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Δ-LACUNARY STRONG A-CONVERGENT VECTOR VALUED DIFFERENCE SEQUENCES WITH RESPECT TO A SEQUENCE OF ORLICZ FUNCTIONS AND SOME INCLUSION RELATIONS

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Abstract: The purpose of this paper is to introduce and study some vector valued difference sequence spaces which are defined by combining sequence of Orlicz functions and using the concepts of lacunary convergence and strong A-convergence, where A=(a_{ik}) is an infinite matrix of complex numbers. We study also some topological properties of these spaces and establish some inclusion relations between these spaces.

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1. Introduction

The space of lacunary strong convergence have been introduced by Freedman et al. [2]. A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0,\ 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1,\ k_r}]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ is defined by

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Freedman et. al. [2] as follows:

$$N_\theta = \Big\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \mid x_i - s \mid = 0, \text{ for some s } \Big\}.$$

The space $\mid \sigma_1 \mid$ of strongly Cesaro summable sequences is

$$\mid \sigma_1 \mid = \Big\{ x = (x_k) : \text{ there exists } L \text{ such that } n^{-1} \sum_{i=1}^{n} \mid x_i - L \mid \to 0 \Big\}.$$

In case, when $\theta = (2^r)$, $N_\theta = \mid \sigma_1 \mid$.

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$, which is continuous, nondecreasing and convex with M(0)=0, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then the function reduces to a modulus function defined and discussed by Ruckle, Maddox and many others.

An Orlicz function M can always be represented (see Kranoselskii and Rutitsky [12]) in the integral form M(x)=$\int_0^x q(t)dt$, where q, known as the Kernel of M, is right differentiable for $t \geq 0$, q(0)=0, $q(t) > 0$ for $t > 0$, $q(t) \to \infty$ as $t \to \infty$.

An Orlicz function is said to satisfy Δ_2-condition (see Kranoselskii and Rutitsky [12]) for all values of x, if there exists a constant K $>$ 0, such that M(2x) $\leq$ KM(x) for all x $\geq$0.

This condition is equivalent to M(Lx)$\leq$ K(L)M(x), for all values of x $\geq$0 and for all L $\geq$ 1. Also an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct the sequence space

$$l_M = \Big\{ x \in \omega : \sum_{k=1}^{\infty} \Big(M(\frac{\mid x_k \mid}{\rho}) \Big) < \infty, \text{ for some } \rho > 0 \Big\}.$$

The space l_M, with the norm

$$\parallel x \parallel = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} \Big(M(\frac{\mid x_k \mid}{\rho}) \Big) \leq 1 \Big\},$$

becomes a Banach space which is called an Orlicz sequence space, where ω is the family of real or complex sequences. Mursaleen and Khan [15], Ahmad [1], Parashar and Choudhary [16], Bektas and Altin [3], Tripathy [18], Bilgin [5], and others used Orlicz function to construct several new sequence spaces.

Recently, the concept of lacunary convergence was generalized by Bilgin [4] as follows:

Let A=(a_{ik}) be an infinite matrix of complex numbers C and Ax= $(A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each i. Let f be a modulus function.Then

$$N_0(A,f) = \Big\{x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f(\mid A_i(x) - s \mid) = 0, \text{ for some s}\Big\}.$$

Later Bilgin [5] generalized lacunary strongly A-convergent sequences with respect to a sequence of modulus functions $F = (f_i)$ as follows:

$$N_0(A,F) = \Big\{x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} f_i(\mid A_i(x) - s \mid) = 0, \text{ for some s }\Big\}.$$

We now introduce the concept of lacunary strongly A-convergent for difference sequences with the elements chosen from a Banach space $(E, \| \,.\, \|)$ over the complex field C, with respect to a sequence of Orlicz functions (M_i).

Let M be the space of sequence of Orlicz functions M=(M_i) such that

$$\lim_{u\to 0} \sup_i M_i\Big(\frac{u}{\rho^{(i)}}\Big) = 0 \text{ for some } \rho^{(i)} > 0$$

Definition 1.1. Let A=(a_{ik}) be an infinite matrix of complex numbers and M=(M_i) be a sequence of Orlicz functions in M. Let $(E, \| \,.\, \|)$ be a Banach space over the complex field. We write $A_i(\Delta x_k) = \sum_{k=1}^{\infty} a_{ik}(x_k - x_{k+1})$, which converges for each i.

We introduce the spaces

$$\triangle N_\theta(E,A,M) = \Big\{x = (x_k) : x_k \in E \text{ and } \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) = 0 \text{ for some } s = (s_1, s_2, \ldots) \in E,\ s_i \in C \text{ and } \rho^{(i)} > 0\Big\}$$

and

$$\triangle N_\theta^0(E,A,M) = \Big\{x = (x_k) : x_k \in E, \text{ and } \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) \|}{\rho^{(i)}}\Big) = 0 \text{ for some} \rho^{(i)} > 0\Big\},$$

where

$$e_i = \begin{cases} 1 & \text{at the i-th place }, \\ 0 & \text{otherwise}. \end{cases}$$

Throughout the article we use this sequence (e_i).

A sequence $x = (x_k)$ is said to be Δ-lacunary strong A-convergent with respect to M if there is a number $s = (s_1, s_2, \dots) \in E$, such that $x = (x_k) \in \triangle N_\theta(E, A, M)$. We denote it as $(x_k) \xrightarrow{\Delta} s$. We also denote $\triangle N_\theta(E, A, M) = \triangle N_\theta(E, A)$ for M(t)=t for $t > 0$.

We have generalized strongly Cesaro-summable sequence space to Δ-strongly Cesaro-summable vector valued sequence space as

$$| \triangle\sigma_1(A) | = \Big\{ x = (x_k) : \text{ there exists } L = (L_1, L_2, \dots) \in E, L_i \in C \\ \text{such that } n^{-1} \sum_{i=1}^{n} \| A_i(\Delta x_k) - L_i e_i \| \to 0 \Big\},$$

where A=(a_{nk}) is a Cesaro matrix, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{if } 1 \le k \le n \ , \\ 0 & \text{if } k > n \, . \end{cases}$$

Then it can be shown that $| \triangle\sigma_1(A) |$ is a paranormed space with respect to the paranorm

$$\| x \|_\star = \sup_n \Big(n^{-1} \sum_{i=1}^{n} \| A_i(\Delta x_k) \| \Big) + \| x_1 \| \ .$$

We have established some simple topological properties of the sequence spaces $\triangle N_\theta^0(E, A, M)$ and $\triangle N_\theta(E, A, M)$.

2. Results

Theorem 2.1. *$\triangle N_\theta(E, A, M)$ is a linear space.*

Proof. Suppose that $x = (x_k), y = (y_k) \in \triangle N_\theta(E, A, M)$ and $(x_k) \xrightarrow{\Delta} s$, $(y_k) \xrightarrow{\Delta} m$, where $s = (s_1, s_2, \dots), m = (m_1, m_2, \dots) \in E$, s_i, $m_i \in C$.

Then

$$\lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i \Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho_1^{(i)}} \Big) = 0$$

and

$$\lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i \Big(\frac{\| A_i(\Delta y_k) - m_i e_i \|}{\rho_2^{(i)}} \Big) = 0.$$

Let a, b are in C, the set of complex numbers.

Without loss of generality we may assume that $\exists\ P_1 > 1, P_2 > 1$ such that $\mid a \mid \leq P_1$and $\mid b \mid \leq P_2$.

Let $\rho^{(i)} = \max(2\rho_1^{(i)},\ 2\rho_2^{(i)})$.

Then

$$
\begin{aligned}
& \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i\Big(\frac{\parallel A_i(a\Delta x_k + b\Delta y_k) - (as_ie_i + bm_ie_i) \parallel}{\rho^{(i)}}\Big) \\
\leq \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} M_i\Big(\frac{\parallel aA_i(\Delta x_k) - as_ie_i \parallel + \parallel bA_i(\Delta y_k) - bm_ie_i) \parallel}{\rho^{(i)}}\Big) \\
\leq \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\parallel a(A_i(\Delta x_k) - s_ie_i) \parallel}{\rho_1^{(i)}}\Big) \\
+ \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\parallel a(A_i(\Delta y_k) - m_ie_i) \parallel}{\rho_2^{(i)}}\Big) \\
= \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\mid a \mid \parallel A_i(\Delta x_k) - s_ie_i \parallel}{\rho_1^{(i)}}\Big) \\
+ \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\mid b \mid \parallel A_i(\Delta y_k) - m_ie_i \parallel}{\rho_2^{(i)}}\Big) \\
\leq \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{P_1 \parallel A_i(\Delta x_k) - s_ie_i \parallel}{\rho_1^{(i)}}\Big) \\
+ \quad & \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{P_2 \parallel A_i(\Delta y_k) - m_ie_i \parallel}{\rho_2^{(i)}}\Big) \\
& \qquad\qquad\qquad\qquad \text{(since } M_i \text{ is non-decreasing)} \\
\leq \quad & K_1 \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\parallel A_i(\Delta x_k) - s_ie_i \parallel}{\rho_1^{(i)}}\Big) \\
+ \quad & K_2 \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} \frac{1}{2} M_i\Big(\frac{\parallel A_i(\Delta y_k) - m_ie_i \parallel}{\rho_2^{(i)}}\Big), \\
& \qquad\qquad \text{where } K_1 \equiv K_1(P_1) \text{ and } K_2 \equiv K_2(P_2)
\end{aligned}
$$

(since M_i satisfies Δ_2 condition and $L_1 > 1, L_2 > 1$).

Therefore, $(\mathrm{a}x_k + by_k) \xrightarrow{\Delta} as + bm$ in $\triangle N_\theta(E, A, M)$.

Similarly it can be shown that $\triangle N_\theta^0(E, A, M)$ is also linear space. □

Theorem 2.2. *$\triangle N_\theta(E,A,M)$ is normal space, when E is normal.*

Proof. Let $x=(x_k)\in\triangle N_\theta(E,A,M)$ and $(x_k)\xrightarrow{\Delta} s$, where $s=(s_1,s_2,\dots)\in E,\ s_i\in C$. Let $\| y_k \|\le\| x_k \|$.

Then

$$\| A_i(\Delta y_k)-s_ie_i \|\le\| A_i(\Delta x_k)-s \| \ .$$

Since M_i is increasing

$$h_r^{-1}\sum_{i\in I_r}M_i\Big(\frac{\| A_i(\Delta y_k)-s_ie_i \|}{\rho^{(i)}}\Big)$$

$$\le h_r^{-1}\sum_{i\in I_r}M_i\Big(\frac{\| A_i(\Delta x_k)-s_ie_i \|}{\rho^{(i)}}\Big)\,.$$

Consequently $y=(y_k)\in\triangle N_\theta(E,A,M)$.

Hence the proof.

Similarly we can prove that $\triangle N_\theta^0(E,A,M)$ is normal. □

Theorem 2.3. *The spaces $\triangle N_\theta(E,A,M)$ and $\triangle N_\theta^0(E,A,M)$ are paranormed spaces, with respect to the paranorm*

$$\| x \|_\triangle=\inf\Big\{\rho^{(i)}>0:M_i\Big(\frac{\| a_{i0}x_1 \|}{\rho^{(i)}}\Big)+\sup_{r\ge1}h_r^{-1}\sum_{i\in I_r}M_i\Big(\frac{\| A_i(\Delta x_k) \|}{\rho^{(i)}}\Big)\le 1,\rho^{(i)}\ge 0\Big\}.$$

Proof. The proof is easy, so we omit it. □

3. Relation between $\triangle N_\theta(E,A)$ and $\triangle N_\theta(E,A,M)$

Theorem 3.1. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers and M=(M_i) be a sequence of Orlicz functions satisfying $\triangle_2$ condition. If $x=(x_k)$ is Δ-lacunary strong A-convergent to s then $x=(x_k)$ Δ-lacunary strong A-convergent to s with respect to M i.e., $\triangle N_\theta(E,A)\subset\triangle N_\theta(E,A,M)$, where (E,$\|\ .\ \|$) is a normal Banach space.*

Proof. Let $x=(x_k)\in\triangle N_\theta(E,A)$ and $(x_k)\xrightarrow{\Delta} s$, where $s=(s_1,s_2,\dots)\in E,\ s_i\in C$. Then

$$\lim_{r\to\infty}h_r^{-1}\sum_{i\in I_r}\frac{\| A_i(\Delta x_k)-s_ie_i \|}{\rho}=0,\ \text{for some } \rho>0.$$

We define two sequences $y = (y_k)$ and $z = (z_k)$ such that

$$\| A_i(\Delta y_k) - s_i e_i \| = \begin{cases} \| A_i(\Delta x_k) - s_i e_i \| & \text{if } \| A_i(\Delta x_k) - s_i e_i \| > 1, \\ \theta & \text{if } \| A_i(\Delta x_k) - s_i e_i \| \leq 1, \end{cases}$$

and

$$\| A_i(\Delta z_k) - s_i e_i \| = \begin{cases} \theta & \text{if } \| A_i(\Delta x_k) - s_i e_i \| > 1, \\ \| A_i(\Delta x_k) - s_i e_i \| & \text{if } \| A_i(\Delta x_k) - s_i e_i \| \leq 1. \end{cases}$$

Hence

$$\| A_i(\Delta x_k) - s_i e_i \| = \| A_i(\Delta y_k) - s_i e_i \| + \| A_i(\Delta z_k) - s_i e_i \| .$$

Now,

$$\| A_i(\Delta y_k) - s_i e_i \| \leq \| A_i(\Delta x_k) - s_i e_i \|$$

and

$$\| A_i(\Delta z_k) - s_i e_i \| \leq \| A_i(\Delta x_k) - s_i e_i \| .$$

Since $\triangle N_\theta(E, A)$ is normal, $y = (y_k), z = (z_k) \in \triangle N_\theta(E, A)$.

Put $\sup_i M_i(2) = T$. Then

$$\begin{aligned}
& h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) - s \|}{\rho^{(i)}}\Big) \\
= \; & h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| A_i(\Delta y_k) - s_i e_i \| + \| A_i(\Delta z_k) - s \|}{\rho^{(i)}}\Big) \\
\leq \; & h_r^{-1} \sum_{i \in I_r} \Big[\frac{1}{2} M_i\Big(2 \frac{\| A_i(\Delta y_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\
+ \; & \frac{1}{2} M_i\Big(2 \frac{\| A_i(\Delta z_k) - s_i e_i \|}{\rho^{(i)}}\Big)\Big] \\
< \; & \frac{1}{2} h_r^{-1} \sum_{i \in I_r} K_1\Big(\frac{\| A_i(\Delta y_k) - s_i e_i \|}{\rho^{(i)}}\Big) M_i\Big(2\Big) \\
+ \; & \frac{1}{2} h_r^{-1} \sum_{i \in I_r} \frac{\| A_i(\Delta z_k) - s_i e_i \|}{\rho^{(i)}} M_i\Big(2\Big), \\
& \qquad\qquad\qquad\qquad (\text{ since } M_i \text{ satisfies } \Delta^2 \text{ condition}) \\
\leq \; & \frac{1}{2} h_r^{-1} \sum_{i \in I_r} K_1\Big(\frac{\| A_i(\Delta y_k) - s_i e_i \|}{\rho^{(i)}}\Big) \sup M_i\Big(2\Big) \\
+ \; & \frac{1}{2} h_r^{-1} \sum_{i \in I_r} \frac{\| A_i(\Delta z_k) - s_i e_i \|}{\rho^{(i)}} \sup M_i\Big(2\Big) \\
\to \; & 0 \text{ as } r \to \infty \text{ (by construction).}
\end{aligned}$$

Hence $x = (x_k) \in \triangle N_\theta(E, A, M)$.

Hence the proof. □

Theorem 3.2. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers and M=(M_i) be a sequence of Orlicz functions satisfying $\triangle_2$ condition. If*

$$\lim_{u\to\infty} inf_i \frac{M_i\Big(\frac{u}{\rho^{(i)}}\Big)}{\frac{u}{\rho^{(i)}}} > 0 \text{ for some } \rho^{(i)} > 0,$$

then $\triangle N_\theta(E, A) = \triangle N_\theta(E, A, M)$.

Proof. If $\lim_{u\to\infty} inf_i \frac{M_i\Big(\frac{u}{\rho^{(i)}}\Big)}{\frac{u}{\rho^{(i)}}} > 0$ then $\exists$ a number $\beta > 0$ such that

$$M_i\Big(\frac{u}{\rho^{(i)}}\Big) \geq \beta\Big(\frac{u}{\rho^{(i)}}\Big) \text{ for all } u > 0 \text{ and some } \rho^{(i)} > 0.$$

Let $x = (x_k) \in \triangle N_\theta(E, A, M)$ and $(x_k) \xrightarrow{\Delta} s$, where $s = (s_1, s_2, \ldots) \in E,\ s_i \in C$.

Then clearly

$$\begin{aligned} h_r^{-1}\sum_{i\in I_r} M_i\Big(\frac{\| A_i(\triangle x_k) - s_i e_i \|}{\rho^{(i)}}\Big) &\geq h_r^{-1}\sum_{i\in I_r} \beta\Big(\frac{\| A_i(\triangle x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ &= \beta h_r^{-1}\sum_{i\in I_r}\Big(\frac{\| A_i(\triangle x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \end{aligned}$$

Hence $x = (x_k) \in \triangle N_\theta(E, A)$. Hence the proof. □

4. Relation between $\mid \triangle\sigma_1(A) \mid$ and $\triangle N_\theta(E, A)$

Lemma 4.1. $\mid \triangle\sigma_1(A) \mid \subset \triangle N_0(E, A)$ *if and only if* $\liminf_r q_r > 1$.

Proof. **The condition is sufficient:**

Let us assume that $\liminf_r q_r > 1$. Then there exist $\delta > 0$ such that $1 + \delta \leq q_r$ for all $r \geq 1$. Let $x \in \mid \triangle\sigma_1(A) \mid^0$.

Then

$$\begin{aligned} h_r^{-1}\sum_{i\in I_r}\frac{\| A_i(\triangle x_k) \|}{\rho} &= h_r^{-1}\sum_{i=1}^{k_r}\frac{\| A_i(\triangle x_k) \|}{\rho} - h_r^{-1}\sum_{i=1}^{k_{r-1}} \| A_i(x_k) \| \\ &= \frac{k_r}{h_r}(k_r^{-1}\sum_{i=1}^{k_r} \| A_i(\triangle x_k) \|) - \frac{k_{r-1}}{h_r}\Big({k_{r-1}}^{-1}\sum_{i=1}^{k_{r-1}} \| A_i(\triangle x_k) \|\Big). \end{aligned}$$

Now, $h_r = k_r - k_{r-1}$.

So we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1} = 1 + \frac{1}{q_r - 1} \leq 1 + \frac{1}{\delta} = \frac{\delta + 1}{\delta}.$$

Also

$$\frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} = \frac{1}{q_r - 1} \leq \frac{1}{\delta}.$$

Since $x \in \mid \triangle\sigma_1(A) \mid^0$, both

$$h_r^{-1} \sum_{i=1}^{k_r} \parallel A_i(\triangle x_k) \parallel \to 0,$$

and

$$h_r^{-1} \sum_{i=1}^{k_{r-1}} \parallel A_i(\Delta x_k) \parallel \to 0,$$

and hence

$$h_r^{-1} \sum_{i \in I_r} \parallel A_i(\triangle x_k) \parallel \to 0$$

i.e., $x = (x_k) \in \triangle N_\theta^0(E, A)$. By linearity, it follows that $\mid \triangle\sigma_1(A) \mid \subset N_\theta(E, A)$.

The condition is necessary:

Assume that $\liminf q_r = 1$.

Since θ is lacunary we can select a subsequence (k_{r_j}) of θ such that

$$\frac{k_{r_j}}{k_{r_j - 1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r_j - 1}}{k_{r_{j-1}}} > j$$

where $r_j \geq r_{j-1} + 2$.

Define $x = (x_i)$ by

$$\triangle x_i = \begin{cases} e_i & \text{if } i \in I_{r_j}, \text{ for some j=1, 2, } \ldots, \\ \theta & \text{otherwise.} \end{cases}$$

where $\parallel e_i \parallel = 1$ and let A=I, then for any L=$(L_1, L_2, \ldots) \in E,\ L_i \in C$,

$$h_{r_j}^{-1} \sum_{i \in I_r} \Big(\frac{\parallel A_i(\triangle x_k) - L \parallel}{\rho} \Big) = \frac{\parallel e_i - L_i e_i \parallel}{\rho} = \frac{\parallel 1 - L_i \parallel}{\rho} \text{ for } j = 1, 2, \ldots$$

and

$$h_r^{-1} \sum_{i \in I_r} \Big(\frac{\parallel A_i(\triangle x_k) \parallel}{\rho} \Big) = \frac{\parallel e_i \parallel}{\rho} = \frac{1}{\rho}.$$

So, $x = (x_k) \notin \triangle N_\theta(E, A)$.

But $x = (x_k)$ is strongly Cesaro-summable , since if t is sufficiently large integer we can find the unique j for which $k_{r_j - 1} < t \leq k_{r_{j+1} - 1}$ and hence

$$t^{-1} \sum_{i=1}^{t} \| A_i(\triangle x_k) \| < \frac{1}{k_{r_j - 1}} \sum_{i=1}^{t} 1 \leq \frac{1}{k_{r_j - 1}} k_{r_j}$$
$$\leq \frac{k_{r_j - 1} + h_{r_j}}{k_{r_j - 1}} < \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

As $t \longrightarrow \infty$, it follows that also $j \longrightarrow \infty$. Hence $x = (x_k) \in | \triangle \sigma_1(A) |^0$. □

Lemma 4.2. $\triangle N_\theta(E, A) \subset | \triangle \sigma_1(A) |$ *if and only if* $\limsup_r q_r < \infty$.

Proof. **The condition is sufficient:**
If $\limsup_r q_r < \infty$, ∃ M >0 such that $q_r < M$ for all $r \geq 1$.
Let $x = (x_k) \in \triangle N_\theta(E, A)$ and $\epsilon > 0$. Then

$$\lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} \frac{\| A_i(\Delta x_k) \|}{\rho} = 0, \text{ for some } \rho > 0.$$

Then we can find $R > 0$ and $K > 0$ such that

$$\sup_{j \geq R} h_j^{-1} \sum_{I_j} \frac{\| A_i(\Delta x_k) \|}{\rho} < \epsilon$$

and

$$h_j^{-1} \sum_{I_j} \frac{\| A_i(\Delta x_k) \|}{\rho} < K \text{ for all } i = 1, 2, \ldots.$$

Then if t is any integer with

$$k_{r-1} \leq t \leq k_r, \text{ where } r > R,$$

then

$$
\begin{aligned}
& t^{-1}\sum_{j=1}^{t}\frac{\| A_i(\Delta x_k) \|}{\rho} \\
\leq\ & k_{r-1}^{-1}\sum_{i=1}^{k_r}\frac{\| A_i(\Delta x_k) \|}{\rho} \\
=& k_{r-1}^{-1}\Big(\sum_{I_1}\frac{\| A_i(\Delta x_k) \|}{\rho}\Big)+\sum_{I_2}\frac{\| A_i(\Delta x_k) \|}{\rho}+\ldots+\sum_{I_r}\frac{\| A_i(\Delta x_k) \|}{\rho} \\
=\ & \frac{k_1}{k_{r-1}}h_1^{-1}\sum_{I_1}\frac{\| A_i(\Delta x_k) \|}{\rho}+\frac{k_2-k_1}{k_{r-1}}h_2^{-1}\sum_{I_2}\frac{\| A_i(\Delta x_k) \|}{\rho}+\ldots \\
+& \frac{k_R-k_{R-1}}{k_{r-1}}h_R^{-1}\sum_{I_R}\frac{\| A_i(\Delta x_k) \|}{\rho}+\frac{k_{R+1}-k_R}{k_{r-1}}h_{R+1}^{-1}\sum_{I_{R+1}}\frac{\| A_i(\Delta x_k) \|}{\rho} \\
+\ & \ldots+\frac{k_r-k_{r-1}}{k_{r-1}}h_r^{-1}\sum_{I_r}\frac{\| A_i(\Delta x_k) \|}{\rho}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{k_R}{k_{r-1}}\sup_{i\geq 1}h_i^{-1}\sum_{I_i}\frac{\| A_i(\Delta x_k) \|}{\rho}+\frac{k_r-k_R}{k_{r-1}}h_r^{-1}\sum_{I_r}\frac{\| A_i(\Delta x_k) \|}{\rho} \\
<\ & K\frac{k_R}{k_{r-1}}+\epsilon\Big(q_r-\frac{k_R}{k_{r-1}}\Big) \\
<\ & K\frac{k_R}{k_{r-1}}+\epsilon q_r \\
<\ & K\frac{k_R}{k_{r-1}}+\epsilon M\,.
\end{aligned}
$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, it follows that

$$
t^{-1}\sum_{j=1}^{t}\frac{\| A_i(\Delta x_k) \|}{\rho}\to 0
$$

and hence $x=(x_k)\in|\triangle\sigma_1|^0$.

The condition is necessary:

We assume that $\limsup_r q_r=\infty$. We construct a sequence in $\triangle N_\theta(X,\ A\ )$ that is not Cesaro Δ-summable.Following the idea of Freedman [2] we can construct a subsequence k_{r_j} of the lacunary sequence $\theta=(k_r)$ such that $q_{r_j} > j$, and then define a bounded difference sequence $x = (x_i)$ by

$$\Delta x_i = \begin{cases} e_i & \text{if } k_{r_j-1} < i < 2k_{r_j-1}\,, \\ \theta & \text{otherwise,} \end{cases}$$

where $\| e_i \|= 1$. Let A=I and $\rho = 1$. Then

$$h_{r_j}{}^{-1} \sum_{I_{r_j}} \| A_i(\Delta x_k) \|= \frac{2k_{r_j-1} - k_{r_j-1}}{k_{r_j} - kr_{j-1}} = \frac{k_{r_j-1}}{k_{r_j} - kr_{j-1}} < \frac{1}{j-1}$$

and if $r \neq r_j$,

$$h_{r_j}{}^{-1} \sum_{I_{r_j}} \| A_i(\Delta x_k) \|= 0.$$

Thus $x = (x_k) \in \triangle N_\theta(E, A)$.

For the above sequence and for i=1,2,$\ldots, k_{r_j}$

$$\begin{aligned} & k_{r_j}{}^{-1} \sum_{i} \| A_i(\Delta x_k) - e_i \| \\ > \;& k_{r_j}{}^{-1}(2k_{r_j-1} - k_{r_j-1}) \\ = \;& 1 - \frac{2}{q_{r_j}} > 1 - \frac{2}{j}\,, \end{aligned}$$

this converges to 1, but for i=1,2,$\ldots, 2k_{r_j-1}$

$$2k_{r_j-1}{}^{-1} \sum_{i} \| A_i(\Delta x_k) \|\geq \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2}\,.$$

It proves that $x = (x_k) \notin \mid \triangle\sigma_1(A) \mid$, since any sequence in $\mid \triangle\sigma_1(A) \mid$ consisting of θ's and e_i's has an limit only 0 or 1. □

Combining these two lemmas we get

Theorem 4.1. *let θ be a lacunary sequence.Then $\mid \triangle\sigma_1(A) \mid = \triangle N_\theta(E, A)$ if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty\,.$$

5. Relation between $\Delta S_\theta(A)$ and $\triangle N_\theta(E, A, M)$

Statistical convergence for real and complex sequences was first introduced by H. Fast [8], Buck [6] and Schoenberg [17] independently. Fast extended the concept of sequential limit which he called statistical convergence. Schoenberg gave some basic properties of statistical convergence and studied the concept as summability method.

The basic concept of statistical convergence is based on the notion of natural density of sets A $\subseteq N = \{1, 2, \ldots, n, \ldots\}$. If A⊆N and A(n)=$|A \cap \{1, 2, \ldots, n, \ldots\}|$, where the vertical bar denotes the cardinality of the set, then

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

is called the natural density of A.

A sequence $x = (x_k)$ of complex numbers is said to converge statistically to L if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{k \le n :\ |x_k - L| \ge \epsilon\Big\}\Big| = 0.$$

Bilgin [5] also introduced the concept of statistical convergence in $N_0(A, F)$ and proved some inclusion relation. We have defined Δ-lacunary A-statistical convergence and proved similar type of inclusion relations in this section.

Definition 5.1. Let θ be a lacunary sequence and $A = (a_{ik})$ be an infinite matrix of complex numbers. Then a sequence $x = (x_k)$ in $\triangle N_\theta(E, A, M)$ is said to be a Δ-lacunary A-statistically convergent to a number $s = (s_1, s_2, \ldots) \in E,\ s_i \in C$ if for any $\epsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1}\Big|\Delta A_0(\epsilon)\Big| = 0\,,$$

where

$$\Delta A_0(\epsilon) = \Big\{i \in I_r : M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \ge \epsilon\Big\}.$$

We denote it as $(x_k) \xrightarrow{\Delta-stat} s$. The vertical bar denotes the cardinality of the set.

The set of all Δ-Lacunary A-statistical convergent sequences is denoted by $\Delta S_\theta(A)$.

Theorem 5.1. *Let $M = (M_i)$ be a sequence of Orlicz functions and (M_i) be pointwise convergent. Then $\triangle N_\theta(E, A, M) \subset \Delta S_0(A)$ if and only if $\lim_i M_i(\frac{u}{\rho^{(i)}}) > 0$ for some u>0, $\rho^{(i)} > 0$.*

Proof. Let $\epsilon > 0$ and $x = (x_k) \in \triangle N_\theta(E, A, M)$.

Let $(x_k) \xrightarrow{\Delta} s$, where $s = (s_1, s_2, \ldots) \in E$, $s_i \in C$. Since $\lim_i M_i(\frac{u}{\rho})$ > 0, there exists a number $c > 0$ such that

$$M_i(\frac{u}{\rho}) \geq c \text{ for } u > \epsilon.$$

Let

$$I_r^1 = \Big\{i \in I_r : M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \geq \epsilon\Big\}.$$

Then

$$\begin{aligned} & h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ \geq \quad & h_r^{-1} \sum_{i \in I_r^1} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ \geq \quad & c h_r^{-1} \Big| \Delta A_0(\epsilon) \Big| . \end{aligned}$$

Hence it follows that $x = (x_k) \in \Delta S_\theta(A)$.

Conversely, let us assume that the condition does not hold good. Then there is a number $u > 0$ such that $\lim_i M_i(\frac{u}{\rho}) = 0$ for some $\rho > 0$. Now, we Select a lacunary sequence $\theta = (k_r)$ such that $M_i(\frac{u}{\rho}) < 2^{-r}$ for any $i > k_r$.

Let A=I, define the sequence $x = (x_k)$ by putting

$$\Delta x_i = \begin{cases} u & \text{if } k_{r-1} < i \leq \frac{k_r + k_{r-1}}{2}, \\ \theta & \text{if } \frac{k_r + k_{r-1}}{2} < i \leq k_r. \end{cases}$$

Therefore,

$$\begin{aligned} & h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| \Delta x_k \|}{\rho^{(i)}}\Big) \\ = \quad & h_r^{-1} \sum_{k_{r-1} < i \leq \frac{k_r + k_{r-1}}{2}} M_i\Big(\frac{u}{\rho^{(i)}}\Big) \\ < \quad & h_r^{-1} \frac{1}{2^{r-1}} \Big[\frac{k_r + k_{r-1}}{2} - k_{r-1}\Big] \\ = \quad & \frac{1}{2^r} \to 0 \text{ as } r \to \infty . \end{aligned}$$

Then we have $x = (x_k) \in \triangle N_\theta^0(E, A, M)$.

But

$$\begin{aligned} & \lim_{r\to\infty} h_r^{-1}\Big|\Big\{i \in I_r : M_i\Big(\frac{\| \Delta x_i \|}{\rho^{(i)}}\Big) \geq \epsilon\Big\}\Big| \\ = & \lim_{r\to\infty} h_r^{-1}\Big|\Big\{i \in (k_{r-1}, \frac{k_r + k_{r-1}}{2}] : M_i\Big(\frac{u}{\rho^{(i)}}\Big) \geq \epsilon\Big\}\Big| \\ = & \lim_{r\to\infty} h_r^{-1}\frac{k_r - k_{r-1}}{2} = \frac{1}{2}\,. \end{aligned}$$

So $x = (x_k) \notin \Delta S_\theta(A)$. ☐

Theorem 5.2. *Let $M = (M_i)$ be a sequence of Orlicz functions. Then $\Delta S_\theta(A) \subset \triangle N_\theta(E, A, M)$ if and only if $\sup_u \sup_i M_i(\frac{u}{\rho}) < \infty$.*

Proof. Let $x = (x_k) \in \Delta S_\theta(A)$ and $(x_k) \stackrel{\Delta-stat}{\longrightarrow} s$.

Suppose $h(u) = \sup_i M_i(\frac{u}{\rho})$ and $h = \sup_u h(u)$.

Let

$$I_r^2 = \Big\{i \in I_r : M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) < \epsilon\Big\}.$$

Now, $M_i(u) \leq h$ for all i $u > 0$. So

$$\begin{aligned} & h_r^{-1}\sum_{i\in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ = & h_r^{-1}\sum_{i\in I_r^1} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ + & h_r^{-1}\sum_{i\in I_r^2} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) \\ \leq & h h_r^{-1}\Big|\Delta A_0(\epsilon)\Big| + h(\epsilon)\,. \end{aligned}$$

Hence as $\epsilon \to 0$, it follows that $x = (x_k) \in \triangle N_\theta(E, A, M)$.

Conversely assume that

$$\sup_u \sup_i M_i(\frac{u}{\rho}) = \infty.$$

Then we have

$$0 < u_1 < u_2 < \ldots < u_{r-1} < u_r < \ldots ,$$

so that $M_{k_r}(\frac{u_r}{\rho}) \geq h_r$ for $r \geq 1$.

Let A=I. We set a sequence $x = (x_i)$ by

$$\Delta x_i = \begin{cases} u_r & \text{if } i = k_r \text{ for some } r = 1, 2, \dots, \\ \theta & \text{otherwise.} \end{cases}$$

Then

$$\lim_{r\to\infty} h_r^{-1} \Big| \Big\{ i \in I_r : M_i \Big(\frac{\| \Delta x_i \|}{\rho^{(i)}} \Big) \geq \epsilon \Big\} \Big| = \lim_{r\to\infty} \frac{1}{h_r} = 0 \,.$$

Hence $(x_k) \stackrel{\Delta - stat}{\longrightarrow} 0$ and hence $x = (x_k) \in \Delta S_\theta(A)$.

But

$$\begin{aligned} & \lim_{r\to\infty} h_r^{-1} \sum_{i \in I_r} M_i \Big(\frac{\| \Delta x_i - s_i e_i \|}{\rho^{(i)}} \Big) \\ = & \lim_{r\to\infty} h_r^{-1} M_{k_r} \Big(\frac{\| u_r - s_i e_i \|}{\rho^{(i)}} \Big) \\ \geq & \lim_{r\to\infty} h_r^{-1} h_r = 1 \,. \end{aligned}$$

So, $x = (x_k) \notin \triangle N_\theta(E, A, M)$. □

References

[1] Z.U. Ahmad, Ahmad H.A. Bataineh, Some new sequences defined by Orlicz function, *The Aligarh Bull of Maths.*, **20**, No. 2 (2001).

[2] Allen R. Freedman, John J. Sember, Marc Raphael, Some Cesaro type summability spaces, *Proc. London Math. Soc.*, **37**, No. 3 (1978), 508-520.

[3] A. Bektas, Yavuz Altin, The sequence space $l_M(p, q, s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **34**, No. 4 (2003), 529-534.

[4] Tunay Bilgin, Lacunary strong A-convergence with respect to a modulus, *Mathematica XLVI*, **4** (2001), 39-46.

[5] Tunay Bilgin, Lacunary strong A-convergence with respect to a sequence of Modulus functions, *Appl. Math. Comput.*, To Appear (2003).

[6] R.C. Buck, Generalized asymptotic density, *Amer. Jour. Math.*, **75** (1953), 335-346.

[7] J. Connor, The statistical and strong p-Cesaro convergence of sequence, *Analysis*, **8** (1988), 47-63.

[8] H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241-244.

[9] J.A. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific Journal Of Mathematics*, **160**, No. 1 (1993), 43-51.

[10] Jinlu Li, Lacunary statistical convergence and inclusion properties between Lacunary methods, *Internat. J. Math. and Math. Sci.*, **23**, No. 3 (2000), 175-180.

[11] P.K. Kamthan, M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, New York (1981).

[12] M.A. Krasnosel'skli , Y.B. Rutickil, *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen, Netherlands (1961).

[13] J. Lindenstrauss , L. Tzafriri, Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[14] I.J. Maddox, *Elements of Functional Analysis*, Cambridge University press, Cambridge (1970).

[15] Mursaleen, Q.A. Khan, T.A. Chishti, Some new convergent sequences spaces defined by Orlicz functions and statistical convergence, *Italian J. Pure. Appl. Math.*, **9** (2001), 25-32.

[16] S.D. Parashar, B. Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure. Appl. Math.*, **25**, No. 4 (1994), 419-428.

[17] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361-375.

[18] B.C. Tripathy, Generalised difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex space, communicated (2003).