r$ $(1 \le r \le 3)$.

Now we prove Theorem C for the case $n = 4t+3$. From (i)-(iv), the Wu class may be nonzero only if $Sq^{4i}(c_2^{t-1-i} e_1 e_2 e_3)$ is nonzero. That is, $v_k = 0$ if $k \not\equiv 0 \pmod 4$ and

$$v_{4i} = \sum_{0 \le \alpha, \beta, \gamma} \binom{t-1-i}{i-(\alpha+\beta+\gamma)} \binom{t+1}{\alpha} \binom{t+1}{\beta} \binom{t+1}{\gamma} c_2^i.$$

Since the right-hand side is the term of c_2^i in

$$(1+c_2)^{t-1-i}(1+c_2)^{t+1}(1+c_2)^{t+1}(1+c_2)^{t+1} = (1+c_2)^{n-1-i},$$

Theorem C holds for $n = 4t+3$. The cases for $n = 4t, 4t+1$ or $4t+2$ can be proved similarly. This completes the proof of Theorem C. □

Alternative Proof of Theorem B (i). Applying the Wu formula ([6, p. 132]) to Theorem C, we have

$$\begin{aligned} w(X_n) &= \sum_{0 \le i,j} \binom{i}{j} \binom{n-1-i}{i} c_2^{i+j} \qquad (4) \\ &= \sum_{0 \le i} \binom{n-1-i}{i} (c_2 + c_2^2)^i. \end{aligned}$$

It is easy to prove the following lemma.

Lemma 2.1. *In* $\mathbf{Z}[x]$, *we have*

$$\sum_{i=0}^{[\frac{\alpha}{2}]} \binom{\alpha - i}{i} x^i = \frac{(1+\sqrt{1+4x})^{\alpha+1} - (1-\sqrt{1+4x})^{\alpha+1}}{2^{\alpha+1}\sqrt{1+4x}}.$$

Putting $\alpha = n-1$ and $x = c_2 + c_2^2$ in Lemma 2.1, we have

$$(4) = \frac{(1+c_2)^n + (-1)^{n+1} c_2^n}{1+2c_2}.$$

Since $c_2^n = 0$ in $H^*(X_n; \mathbf{Z}/2)$, Theorem B (i) follows. □

References

[1] A. Borel, Kählerian coset spaces of semisimple Lie groups, *Proc. Nat. Acad. Sci. U.S.A.*, **40** (1954), 1147–1151.

[2] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces, I, *Amer. J. Math.*, **80** (1958), 458–538.

[3] R. Bott, An application of the Morse theory to the topology of Lie-groups, *Bull. Soc. Math. France*, **84** (1956), 251–281.

[4] C. P. Boyer, B. M. Mann, D. Waggoner, On the homology of SU (n) instantons, *Trans. Amer. Math. Soc.*, **323** (1991), 529–561.

[5] Y. Kamiyama, A. Kono and M. Tezuka, Cohomology of the moduli space of SO (n)-instantons with instanton number 1, *Proceedings of International Conference on Topology in Matsue (Matsue, 2002), Topology Appl.*, To Appear.

[6] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton Univ. Press (1974).