International Journal of Pure and Applied Mathematics
Volume 11 No. 4 2004, 377-399

STOCHASTIC DELAY POPULATION DYNAMICS

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Abstract: In this paper we stochastically perturb the delay Lotka-Volterra model $\dot{x}(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t) + Bx(t-\tau)]$ into the stochastic delay differential equation $dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))\ [(b + Ax(t) + Bx(t-\tau))dt + \beta dw(t)]$, and show that under certain conditions,the original delay equation and the associated stochastic delay equation behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will also show, under some other conditions, that although the solution to the original delay equation may be persistent, the solution to the associated stochastic delay equation will become extinct with probability one. This reveals the important fact that the environmental noise may make the population extinct.

AMS Subject Classification: 60J65, 60H10, 34K40
Key Words: Brownian motion, stochastic differential delay equation, ultimate boundedness, extinction, Itô formula

Received: December 9, 2003

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1. Introduction

The delay differential equation

$$\frac{dx(t)}{dt} = x(t)[\mu + \alpha x(t) + \delta x(t-\tau)] \tag{1.1}$$

has been used to model the population growth of certain species and is known as the delay Lotka-Volterra model or the delay logistic equation. The delay Lotka-Volterra model for n interacting species is described by the n-dimensional delay differential equation

$$\frac{dx(t)}{dt} = \text{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t) + Bx(t-\tau)], \tag{1.2}$$

where

$$x = (x_1, \cdots, x_n)^T, \quad b = (b_1, \cdots, b_n)^T, \quad A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n}.$$

There is an extensive literature concerned with the dynamics of this delay model and we here only mention Ahmad and Rao [1], Bereketoglu and Gyori [2], Freedman and Ruan [3], He and Gopalsamy [5], Kuang and Smith [8], Teng and Yu [14] among many others. In particular, the books by Gopalsamy [4], Kolmanovskii and Myshkis [6] as well as Kuang [7] are good references in this area.

In the equations above, the state $x(t)$ denotes the population sizes of the species. We are therefore not only interested in the positive solutions but also require the solutions not to explode at a finite time. To guarantee the positive solutions without explosion (i.e. the global positive solutions), some conditions are in general needed to impose on the system parameters. For example, it is generally assumed that $\mu > 0$, $\alpha < 0$ and $\delta < |\alpha|$ for equation (1.1) while much more complicated conditions are required on matrices A and B for equation (1.2) (see e.g. (2.3) and (2.16) below or [4, 6, 7] for more).

On the other hand, population systems are often subject to environmental noise. It is therefore useful to reveal how the noise affects on the delay population systems. It has been well known in the control theory that noise can not only have a destabilising effect but can also have a stabilising effect (see e.g. Mao [11]). It has also been revealed recently by Mao, Marion and Renshaw [13] that the environmental noise can suppress a potential population explosion. These indicate clearly that different structures of environmental noise may have different effects on the population systems. In this paper we consider the simple situation of the parameter perturbation. Recall that the parameter

b_i represents the intrinsic growth rate of species i. In practice we usually estimate it by an average value plus an error term. According to the well-known central limit theorem, the error term follows a normal distribution. In terms of mathematics, we can therefore replace the rate b_i by

$$b_i + \beta_i \dot{w}(t),$$

where $\dot{w}(t)$ is a white noise (i.e. $w(t)$ is a Brownian motion) and $\beta_i \geq 0$ represents the intensity of noise. As a result, equation (1.2) becomes the Itô stochastic differential delay equation

$$dx(t)=\text{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t) + Bx(t-\tau))dt + \beta dw(t)], \qquad (1.3)$$

where $\beta = (\beta_1, \cdots, \beta_n)^T$. Since this equation describes a stochastic population dynamics, it is critical to find out whether or not the solution:

- will remain positive or never become negative,
- will not explode to infinity in a finite time,
- will be ultimately bounded,
- will become extinct.

In this paper we will discuss these problems one by one. It is interesting to see from our results that under certain conditions, the original delay equation (1.2) and the associated stochastic delay equation (1.3) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will also show, under certain other conditions, that although the solution to the original delay equation is persistent, the solution to the associated stochastic delay equation will become extinct with probability one. This reveals the important fact that the environmental noise may make the population extinct.

2. Positive and Global Solutions

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). Let $w(t)$ denote a scalar Brownian motion defined on this probability space. We also denote by R^n_+ the positive cone in R^n, that is $R^n_+ = \{x \in R^n : x_i > 0 \text{ for all } 1 \le i \le n\}$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R^n_+)$ the family of continuous functions from $[-\tau, 0]$ to R^n_+. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$.

In this paper we will use a lot of quadratic functions of the form $x^T Ax$ for the state $x \in R^n_+$ only. Therefore, for a symmetric $n \times n$ matrix A, we naturally introduce the following definition

$$\lambda^+_{\max}(A) = \sup_{x \in R^n_+, |x|=1} x^T Ax.$$

Let us emphasise that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A but we will show in Appendix that $\lambda^+_{\max}(A)$ has some similar properties as $\lambda_{\max}(A)$ has. In particular, we often require $\lambda^+_{\max}(A) \le 0$ or $\lambda^+_{\max}(A) < 0$ in this paper so we will give some sufficient conditions for them in Appendix, too.

In this paper we consider the delay Lotka-Volterra model for a system with n interacting species, namely

$$\dot{x}_i(t) = x_i(t)\Bigg(b_i + \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}x_j(t-\tau)\Bigg) \quad (1 \le i \le n) \ ,$$

on $t \ge 0$. This takes the matrix form

$$\dot{x}(t) = \text{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t) + Bx(t-\tau)] \ , \tag{2.1}$$

where

$$x = (x_1, \cdots, x_n)^T, \quad b = (b_1, \cdots, b_n)^T, \quad A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n}.$$

Given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, equation (2.1) will have a unique local or global solution dependent on the parameter matrices A and B (see [4, 6, 7]).

In this classical model, the parameter b_i represents the intrinsic growth rate of species i. If one replaces this rate by an average value plus a random fluctuation term

$$b_i + \beta_i \dot{w}(t),$$

then equation (2.1) becomes a stochastic differential delay equation

$$dx(t) = \text{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t) + Bx(t-\tau))dt + \beta dw(t)], \tag{2.2}$$

where $\beta = (\beta_1, \cdots, \beta_n)^T$. As the i-th state $x_i(t)$ of equation (2.2) is the size of the i-th species in the system, it should be nonnegative. Moreover, in order for a stochastic differential delay equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Mao [9, 12]). However, the coefficients of equation (2.2) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of equation (2.2) may explode at a finite time. It is therefore useful to establish some conditions under which the solution of equation (2.2) is not only positive but will also not explode to infinite at any finite time.

Theorem 2.1. *Assume that there are positive numbers $c_1, \cdots, c_n$ and θ such that*

$$\lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C}) + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\Big) \le 0\,, \tag{2.3}$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$ and I is the $n \times n$ identity matrix. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, there is a unique solution $x(t)$ to equation (2.2) on $t \ge -\tau$ and the solution will remain in R^n_+ with probability 1, namely $x(t) \in R^n_+$ for all $t \ge -\tau$ almost surely.

In order to prove the theorem let us present a lemma.

Lemma 2.2. *The following inequality holds*

$$u \le 2(u - 1 - \log u) + 2\,, \quad \forall u > 0.$$

Proof. The inequality holds for $u \in (0, 2)$ because we always have $u - 1 - \log u \ge 0$ for $u > 0$. To show the inequality for $u \ge 2$, let us define

$$f(u) = u - 2\log u.$$

Note

$$\frac{df(u)}{du} = 1 - \frac{2}{u} \ge 0.$$

So $f(u)$ is nondecreasing on $u \ge 2$ and hence

$$2(u - 1 - \log u) + 2 - u = u - 2\log u = f(u) \ge f(2) = 2 - 2\log 2 > 0.$$

In other words, the desired inequality holds for $u \ge 2$ as well. □

Let us now begin to prove Theorem 2.1.

Proof of Theorem 2.1. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$

there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time (cf. Mao [10], p. 95). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau \le t \le 0} |x(t)| \le \max_{-\tau \le t \le 0} |x(t)| < k_0.$$

For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : x_i(t) \notin (1/k,\ k) \text{ for some } i = 1, \cdots, n\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \ge 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. To show this statement, let us define a C^2-function $V : R_+^n \to R_+$ by

$$V(x) = \sum_{i=1}^n c_i \Big[x_i - 1 - \log(x_i) \Big].$$

The non-negativity of this function can be seen from

$$u - 1 - \log(u) \ge 0 \quad \text{on } u > 0.$$

Let $k \ge k_0$ and $T > 0$ be arbitrary. For $0 \le t \le \tau_k \wedge T$, it is not difficult to show by the Itô formula that

$$dV(x(t)) = LV(x(t), x(t-\tau))dt + (x^T(t)\bar{C} - C)\beta dw(t), \tag{2.4}$$

where $C = (c_1, \cdots, c_n)$ and $LV : R_+^n \times R_+^n \to R$ is defined by

$$LV(x,y) = x^T\bar{C}b + x^T\bar{C}Ax + x^T\bar{C}By - C(b + Ax + By) + \frac{1}{2}\beta^T\bar{C}\beta. \tag{2.5}$$

Using condition (2.3) we compute

$$\begin{aligned} x^T\bar{C}Ax + x^T\bar{C}By &\le \frac{1}{2}x^T(\bar{C}A + A^T\bar{C})x + \frac{1}{4\theta}x^T\bar{C}BB^T\bar{B}x + \theta|y|^2 \\ &= x^T\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C}) + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\Big)x - \theta|x|^2 + \theta|y|^2 \\ &\le -\theta|x|^2 + \theta|y|^2. \end{aligned} \tag{2.6}$$

Moreover, there is clearly a constant $K_1 > 0$ such that

$$x^T \bar{C} b - C(b + Ax + By) + \frac{1}{2}\beta^T \bar{C} \beta \le K_1(1 + |x| + |y|).$$

Substituting these into (2.5) yields

$$LV(x, y) \le K_1(1 + |x| + |y|) - \theta|x|^2 + \theta|y|^2. \tag{2.7}$$

But for $x \in R^n_+$, we compute by Lemma 2.2 that

$$\begin{aligned} |x| &\le \sum_{i=1}^n x_i \le \sum_{i=1}^n [2(x_i - 1 - \log x_i) + 2] \\ &\le 2n + \frac{2}{\min_{1\le i\le n} c_i} \sum_{i=1}^n c_i(x_i - 1 - \log x_i) = 2n + \frac{2}{\min_{1\le i\le n} c_i} V(x). \end{aligned}$$

We therefore obtain from (2.7) that

$$LV(x, y) \le K_2(1 + V(x) + V(y)) - \theta|x|^2 + \theta|y|^2, \tag{2.8}$$

where K_2 is a positive constant. Substituting these into (2.4) yields

$$\begin{aligned} dV(x(t)) \le \big[&K_2(1 + V(x(t)) + V(x(t-\tau))) \\ &- \theta|x(t)|^2 + \theta|x(t-\tau)|^2\big]dt + (x^T(t)\bar{C} - C)\beta dw(t). \end{aligned} \tag{2.9}$$

Now, for any $t_1 \in [0, T]$, we can integrate both sides of (2.9) from 0 to $\tau_k \wedge t_1$ and then take the expectations to get

$$\begin{aligned} EV(x(\tau_k \wedge t_1)) \le V(x(0)) + E\int_0^{\tau_k\wedge t_1} \Big[&K_2(1 + V(x(t)) \\ &+ V(x(t-\tau))) - |x(t)|^2 + |x(t-\tau)|^2\Big]dt. \end{aligned} \tag{2.10}$$

Compute

$$\begin{aligned} E\int_0^{\tau_k\wedge t_1} V(x(t-\tau))dt &= E\int_{-\tau}^{\tau_k\wedge t_1 - \tau} V(x(t))dt \\ &\le \int_{-\tau}^0 V(x(t))dt + E\int_0^{\tau_k\wedge t_1} V(x(t))dt \end{aligned}$$

and, similarly

$$E\int_0^{\tau_k\wedge t_1} |x(t-\tau)|^2 dt \le \int_{-\tau}^0 |x(t)|^2 dt + E\int_0^{\tau_k\wedge t_1} |x(t)|^2 dt.$$

Substituting these into (2.10) gives

$$\begin{aligned} EV(x(\tau_k \wedge t_1)) &\leq K_3 + 2K_2 E \int_0^{\tau_k \wedge t_1} V(x(t))dt \\ &\leq K_3 + 2K_2 E \int_0^{t_1} V(x(\tau_k \wedge t))dt \\ &\leq K_3 + 2K_2 \int_0^{t_1} EV(x(\tau_k \wedge t))dt, \end{aligned} \tag{2.11}$$

where

$$K_3 = V(x(0) + K_2 T + K_2 \int_{-\tau}^0 V(x(t))dt + \theta \int_{-\tau}^0 |x(t)|^2 dt.$$

By the Gronwall inequality we then obtain that

$$EV(x(\tau_k \wedge T)) \leq K_3 e^{2TK_2}. \tag{2.12}$$

Note that for every $\omega \in \{\tau_k \leq T\}$, there is some i such that $x_i(\tau_k, \omega)$ equals either k or $1/k$, and hence $V(x(\tau_k, \omega))$ is no less than either

$$\hat{c}\Big[k - 1 - \log(k)\Big],$$

or

$$\hat{c}\Big[\frac{1}{k} - 1 - \log(\frac{1}{k})\Big] = \hat{c}\Big[\frac{1}{k} - 1 + \log(k)\Big],$$

where $\hat{c} = \min_{1 \leq i \leq n} c_i > 0$. Consequently,

$$V(x(\tau_k, \omega)) \geq \hat{c}\Big(\Big[k - 1 - \log(k)\Big] \wedge \Big[\frac{1}{k} - 1 + \log(k)\Big]\Big).$$

It then follows from (2.12) that

$$\begin{aligned} K_3 e^{2TK_2} &\geq E\Big[1_{\{\tau_k \leq T\}}(\omega) V(x(\tau_k, \omega))\Big] \\ &\geq \hat{c} P\{\tau_k \leq T\}\Big(\Big[k - 1 - \log(k)\Big] \wedge \Big[\frac{1}{k} - 1 + \log(k)\Big]\Big), \end{aligned}$$

where $1_{\{\tau_k \leq T\}}$ is the indicator function of $\{\tau_k \leq T\}$. Letting $k \to \infty$ gives

$$\lim_{k \to \infty} P\{\tau_k \leq T\} = 0$$

and hence

$$P\{\tau_\infty \leq T\} = 0.$$

Since $T > 0$ is arbitrary, we must have

$$P\{\tau_\infty < \infty\} = 0,$$

so $P\{\tau_\infty = \infty\} = 1$ as required. □

Let us now establish a useful corollary which we will need in Section 4 below.

Corollary 2.3. *Assume that there are positive numbers $c_1, \cdots, c_n$ such that*

$$\lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big) \le -\|\bar{C}B\|\,, \tag{2.13}$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$ as before. Then the conclusion of Theorem 2.1 holds.

Proof. By condition (2.13) we compute

$$\begin{aligned}
&\lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C}) + \frac{1}{2\|\bar{C}B\|}\bar{C}BB^T\bar{C} + \frac{1}{2}\|\bar{C}B\|I\Big)\\
&\quad\le \lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big) + \frac{1}{2\|\bar{C}B\|}\lambda^+_{\max}(\bar{C}BB^T\bar{C}) + \frac{1}{2}\|\bar{C}B\|\\
&\quad\le -\|\bar{C}B\| + \frac{1}{2}\|\bar{C}B\| + \frac{1}{2}\|\bar{C}B\| = 0.
\end{aligned}$$

That is, condition (2.3) holds with $\theta = \frac{1}{2}\|\bar{C}B\|$. The corollary hence follows from Theorem 2.1 directly. □

More usefully, we observe from the proof above that condition (2.3) is used to derive (2.7) from (2.5). But there are several different ways to estimate (2.5) which will lead to different alternative conditions for the positive global solution. For example, we know that

$$x^T\bar{C}By \le \frac{1}{2\theta}x^T\bar{C}x + \frac{\theta}{2}y^TB^T\bar{C}By$$

holds for any $\theta > 0$ so

$$\begin{aligned}
x^T\bar{C}Ax + x^T\bar{C}By &\le \frac{1}{2}x^T(\bar{C}A + A^T\bar{C})x + \frac{1}{2\theta}x^T\bar{C}x + \frac{\theta}{2}y^TB^T\bar{C}By\\
&= \frac{1}{2}x^T\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta B^T\bar{C}B\Big)x - \frac{\theta}{2}x^TB^T\bar{C}Bx + \frac{\theta}{2}y^TB^T\bar{C}By.
\end{aligned} \tag{2.14}$$

If we assume that

$$\lambda_{\max}^{+}\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta B^T\bar{C}B\Big) \le 0,$$

we will then have

$$x^T\bar{C}Ax + x^T\bar{C}By \le -\frac{\theta}{2}x^TB^T\bar{C}Bx + \frac{\theta}{2}y^TB^T\bar{C}By,$$

whence

$$LV(x,y) \le K_1(1+|x|+|y|) - \frac{\theta}{2}x^TB^T\bar{C}Bx + \frac{\theta}{2}y^TB^T\bar{C}By. \tag{2.15}$$

From this can we show in the same way as in the proof of Theorem 2.1 that the solution of equation (2.2) is positive and global. In other words, the arguments above give us an alternative result which we describe as a theorem below.

Theorem 2.4. *Assume that there are positive numbers $c_1, \cdots, c_n$ and θ such that the symmetric matrix*

$$\lambda_{\max}^{+}\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta B^T\bar{C}B\Big) \le 0, \tag{2.16}$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$ as before. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+^n)$, there is a unique solution $x(t)$ to equation (2.2) on $t \ge -\tau$ and the solution will remain in R_+^n with probability 1, namely $x(t) \in R_+^n$ for all $t \ge -\tau$ almost surely.

We leave the other alternatives to the reader. It is interesting to observe that the conditions imposed either in Theorem 2.1 or Theorem 2.4 are independent of the noise intensity vector β. This shows clearly that the property of the positive global solution of equation (2.2) will not change no matter the environmental noise is large or small. In other words, the property of the positive global solution of equation (2.2) is very robust under the noise.

3. Ultimate Boundedness

One of the important properties in a population system is the ultimate boundedness. For the solution $x(t)$ of the deterministic delay equation (2.1), the ultimate boundedness means that there is a positive constant K independent of the initial data such that

$$\limsup_{t\to\infty} |x(t)| \le K.$$

The most natural analogue for the stochastic delay differential equation (2.2) is

$$\limsup_{t\to\infty} E|x(t)| \le K.$$

In this case, equation (2.2) is said to be ultimately bounded in mean. The following theorem gives a criterion for this property.

Theorem 3.1. *Assume that there are positive numbers $c_1, \cdots, c_n$ and θ such that*

$$-\lambda := \lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C}) + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\Big) < 0, \tag{3.1}$$

where $\bar{C}$ and I are the same as in Theorem 2.1. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t)$ of equation (2.2) has the properties that

$$\limsup_{t\to\infty} E|x(t)| \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\gamma\lambda \min_{1\le i\le n} c_i} \tag{3.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(s)|^2 ds \le \frac{|\bar{C}b|^2}{\lambda^2}, \tag{3.3}$$

where $C = (c_1, \cdots, c_n)$ and

$$\gamma = \frac{1}{\tau}\log\Big(\frac{\lambda + 2\theta}{2\theta}\Big) > 0.$$

In particular, equation (2.2) is ultimately bounded in mean.

Proof. By Theorem 2.1, the solution $x(t)$ will remain in R^n_+ for all $t \ge -\tau$ with probability 1. Define

$$V(x) = Cx = \sum_{i=1}^n c_i x_i \quad \text{for } x \in R^n_+.$$

By the Itô formula, we have

$$dV(x(t)) = x^T(t)\bar{C}\big[(b + Ax(t) + Bx(t-\tau))dt + \beta dw(t)\big]. \tag{3.4}$$

By (2.6) we have

$$x^T(t)\bar{C}(Ax(t) + Bx(t-\tau)) \le -(\lambda + \theta)|x(t)|^2 + \theta|x(t-\tau)|^2.$$

It follows therefore from (3.4) that

$$dV(x(t)) \le \Big(|\bar{C}b||x(t)| - (\lambda+\theta)|x(t)|^2 + \theta|x(t-\tau)|^2\Big)dt \\ + x^T(t)\bar{C}\beta dw(t). \quad (3.5)$$

By the Itô formula once again, we have

$$\begin{aligned} d[e^{\gamma t}V(x(t))] &= e^{\gamma t}[\gamma V(x(t))dt + dV(x(t))] \\ &\le e^{\gamma t}\Big[(\gamma|C| + |\bar{C}b|)|x(t)| - (\lambda+\theta)|x(t)|^2 + \theta|x(t-\tau)|^2\Big]dt \\ &\quad + e^{\gamma t}x^T(t)\bar{C}\beta dw(t). \end{aligned}$$

This implies

$$e^{\gamma t}EV(x(t)) \le V(x(0)) + E\int_0^t e^{\gamma s}\Big[(\gamma|C| + |\bar{C}b|)|x(s)| \\ - (\lambda+\theta)|x(s)|^2 + \theta|x(s-\tau)|^2\Big]ds. \quad (3.6)$$

But

$$\begin{aligned} \int_0^t e^{\gamma s}|x(s-\tau)|^2 ds &= e^{\gamma\tau}\int_0^t e^{\gamma(s-\tau)}|x(s-\tau)|^2 ds \\ &= e^{\gamma\tau}\int_{-\tau}^{t-\tau} e^{\gamma(s)}|x(s)|^2 ds \le e^{\gamma\tau}\int_{-\tau}^0 |x(s)|^2 ds + e^{\gamma\tau}\int_0^t e^{\gamma(s)}|x(s)|^2 ds. \end{aligned}$$

Substituting this into (3.6) and noting from the definition of γ that

$$\lambda + \theta - \theta e^{\gamma\tau} = \lambda + \theta - \frac{\lambda + 2\theta}{2} = \frac{\lambda}{2},$$

we obtain that

$$e^{\gamma t}EV(x(t)) \le H + E\int_0^t e^{\gamma s}\Big[(\gamma|C| + |\bar{C}b|)|x(s)| - \frac{\lambda}{2}|x(s)|^2\Big]ds, \quad (3.7)$$

where

$$H = V(x(0)) + \theta e^{\gamma\tau}\int_{-\tau}^0 |x(s)|^2 ds.$$

Noting that

$$(\gamma|c| + |\bar{C}b|)|x(s)| - \frac{\lambda}{2}|x(s)|^2 \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\lambda},$$

we therefore have

$$\begin{aligned} e^{\gamma t} EV(x(t)) &\le H + \frac{(\gamma|C| + |\bar{C}b|)^2}{2\lambda} \int_0^t e^{\gamma s} ds \\ &= H + \frac{(\gamma|C| + |\bar{C}b|)^2}{2\gamma\lambda} \big[e^{\gamma t} - 1\big]. \end{aligned}$$

This yields

$$\limsup_{t\to\infty} EV(x(t)) \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\gamma\lambda}.$$

However

$$|x(t)| \le \sum_{i=1}^n x_i(t) \le \frac{V(x(t))}{\min_{1\le i\le n} c_i}.$$

Hence

$$\limsup_{t\to\infty} E|x(t)| \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\gamma\lambda \min_{1\le i\le n} c_i},$$

which is the required assertion (3.2).

To show the other assertion (3.3) we derive from (3.5) that

$$\begin{aligned} 0 \le EV(x(t)) &\le V(x(0)) \\ &\quad + E\int_0^t \Big(|\bar{C}b||x(s)| - (\lambda+\theta)|x(s)|^2 + \theta|x(s-\tau)|^2 \Big) ds. \end{aligned}$$

But

$$\int_0^t |x(s-\tau)|^2 ds \le \int_{-\tau}^0 |x(s)|^2 ds + \int_0^t |x(s)|^2 ds.$$

Hence

$$0 \le H_1 + E\int_0^t \Big(|\bar{C}b||x(s)| - \lambda|x(s)|^2 \Big) ds.$$

where $H_1 = V(x(0)) + \theta \int_{-\tau}^0 |x(s)|^2 ds$. This implies

$$\frac{\lambda}{2} \int_0^t E|x(s)|^2 ds \le H_1 + E\int_0^t \Big(|\bar{C}b||x(s)| - \frac{\lambda}{2}|x(s)|^2 \Big) ds.$$

Noting

$$|\bar{C}b||x(s)| - \frac{\lambda}{2}|x(s)|^2 \le \frac{|\bar{C}b|^2}{2\lambda},$$

we obtain

$$\frac{\lambda-\theta}{2} \int_0^t E|x(s)|^2 ds \le H_1 + \frac{|\bar{C}b|^2 t}{2\lambda}.$$

This implies immediately that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(s)|^2 ds \le \frac{|\bar{C}b|^2}{\lambda^2},$$

which is the desired assertion (3.3). □

Theorem 3.2. *Assume that there are positive numbers $c_1, \cdots, c_n$ and θ such that the symmetric matrix*

$$\lambda^+_{\max}\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta B^T\bar{C}B\Big) < 0, \tag{3.8}$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$ as before. Then there are two positive constants K_1 and K_2, which are independent of the noise intensity vector β, such that for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t)$ of equation (2.2) has the properties that

$$\limsup_{t\to\infty} E|x(t)| \le K_1 \tag{3.9}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(s)|^2 ds \le K_2. \tag{3.10}$$

In particular, equation (2.2) is ultimately bounded in mean.

Proof. By Theorem 2.4, the solution $x(t)$ will remain in R^n_+ for all $t \ge -\tau$ with probability 1. Let $V : R^n_+ \to R_+$ be the same as defined in the proof of Theorem 3.1 so we have (3.4). By condition (3.8) we can find a positive constant γ sufficiently small for

$$-\bar{\lambda} := \frac{1}{2}\lambda^+_{\max}\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta e^{\gamma\tau}B^T\bar{C}B\Big) < 0.$$

Note from (2.14) that, for any $(x, y) \in R^n_+ \times R^n_+$,

$$\begin{aligned} &x^T\bar{C}Ax + x^T\bar{C}By \\ &\quad \le \frac{1}{2}x^T\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta e^{\gamma\tau}B^T\bar{C}B\Big)x - \frac{\theta e^{\gamma\tau}}{2}x^TB^T\bar{C}Bx \\ &\qquad + \frac{\theta}{2}y^TB^T\bar{C}By \le -\bar{\lambda}|x|^2 - \frac{\theta e^{\gamma\tau}}{2}x^TB^T\bar{C}Bx + \frac{\theta}{2}y^TB^T\bar{C}By. \end{aligned}$$

Substituting this into (3.4) yields

$$\begin{aligned} dV(x(t)) &\le \Big[|\bar{C}b||x(t)| - \bar{\lambda}|x(t)|^2 - \frac{\theta e^{\gamma\tau}}{2}x^T(t)B^T\bar{C}Bx(t) \\ &+ \frac{\theta}{2}x^T(t-\tau)B^T\bar{C}Bx(t-\tau)\Big]dt + x^T(t)\bar{C}\beta dw(t). \end{aligned}$$

By the Itô formula, we have

$$
\begin{aligned}
& d[e^{\gamma t}V(x(t))] = e^{\gamma t}[\gamma V(x(t))dt + dV(x(t))] \\
\leq \ & e^{\gamma t}\Big[(\gamma|c| + |\bar{C}b|)|x(t)| - \bar{\lambda}|x(t)|^2 - \frac{\theta e^{\gamma\tau}}{2}x^T(t)B^T\bar{C}Bx(t) \\
+ \ & \frac{\theta}{2}x^T(t-\tau)B^T\bar{C}Bx(t-\tau)\Big]dt + e^{\gamma t}x^T(t)\bar{C}\beta dw(t).
\end{aligned}
$$

This implies

$$
\begin{aligned}
e^{\gamma t}EV(x(t)) \leq V(x(0)) + E\int_0^t e^{\gamma s}\Big[(\gamma|C| + |\bar{C}b|)|x(s)| - \bar{\lambda}|x(s)|^2 \\
- \frac{\theta e^{\gamma\tau}}{2}x^T(s)B^T\bar{C}Bx(s) + \frac{\theta}{2}x^T(s-\tau)B^T\bar{C}Bx(s-\tau)\Big]ds. \quad (3.11)
\end{aligned}
$$

But

$$
\begin{aligned}
& \int_0^t e^{\gamma s}x^T(s-\tau)B^T\bar{C}Bx(s-\tau)ds \\
& \qquad \leq e^{\gamma\tau}\int_{-\tau}^0 x^T(s)B^T\bar{C}Bx(s)ds + e^{\gamma\tau}\int_0^t e^{\gamma(s)}x^T(s)B^T\bar{C}Bx(s)ds.
\end{aligned}
$$

Hence

$$
e^{\gamma t}EV(x(t)) \quad \leq \quad H_2E\int_0^t e^{\gamma s}\Big[(\gamma|C| \ + \ |\bar{C}b|)|x(s)| \ - \ \bar{\lambda}|x(s)|^2\Big]ds, \quad (3.12)
$$

where

$$
H_2 = V(x(0)) + \frac{\theta}{2}e^{\gamma\tau}\int_{-\tau}^0 x^T(s)B^T\bar{C}Bx(s)ds.
$$

In the same way as in the proof of Theorem 3.1 we can then show

$$
\limsup_{t\to\infty} E|x(t)| \leq \frac{(\gamma|C| + |\bar{C}b|)^2}{4\gamma\bar{\lambda}\min_{1\leq i\leq n} c_i},
$$

and the required assertion (3.9) follows by setting K_1 to be the right-hand side term of the inequality above.

To show the other assertion (3.10), we derive from (3.4) and (2.14) that

$$
\begin{aligned}
0 \quad \leq \quad & V(x(0)) + E\int_0^t \Big(|\bar{C}b||x(s)| - \tilde{\lambda}|x(s)|^2 \\
- \quad & \frac{\theta}{2}x^T(t)B^T\bar{C}Bx(t) \ + \frac{\theta}{2}x^T(t-\tau)B^T\bar{C}Bx(t-\tau)\Big]\Big)ds,
\end{aligned}
$$

where

$$-\tilde{\lambda} = \frac{1}{2}\lambda^+_{\max}\Big(\bar{C}A + A^T\bar{C} + \theta^{-1}\bar{C} + \theta B^T\bar{C}B\Big) < 0.$$

But

$$\int_0^t x^T(s-\tau)B^T\bar{C}Bx(s-\tau)ds$$
$$\leq \int_{-\tau}^0 x^T(s)B^T\bar{C}Bx(s)ds + \int_0^t e^{\gamma(s)}x^T(s)B^T\bar{C}Bx(s)ds.$$

Hence

$$0 \leq H_3 + E\int_0^t \Big(|\bar{C}b||x(s)| - \tilde{\lambda}|x(s)|^2\Big)ds,$$

where

$$H_3 = V(x(0)) + \frac{\theta}{2}\int_{-\tau}^0 x^T(s)B^T\bar{C}Bx(s)ds.$$

In the same way as in the proof of Theorem 3.1 we can then show that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E|x(s)|^2ds \leq \frac{|\bar{C}b|^2}{\tilde{\lambda}^2},$$

and the desired assertion (3.10) follows by setting $K_2 = \frac{|\bar{C}b|^2}{\tilde{\lambda}^2}$. □

It is interesting to observe that the conditions and assertions in both Theorem 3.1 and Theorem 3.2 are independent of the noise intensity vector β. In particular, under the conditions imposed either in Theorem 3.1 or Theorem 3.2, the property of the ultimate boundedness in mean of equation (2.2) will not change no matter the environmental noise is large or small. In other words, the property of this boundedness is very robust under the noise.

4. Extinction

In the previous sections we have showed that under certain conditions, the original delay equation (2.1) and the associated stochastic delay equation (2.2) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to the associated stochastic delay equation (2.2) will become extinct with probability one, although the solution to the original delay equation (2.1) may be persistent. For example, recall equation (1.1), namely the scalar delay equation

$$\frac{dx(t)}{dt} = x(t)[\mu + \alpha x(t) + \delta x(t-\tau)].$$

It is well-known that if $\mu > 0$, $\alpha < 0$ and $0 < \delta < |\alpha|$, then its solution $x(t)$ is persistent, namely

$$\liminf_{t\to\infty} x(t) > 0.$$

However, consider its associated stochastic delay equation

$$dx(t) = x(t)\Big([\mu + \alpha x(t) + \delta x(t-\tau)]dt + \sigma dw(t)\Big),$$

where $\sigma > 0$. We will see from the following theorem that if $\sigma^2 > 2\mu$, then the solution to this stochastic delay equation will become extinct with probability one, namely

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.}$$

In other words, the following theorem reveals the important fact that the environmental noise may make the population extinct.

Theorem 4.1. *Assume that there are positive numbers $c_1, \cdots, c_n$ such that*

$$\lambda^+_{\max}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big) \le -\frac{|C|}{\hat{c}}\|\bar{C}B\|, \tag{4.1}$$

where $\bar{C} = \text{diag}(c_1, \cdots, c_n)$ and $C = (c_1, \cdots, c_n)$ as before while $\hat{c} = \min_{1\le i\le n} c_i$. Assume moreover that the noise intensities β_i are sufficiently large in the sense that

$$\beta_i\beta_j > b_i + b_j, \quad 1 \le i, j \le n. \tag{4.2}$$

Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t)$ of equation (2.2) has the property that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\varphi}{2} \quad \text{a.s.}, \tag{4.3}$$

where

$$\varphi = \min_{1\le i,j\le n}(\beta_i\beta_j - b_i - b_j) > 0.$$

That is, the population will become extinct exponentially with probability one.

Proof. Clearly, $|C| > \hat{c}$ so condition (4.1) implies

$$\lambda_{\max}^{+}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big) \le -\|\bar{C}B\|.$$

By Corollary 2.3, the solution $x(t)$ will remain in R_+^n for all $t \ge -\tau$ with probability 1. Define

$$V(x) = Cx = \sum_{i=1}^{n} c_i x_i \quad \text{for } x \in R_+^n.$$

By the Itô formula, we have

$$d[\log(V(x(t)))] = \frac{1}{V(x(t))} x^T(t)\bar{C}\big[(b + Ax(t) + Bx(t-\tau))dt \\ + \beta dw(t)\big] - \frac{1}{2V^2(x(t))}|x^T(t)\bar{C}\beta|^2 dt. \quad (4.4)$$

Using the elementary inequalities

$$\hat{c}|x| \le V(x) \le |C||x| \quad \forall x \in R_+^n,$$

and setting $\mu = \frac{\|\bar{C}B\|}{\hat{c}}$, we compute

$$\frac{1}{V(x(t))} x^T(t)\bar{C}Ax(t) \le \lambda_{\max}^{+}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big)\frac{|x(t)|^2}{V(x(t))} \\ \le \lambda_{\max}^{+}\Big(\frac{1}{2}(\bar{C}A + A^T\bar{C})\Big)\frac{|x(t)|}{|C|} \le -\mu|x(t)|,$$

where condition (4.1) is used in the last step. Compute also

$$\frac{1}{V(x(t))} x^T(t)\bar{C}Bx(t-\tau) \le \frac{|x(t)|}{V(x(t))}\|\bar{C}B\||x(t-\tau)| \\ \le \frac{\|\bar{C}B\|}{\hat{c}}|x(t-\tau)| = \mu|x(t-\tau)|.$$

Moreover

$$\frac{1}{V(x(t))} x^T(t)\bar{C}b - \frac{1}{2V^2(x(t))}|x^T(t)\bar{C}\beta|^2 \\ = \frac{1}{2V^2(x(t))}\Big[2x^T(t)\bar{C}bCx(t) - x^T(t)\bar{C}\beta\beta^T\bar{C}x(t)\Big]$$

$$= \frac{1}{2V^2(x(t))}\Big[2x^T(t)\bar{C}b\vec{1}\bar{C}x(t) - x^T(t)\bar{C}\beta\beta^T\bar{C}x(t)\Big]$$
$$= \frac{1}{2V^2(x(t))}\Big[x^T(t)\bar{C}(b\vec{1} + \vec{1}^T b)\bar{C}x(t) - x^T(t)\bar{C}\beta\beta^T\bar{C}x(t)\Big]$$
$$= -\frac{1}{2V^2(x(t))}x^T(t)\bar{C}Q\bar{C}x(t),$$

where $\vec{1} = (1, \cdots, 1)$ and $Q = \beta\beta^T - (b\vec{1} + \vec{1}^T b)$. Substituting these into (4.4)

$$\begin{aligned} d[\log(V(x(t)))] &\leq \Big[-\frac{x^T(t)\bar{C}Q\bar{C}x(t)}{2V^2(x(t))} - \mu|x(t)| + \mu|x(t-\tau)|\Big]dt \\ &+ \frac{x^T(t)\bar{C}\beta}{V(x(t))}dw(t). \end{aligned} \tag{4.5}$$

Note that the ij-th element of the matrix Q is

$$\beta_i\beta_j - b_i - b_j$$

which is positive by condition (4.2). It is therefore easy to verify

$$x^T(t)\bar{C}Q\bar{C}x(t) \geq \varphi V^2(x(t)),$$

since $x(t) \in R_+^n$, where φ has been defined in the statement of the theorem. Substituting this into (4.5) yields

$$d[\log(V(x(t)))] \leq \Big[-\frac{\varphi}{2} - \mu|x(t)| + \mu|x(t-\tau)|\Big]dt + \frac{x^T(t)\bar{C}\beta}{V(x(t))}dw(t).$$

This implies

$$\begin{aligned} \log(V(x(t))) &\leq \log(V(x(0))) - \frac{\varphi t}{2} + \int_0^t [-\mu|x(s)| + \mu|x(s-\tau)|]ds + M(t) \\ &\leq \log(V(x(0))) + \mu\int_{-\tau}^0 |x(s)|ds - \frac{\varphi t}{2} + M(t), \end{aligned} \tag{4.6}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \frac{x^T(s)\bar{C}\beta}{V(x(s))}dw(s).$$

The quadratic variation of this martingale is

$$\langle M, M\rangle_t = \int_0^t \frac{|x^T(s)\bar{C}\beta|^2}{V^2(x(s))}ds \leq \frac{|\bar{C}\beta|^2 t}{\hat{c}^2}.$$

Hence

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} \le \frac{|\bar{C}\beta|^2}{\hat{c}^2} \quad \text{a.s.}$$

By the strong law of large numbers for martingales (see e.g. [12, Theorem 1.3.4 on p. 12]) we therefore have

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.}$$

It finally follows from (4.6) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty} \frac{1}{t} \log(V(x(t))) \le -\frac{\varphi}{2} \quad \text{a.s.},$$

which is the required assertion (4.3). □

Acknowledgements

The first author would like to thank the University of Technology Malaysia for the financial support.

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A. Appendix

In this appendix we let $A = (a_{ij})_{n\times n}$ be a symmetric matrix, namely

$$a_{ij} = a_{ji}\,, \quad \forall i \neq j.$$

In Section 2 we defined

$$\lambda^+_{\max}(A) = \sup_{x\in R^n_+, |x|=1} x^T A x.$$

We also pointed out there that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A. To see this more clearly, let us recall the nice property of the largest eigenvalue:

$$\lambda_{\max}(A) = \sup_{x\in R^n, |x|=1} x^T A x.$$

It is therefore clear that we always have

$$\lambda^+_{\max}(A) \le \lambda_{\max}(A).$$

In many situations we even have $\lambda^+_{\max}(A) < \lambda_{\max}(A)$. For example, for

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

we have $\lambda^+_{\max}(A) = -1 < \lambda_{\max}(A) = 0$. On the other hand, $\lambda^+_{\max}(A)$ does have many similar properties as $\lambda_{\max}(A)$ has. For example, it follows straightforward from the definition that

$$x^T Ax \le \lambda^+_{\max}(A)|x|^2 \quad \forall x \in R^n_+ \text{ and } \lambda^+_{\max}(A) \le \|A\|.$$

Moreover

$$\lambda^+_{\max}(A + B) \le \lambda^+_{\max}(A) + \lambda^+_{\max}(B)$$

if B is another symmetric $n \times n$ matrix.

The results established in this paper require to verify $\lambda^+_{\max}(A) \le 0$ or $\lambda^+_{\max}(A) < 0$. It is easy to see from the definition that $\lambda^+_{\max}(A) \le 0$ if

$$a_{ij} \le 0 \quad \text{for all } 1 \le i, j \le n.$$

But the following two theorems give better results.

Theorem A.1. *We always have*

$$\lambda^+_{\max}(A) \le \max_{1\le i\le n} \Big(a_{ii} + \sum_{j\ne i} (0 \vee a_{ij}) \Big). \tag{A.1}$$

Consequently, $\lambda^+_{\max}(A) \le 0$ *if*

$$a_{ii} \le -\sum_{j\ne i} (0 \vee a_{ij}), \quad 1 \le i \le n;$$

while $\lambda^+_{\max}(A) < 0$ *if*

$$a_{ii} < -\sum_{j\ne i} (0 \vee a_{ij}), \quad 1 \le i \le n.$$

Proof. For any $x \in R_+^n$ with $|x| = 1$, compute

$$x^T Ax = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \le \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i=1}^n \sum_{j\neq 1} (0 \vee a_{ij}) x_i x_j$$

$$\le \sum_{i=1}^n a_{ii} x_i^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j\neq 1} (0 \vee a_{ij})(x_i^2 + x_j^2) = \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i=1}^n \Big(\sum_{j\neq 1} (0 \vee a_{ij}) \Big) x_i^2$$

$$= \sum_{i=1}^n \Big(a_{ii} + \sum_{j\neq 1} (0 \vee a_{ij}) \Big) x_i^2 \le \max_{1\le i\le n} \Big(a_{ii} + \sum_{j\neq i} (0 \vee a_{ij}) \Big).$$

Thus the required assertion (A.1) follows. □

Theorem A.2. *Given a symmetric matrix* $A = (a_{ij})_{n\times n}$, *we define its associated matrix* $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ *by*

$$\tilde{a}_{ii} = a_{ii}, \quad 1 \le i \le n\,,$$

while

$$\tilde{a}_{ij} = 0 \vee a_{ij}, \quad 1 \le i, \quad j \le n, \quad i \neq j.$$

Then $\lambda_{\max}^+(A) \le \lambda_{\max}(\tilde{A})$.

Proof. Clearly,

$$a_{ij} \le \tilde{a}_{ij}, \quad 1 \le i, j \le n.$$

By definition, it is easy to see that

$$\lambda_{\max}^+(A) \le \lambda_{\max}^+(\tilde{A}).$$

But we always have

$$\lambda_{\max}^+(\tilde{A}) \le \lambda_{\max}(\tilde{A}).$$

The required assertion hence follows. □