International Journal of Pure and Applied Mathematics
Volume 11 No. 4 2004, 421-441

CONFORMAL SCREEN ON LIGHTLIKE HYPERSURFACES

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Abstract: We study some properties of a lightlike hypersurface M, of a Lorentzian manifold, whose shape operator is conformal to the shape operator of its screen distribution. We prove that some specified aspects of the null geometry of M reduce to the Riemannian geometry of a leaf of its screen distribution. As a physical relevance, we show that there exists such a class of screen globally conformal lightlike hypersurfaces of 4-dimensional stationary non-flat spacetimes which admit a Killing horizon.

AMS Subject Classification: 53C20, 53C50, 83C40
Key Words: lightlike hypersurface, conformal screen, stationary spacetime

1. Introduction

There are three types of submanifolds in a semi-Riemannian manifold, namely, Riemannian, semi-Riemannian and lightlike. The geometry of lightlike submanifolds is different and rather difficult since (contrary to the other two types) its normal vector bundle intersects with the tangent bundle.

Received: January 9, 2004

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Thus, one can not use, in the usual way, the theory of non-degenerate submanifolds to define any induced object on a lightlike submanifold. In 1996, Duggal-Bejancu [7] published a book on general theory of lightlike submanifolds of semi-Riemannian manifolds. They introduced a non-degenerate screen distribution to construct a lightlike transversal vector bundle which is non-intersecting to its lightlike tangent bundle. It is now well-known that a suitable choice of screen distribution has produced substantial results in lightlike geometry. On the other hand, we know that the shape operator plays a key role in studying geometry of submanifolds. Motivated by above line of direction, the objective of this paper is to study a class of lightlike hypersurfaces M, of Lorentzian manifolds, whose shape operators are conformal to shape operators of their corresponding screen distributions. In Section 2, we brief basic information needed for the rest of this paper. In Section 3, we define screen conformal lightlike hypersurfaces (see Definition 1) and prove two results on their existence. In Section 4, we prove that some essential aspects of the null geometry of M reduce to the Riemannian geometry of a leaf of its integrable screen distribution. Recently, the second author (Duggal) has done some study on global null geometry [4, 5, 6] by introducing a new class of lightlike manifolds, called globally null manifolds (see Definition 2). In Section 5, we first show a relation between screen conformal hypersurfaces and globally null manifolds. Then, we present a physical model of screen globally conformal lightlike hypersurfaces of 4-dimensional stationary non-flat spacetimes having Killing horizons. Since the screen distribution of any lightlike manifold is not unique, in Section 6, we study the behavior of the screen conformality with respect to a change in screen distribution.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(n+2)$-dimensional semi-Riemannian manifold, where $n \geq 1$ and $\bar{g}$ is a non-degenerate tensor field of type $(0,2)$ and of constant index $q \geq 1$, on $\bar{M}$. Consider a submanifold M of $\bar{M}$ of codimension p with $p \geq 1$. The metric $\bar{g}$ might be nondegenerate or degenerate on the tangent bundle TM of M. The case $\bar{g}$ is nondegenerate has many similarities with the Riemannian submanifolds [9]. In the degenerate case, basic differences occur mainly due to the fact that the normal vector bundle $TM^{\perp}$ intersect with the tangent bundle along a non-zero differentiable distribution called radical distribution of M and denoted by $\mathrm{Rad}\,(TM)$

$$x \longrightarrow \mathrm{Rad}\,(T_xM) = T_xM \cap T_xM^{\perp}. \tag{1}$$

The dimension r ($r > 0$) of fibres of $\mathrm{Rad}\,(TM)$ is called the degree of nullity of M. Given an integer $r > 0$, the submanifold M is said to be r-lightlike if $\mathrm{rank}\,(\mathrm{Rad}\,(T_xM)) = r$ everywhere [7]. Lightlike hypersurfaces M of $\bar{M}$ are co-dimension one submanifolds for which $r = 1$. Given an $(n+1)$-dimensional lightlike hypersurface M, a screen distribution, denoted $S(TM)$, on M is a subbundle of TM which is complementary to the radical distribution. It is non-degenerate and has constant rank n. It plays an important role in the study of induced geometric objects on M. Lightlike hypersurfaces of Lorentzian manifolds endowed with a conformal structure of Lorentzian signature [1] have their degenerate metric of signature $(0, +, \cdots, +)$. For this case, the induced metric on their screen distribution $S(TM)$ is positive definite. For the lightlike hypersurfaces, (1) becomes $\mathrm{Rad}\,(TM) = TM^{\perp}$ and we have the orthogonal direct sum

$$TM = TM^{\perp} \oplus S(TM). \tag{2}$$

Throughout this paper we denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and $\Gamma(E)$ the $\mathcal{F}(M)$-module of differentiable sections of a vector bundle E over M. The manifolds we consider are supposed to be paracompact, smooth and connected. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM \,/\, \mathrm{Rad}\, TM$. The existence of $S(TM)$ is secured since M is paracompact. Our results will be based on a choice of $S(TM)$. The following normalization result is known.

Theorem. (Duggal-Bejancu, see [7, page 79]) *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a unique vector bundle tr(TM) of* rank 1 *over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of tr(TM) on $\mathcal{U}$ such that*

$$\begin{aligned} \bar{g}(\xi, N) &= 1, \qquad \bar{g}(N, N) \;=\; \bar{g}(N, W) = 0, & (3)\\ N &= \frac{1}{\bar{g}(\xi, V)}\Big\{ V - \frac{\bar{g}(V, V)}{2\,\bar{g}(\xi, V)}\,\xi \Big\}, \quad \forall\;\; W \in \Gamma(S(TM|_{\mathcal{U}})), & (4)\end{aligned}$$

where V is a non-zero vector field of $\bar{M}$ such that $\bar{g}(\xi, V)$ is non-zero.

Consider $\bar{\nabla}$ the Levi-Civita connection of $\bar{M}$ and ∇ the induced connection on the lightlike hypersurface (M, g) where g is the induced metric on M by $\bar{g}$. With the decomposition (2) and (3), we have

$$T\bar{M}|_M = S(TM) \perp (TM^{\perp} \oplus \mathrm{tr}\,(TM)) = TM \oplus \mathrm{tr}\,(TM). \tag{5}$$

Gauss and Weingarten formulae provide (see details in [7])

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall \;\; X, Y \in \Gamma(TM),$$
$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall \;\; X \in \Gamma(TM), \quad \forall \;\; V \in \Gamma(\mathrm{tr}\,(TM)), \tag{6}$$

where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(\mathrm{tr}\,(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, A_V is an $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on $\mathrm{tr}\,(TM)$. In general, induced connection ∇ is not unique and depends on the triplet $(M, g, S(TM))$. Define a symmetric $\mathcal{F}(M)$-bilinear form B and a $1-$form τ on the coordinate neighborhood $\mathcal{U}$ by

$$B(X,Y) \;=\; \bar{g}(h(X,Y), \xi), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{7}$$
$$\tau(X) \;=\; \bar{g}(\nabla^t_X N, \xi), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{8}$$

It follows that,

$$\bar{\nabla}_X Y \;=\; \nabla_X Y + B(X,Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{9}$$
$$\bar{\nabla}_X N \;=\; -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{10}$$

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2). We obtain

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \forall X, Y \in \Gamma(TM), \tag{11}$$
$$\nabla_X U = -\overset{\star}{A}_U X + \nabla^{*t}_X U, \; \forall \; X \in \Gamma(TM), \; \forall U \in \Gamma(TM^{\perp}), \tag{12}$$

where $\nabla^*_X Y$ and $\overset{\star}{A}_U X$ belong to $\Gamma(S(TM))$, ∇ and ∇^{*t} are linear connections on $\Gamma(S(TM))$ and $TM^{\perp}$ respectively, h^* is a $\Gamma(TM^{\perp})$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and $\overset{\star}{A}_U$ is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$. They are the second fundamental form and shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$

$$C(X, PY) \;=\; \bar{g}(h^*(X, PY), N), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{13}$$
$$\varepsilon(X) \;=\; \bar{g}(\nabla^{*t}_X \xi, N), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{14}$$

One can show that $\varepsilon(X) = -\tau(X)$. Thus, locally we obtain

$$\nabla_X PY \;=\; \nabla^*_X PY + C(X, PY)\xi, \quad \forall X, Y \in \Gamma(TM), \tag{15}$$
$$\nabla_X \xi \;=\; -\overset{\star}{A}_\xi X - \tau(X)\xi, \qquad \forall X \in \Gamma(TM). \tag{16}$$

The linear connection ∇^* is a metric connection on $\Gamma(S(TM))$. Hence if $S(TM)$ is integrable and S denotes a leaf of $S(TM)$ then ∇^* represents the Levi-Civita connection of S, intrinsically connected to its geometry. In general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y),\ \forall X,Y \in \Gamma(TM|_{\mathcal{U}})\,, \qquad (17)$$

where

$$\eta(X) = \bar{g}(X,N), \qquad \forall X \in \Gamma(TM|_{\mathcal{U}}). \qquad (18)$$

Finally, it is easy to check that

$$B(X,\xi) = 0,\ \ \forall\ X \in \Gamma(TM|_{\mathcal{U}}). \qquad (19)$$

3. Screen Conformal Lightlike Hypersurfaces

It is well known that the second fundamental form and the shape operator of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (7) and (13) that in case of lightlike hypersurface, there are interrelations between these geometric objects and those of its screen distribution. More precisely, the second fundamental forms of the lightlike hypersurface M and its screen distribution $S(TM)$ are related to their respective shape operator A_N and $\overset{\star}{A}_\xi$ by

$$\begin{aligned} B(X,Y) &= g(\overset{\star}{A}_\xi\, X,Y),\ \ \forall X,Y \in \Gamma(TM|_{\mathcal{U}}), && (20)\\ C(X,PY) &= g(A_N X,PY),\ \ \forall X,Y \in \Gamma(TM|_{\mathcal{U}}). && (21)\end{aligned}$$

This consolidates the fact that the geometry of a lightlike hypersurface depends on a choice of screen distribution, and that the latter plays an important role in studying differential geometry of lightlike hypersurfaces. Moreover, $S(TM)$ being non-degenerate, its geometry is classical. Thus, one may propose the following problem:

Find a class of lightlike hypersurfaces, whose geometry is essentially the same as that of their chosen screen distribution.

As the shape operator is an information tool in studying geometry of submanifolds, we are lead to consider lightlike hypersurfaces whose shape operators are the same as the one of their screen distribution up to a conformal non-vanishing smooth factor in $\mathcal{F}(M)$. More precisely

Definition 1. A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold, is screen locally conformal if the shape operators A_N and $\overset{\star}{A}_\xi$ of M and its screen distribution $S(TM)$ are related by

$$A_N = \varphi \overset{\star}{A}_\xi \,, \tag{22}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M.

In order to avoid trivial ambiguities, we will consider $\mathcal{U}$ to be connected and maximal in the sense that there is no larger domain $\mathcal{U}' \supset \mathcal{U}$ on which equation (22) holds. In case $\mathcal{U} = M$ the screen conformality is said to be global.

Examples. (a) *The lightlike cone Λ_0^{n+1} of $\mathbf{R}_1^{n+2}$.* Let $\mathbf{R}_1^{n+2}$ be the space $\mathbf{R}^{n+2}$ endowed with the semi-Euclidean metric

$$\bar{g}(x, y) = -x^0 y^0 + \sum_{a=1}^{n+1} x^a y^a, \qquad (x = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}).$$

The light cone Λ_0^{n+1} is given by the equation $-(x^0)^2 + \sum_{a=1}^{n+1}(x^a)^2 = 0$, $x \neq 0$. It is known that Λ_0^{n+1} is a lightlike hypersurface of $\mathbf{R}_1^{n+2}$ and the radical distribution is spanned by a global vector field

$$\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A} \tag{23}$$

on Λ_0^{n+1}. The unique section satisfying (3) is given by

$$N = \frac{1}{2(x^0)^2}\{-x^0 \frac{\partial}{\partial x^0} + \sum_{a=1}^{n+1} x^a \frac{\partial}{\partial x^a}\}\,, \tag{24}$$

and is also globally defined. As ξ is the position vector field we get

$$\bar{\nabla}_X \xi = \nabla_X X = X, \quad \forall X \in \Gamma(TM).$$

Then, $\overset{\star}{A}_\xi X + \tau(X)\xi + X = 0$. As $\overset{\star}{A}_\xi$ is $\Gamma(S(TM))-$valued we obtain

$$\overset{\star}{A}_\xi X = -PX, \qquad \forall X \in \Gamma(TM). \tag{25}$$

Next, any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ is expressed by $X = \sum_{a=1}^{n+1} X^a \frac{\partial}{\partial x^a}$, where $(X^1, \cdots, X^{n+1})$ satisfy

$$\sum_{a=1}^{n+1} x^a X^a = 0 \tag{26}$$

and then

$$\begin{aligned}\nabla_\xi X &= \bar{\nabla}_\xi X = \sum_{A=0}^{n+1}\sum_{a=1}^{n+1} x^A \frac{\partial X^a}{\partial x^A}\frac{\partial}{\partial x^a},\\ \bar{g}(\nabla_\xi X,\xi) &= \sum_{A=0}^{n+1}\sum_{a=1}^{n+1} x^a x^A \frac{\partial X^a}{\partial x^A} = -\sum_{a=1}^{n+1} x^a X^a = 0,\end{aligned} \tag{27}$$

where (26) is differentiated with respect to each x^A. From (26) and (27) we obtain $\nabla_\xi X \in \Gamma(S(T\Lambda_0^{n+1}))$, that is, $A_N\xi = 0$. Compute $A_N X$ for $X \in \Gamma(S(T\Lambda_0^{n+1}))$. Let $X,\ Y \in \Gamma(S(T\Lambda_0^{n+1}))$. Using (24) and (26) we obtain

$$C(X,Y) = g(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\frac{1}{2(x^0)^2} g(X,Y),$$

that is,

$$g(A_N X, Y) = -\frac{1}{2(x^0)^2} g(X,Y) \quad X,\ Y \in \Gamma(S(T\Lambda_0^{n+1})).$$

Therefore, we have

$$A_N X = -\frac{1}{2(x^0)^2} PX \quad X \in \Gamma(T\Lambda_0^{n+1}). \tag{28}$$

Taking into account (25) and (28) we infer the following relation

$$A_N X = \frac{1}{2(x^0)^2} \overset{\star}{A}_\xi X, \quad X \in \Gamma(T\Lambda_0^{n+1}),$$

that is, Λ_0^{n+1} is screen globally conformal lightlike hypersurface of $\mathbf{R}_1^{n+2}$ with positive conformal function $\varphi = \frac{1}{2(x^0)^2}$ globally defined on Λ_0^{n+1}.

(b) *Lightlike Monge hypersurfaces of* $\mathbf{R}_q^{n+2}$. Consider a smooth function $F : \Omega \to \mathbf{R}$, where Ω is an open set of $\mathbf{R}^{n+1}$, then

$$M = \{(x^0, \cdots, x^{n+1}) \in \mathbf{R}_q^{n+2}, x^0 = F(x^1, \cdots, x^{n+1})\}$$

is called a Monge hypersurface [7]. Such a hypersurface is lightlike if and only if F is a solution of the partial differential equation

$$1 + \sum_{i=1}^{q-1} (F'_{x^i})^2 = \sum_{a=q}^{n+1} (F'_{x^a})^2.$$

The radical distribution is spanned by a global vector field

$$\xi = \frac{\partial}{\partial x^0} - \sum_{i=1}^{q-1} F'_{x^i} \frac{\partial}{\partial x^i} + \sum_{a=q}^{n+1} F'_{x^a} \frac{\partial}{\partial x^a}.$$

Along M consider the constant timelike section $V^* = \frac{\partial}{\partial x^0}$ of $\mathbf{R}_q^{n+2}$. The vector bundle H^* =span$\{V^*, \xi\}$ is nondegenerate on M. The complementary orthogonal vector bundle $S^*(TM)$ to H^* in $T\mathbf{R}_q^{n+2}$ is an integrable screen distribution on M called the natural screen distribution on M. The transversal bundle $\mathrm{tr}^*(TM)$ is spanned by $N = -V^* + \frac{1}{2}\xi$ and $\tau(X) = 0$, $\forall X \in \Gamma(TM)$; indeed, $\tau(X) = \bar{g}(\bar{\nabla}_X N, \xi) = \frac{1}{2}\bar{g}(\bar{\nabla}_X \xi, \xi) = 0$. Therefore, Weingarten equations take simplified forms $\bar{\nabla}_X N = -A_N X$ and $\nabla_X \xi = -\overset{\star}{A}_\xi X$. We deduce

$$A_N X = \frac{1}{2} \overset{\star}{A}_\xi X =, \quad \forall X \in \Gamma(TM).$$

Hence, any lightlike Monge hypersurfac in $\mathbf{R}_q^{n+2}$ is screen globally conformal with constant positive conformal function $\varphi(x) = \frac{1}{2}$.

Remark 1. In case $q = 1$, the natural and the canonical (see [7]) screen distributions coincide on Lightlike Monge hypersurfaces. This is to say that, endowed with the canonical screen distribution, lightlike Monge hypersurfaces are screen globally conformal in Lorentz space. Moreover, it has been proved in [7, p. 133] that the geometry of a Monge lightlike hypersurface of $\mathbf{R}_1^4$ essentially reduces to the geometry of a leaf of its canonical screen distribution. This result can be proved for $\mathbf{R}_1^{n+1}$. Thus, for this class of lightlike Monge hypersurfaces, our proposed problem is completely solved.

Proposition 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Suppose $S(TM)$ is integrable and any leaf of $S(TM)$ is totally umbilical immersed in $\bar{M}$ as a codimension 2 non-degenerate submanifold with nowhere vanishing spacelike mean curvature vector field. If the screen distribution is parallel along integral curves of theradical distribution, then M is screen locally conformal.*

Proof. Let us denote M' a leaf of $S(TM)$. We have

$$\bar{\nabla}_X Y = \overset{\star}{\nabla}_X Y + C(X,Y)\xi + B(X,Y)N, \quad \forall X, Y \in \Gamma(TM'). \tag{29}$$

The mean curvature vector field of M', say $H^\star$, is a vector field of the rank 2 bundle $(TM^\perp \oplus \mathrm{ltr}(TM))$, that is the normal bundle of M' in $\bar{M}$. Therefore, there exist two smooth functions α and ρ such that $H^\star = \alpha\xi + \rho N$. Since M' is totally umbilical immersed in $(\bar{M}, \bar{g})$, we have

$$C(X,Y)\xi + B(X,Y)N = g(X,Y)(\alpha\xi + \rho N), \quad \forall X, Y \in \Gamma(TM').$$

Therefore, it follows that

$$B(X,Y) = \rho g(X,Y) \text{ and } C(X,Y) = \alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM'). \tag{30}$$

Also, as $\langle H^\star, H^\star\rangle = 2\alpha\rho$ and $H^\star$ is nowhere vanishing spacelike, we get $\alpha\rho > 0$ on M. From (30), $C(X,Y) = \frac{\alpha}{\rho}B(X,Y), \quad \forall X, Y \in \Gamma(TM')$ with $\frac{\alpha}{\rho} > 0$ on M'. This is equivalent to $A_N X = \frac{\alpha}{\rho} \overset{\star}{A}_\xi X$, $\forall X \in \Gamma(TM')$. Since $S(TM)$ is parallel along integral curves of Rad TM, we have $A_N\xi = 0 = \frac{\alpha}{\rho} \overset{\star}{A}_\xi \xi$. Thus,

$$A_N X = \frac{\alpha}{\rho} \overset{\star}{A}_\xi X, \quad \forall X \in \Gamma(TM)$$

which implies that M is screen locally conformal with $\varphi = \frac{\alpha}{\rho}$. □

Remark 2. If $\bar{M}$ is of constant sectional curvature c, then for $c \neq 0$ the non-vanishing condition on H^* is not necessary, for in this case, we always have $\alpha \neq 0$ at every point.

Theorem 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface in a Lorentzian manifold $(\bar{M}, \bar{g})$. The following assertions are equivalent:*

(a) $(M, g, S(TM))$ *is screen locally conformal.*

(b) There is a (maximal) domain $\mathcal{U}$ in M on which M and its screen distribution have commutative shape operators. Moreover, their corresponding principal curvatures are the same up to a nowhere vanishing smooth function φ on $\mathcal{U} \subset M$.

Proof. If $(M, g, S(TM))$ is screen locally conformal, then on the conformality domain $\mathcal{U}$, there exist a nowhere vanishing smooth function φ such that $A_N X = \varphi \overset{\star}{A}_\xi X$ for all X tangent to $\mathcal{U} \subset M$. Then, the commutativity of the shape operators A_N and $\overset{\star}{A}_\xi$ is immediate on $\mathcal{U}$. Hence, there is a local frame field with respect to which $\overset{\star}{A}_\xi$ and A_N are simultaneously diagonal. Therefore, consider on $\mathcal{U} \subset M$ an eigen frame field $(E_0, \cdots, E_n)$ both for A_N and $\overset{\star}{A}_\xi$. If μ_i and λ_i denote the principal curvatures corresponding to E_i with respect to A_N and $\overset{\star}{A}_\xi$ respectively, then by (22), $\mu_i = \varphi\lambda_i$ and the last assertion in (b) follows.

Conversely, assume (b). The commutativity of the shape operators A_N and $\overset{\star}{A}_\xi$ implies the existence on $\mathcal{U}$ of a frame field $(E_0, \cdots, E_n)$ with respect to which A_N and $\overset{\star}{A}_\xi$ are simultaneously diagonal. Let μ_i and λ_i denote the principal curvatures corresponding to E_i with respect to A_N and $\overset{\star}{A}_\xi$. The last assertion in (b) requires that there exists on $\mathcal{U}$, a nowhere vanishing smooth function φ such that $\mu_i = \varphi\,\lambda_i$, $0 \le i \le n$. Decompose a tangent vector $X \in T\mathcal{U}$ on the frame field $(E_0, \cdots, E_n)$ such that $X = X^i E_i$. Then,

$$\begin{aligned} A_N X &= A_N(X^i E_i) = X^i A_N E_i = X^i \mu_i E_i = X^i \varphi \lambda_i E_i \\ &= \varphi\left(X^i \lambda_i E_i\right) = \varphi X^i \overset{\star}{A}_\xi\, E_i = \varphi\, \overset{\star}{A}_\xi \left(X^i E_i\right) = \varphi\, \overset{\star}{A}_\xi\, X. \end{aligned}$$

Thus, (22) holds for a non-vanishing smooth function φ on $\mathcal{U} \subset M$. □

Remark 3. On the eigenspace, say $\mathcal{N}_0$, of the zero eigenvalue of $\overset{\star}{A}_\xi$, we note the following. Let $\mathcal{S}$ denote a subbundle of $S(TM)$ defined by

$$\mathcal{S} = \operatorname{span}\{\overset{\star}{A}_\xi\, Y, Y \in \Gamma(TM)\}.$$

Then,

$$\mathcal{N}_0 = \operatorname{span}\{\overset{\star}{A}_\xi\, Y, Y \in \Gamma(TM)\}^{\perp_M} = \mathcal{S}^{\perp_M},$$

where $\perp_M$ is the orthogonality symbol in M. In particular, if $\mathcal{S}$ coincides with $S(TM)$, then any eigenvector field in $\mathcal{N}_0$ is a multiple of ξ.

4. Geometry of Lightlike Hypersurfaces

In general the screen distribution is not integrable. In case of the screen locally conformal lightlike hypersurfaces we have the following theorem.

Theorem 2. *Let $(M, g, S(TM))$ be a screen locally conformal lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the screen distribution is integrable. Moreover, M is totally geodesic, totally umbilical or minimal in $\bar{M}$ if and only if any leaf M' of $S(TM)$ is so immersed in $\bar{M}$ as a codimension 2 nondegenerate submanifold.*

Proof. It is well-known that the screen distribution is integrable if and only if the shape operator of M is symmetric with respect to the induced metric tensor g. The integrability assertion follows relation $A_N = \varphi\, \overset{\star}{A}_\xi$ and the symmetry of $\overset{\star}{A}_\xi$ with respect to g. For the last assertion of the theorem, let X, Y be tangent vector fields of the leaf M' of a screen distribution and h' be its second fundamental form in $\bar{M}$ as a codimension 2 nondegenerate submanifold. We have

$$\bar{\nabla}_X Y = \nabla^\star_X Y + C(X,Y)\xi + B(X,Y)N\,,$$

which leads to

$$\bar{\nabla}_X Y = \nabla^\star_X Y + g(\overset{\star}{A}_\xi\, X, Y)(\varphi\xi + N), \forall X, Y \in \Gamma(TM'|_{\mathcal{U}})\,,$$

that is

$$h'(X,Y) = B(X,Y)(\varphi\xi + N) = \sqrt{2|\varphi|}B(X,Y)\left(\frac{\varphi}{\sqrt{2|\varphi|}}\xi + \frac{1}{\sqrt{2|\varphi|}}N\right)\,,$$

$\forall X, Y \in \Gamma(TM'|_{\mathcal{U}})$, where $\frac{\varphi}{\sqrt{2|\varphi|}}\xi + \frac{1}{\sqrt{2|\varphi|}}N$ is a unit normal vector field on M'. As $\sqrt{2|\varphi|}$ is nowhere zero and $B(X,\xi) = 0$, $\forall X \in \Gamma(TM')$ the last assertion in the above stated theorem follows. □

It is known [7] that, in general, the induced Ricci tensor on any lightlike submanifold is not symmetric. For a screen lightlike hypersurface we prove the following theorem.

Theorem 3. *Let $(M, g, S(TM))$ be a screen locally (or globally) lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}^{n+2}(c), \bar{g})$ of constant sectional curvature c. Then the induced Ricci tensor on M is symmetric.*

Proof. Using equation (3.1) on p. 93 of Duggal-Bejancu [7] one can show that the general expression of the Ricci tensor of M is as follows:

$$\begin{aligned}\mathrm{Ric}\,(X,Y) &= \bar{\mathrm{R}}\mathrm{ic}\,(X,Y) - \bar{g}(\bar{R}(\xi,Y)X, N)\\ &\quad + B(X,Y)\mathrm{tr}\,A_N - g(A_N X, \overset{\star}{A}_\xi\, Y).\end{aligned} \tag{31}$$

As $\bar{M}^{n+2}(c)$ is nondegenerate we have $\bar{\mathrm{R}}\mathrm{ic}\,(\xi,Y)X = \pm\, c\bar{g}(X,Y)\xi$, where one can take either the sign $+$ or $-$, depending on the chosen definition of the curvature tensor. Thus, taking $-$ sign, (31) reduces to

$$\mathrm{Ric}\,(X,Y) = \bar{\mathrm{R}}\mathrm{ic}\,(X,Y) + c\bar{g}(X,Y) + B(X,Y)\mathrm{tr}\,A_N - g(A_N X, \overset{\star}{A}_\xi\, Y).$$

Since g and B are symmetric, it follows that

$$\mathrm{Ric}\,(X,Y) - \mathrm{Ric}\,(Y,X) = g(A_N Y, \overset{\star}{A}_\xi\, X) - g(A_N X, \overset{\star}{A}_\xi\, Y).$$

Finally, by equation (22) $\mathrm{Ric}\,(X,Y) - \mathrm{Ric}\,(Y,X) = \varphi g([\overset{\star}{A}_\xi, \overset{\star}{A}_\xi]Y, X) = 0$, which completes the proof. □

5. A Physical Model

In this section we present a physical model of screen globally conformal lightlike hypersurfaces. For this purpose, we first recall the following definition.

Definition 2. (Duggal [4]) An $(n+1)$-dimensional $(n>1)$ lightlike manifold (M, g) is said to be a globally null manifold if it admits a global null vector field and a complete Riemannian hypersurface.

An up-to-date information on globally null manifolds, with some physical applications, is available in [4, 5, 6].

Theorem. (Duggal [4]) *Let $(M,\, g)$ be an $(n+1)$-dimensional $(n>1)$ globally null manifold. Then, the following assertions are equivalent:*

(a) *The screen distribution $S(TM)$ is integrable.*

(b) *$M = M' \times C'$ is a global product manifold, where M' is a leaf of $S(TM)$ and C' is a leaf of* Rad TM *in M.*

(c) *$S(TM)$ is parallel with respect to the metric connection ∇ on M.*

Consider a special class of globally null manifolds, denoted by $(M, g,\, G)$, such that each M carries a smooth 1-parameter group G of isometries whose orbits are global null curves in M.

Proposition 2. *An $(n+1)$-dimensional $(n>1)$ special globally null manifold (M, g, G) is a global product manifold $M = M' \times C'$, where M' and C' are leafs of a screen distribution $S(TM)$ and the* Rad TM *of M. Moreover, $S(TM)$ is integrable.*

Proof. Let M' be the orbit space of the action $G \approx C'$, where C' is a 1-dimensional null leaf of Rad TM in M. Then, M' is a smooth Riemannian hypersurface of M and the projection $\pi : M \to M'$ is a principle C'-bundle, with null fiber G. The global existence of null vector field, of M, implies that M' is Hausdorff and paracompact. The infinitesimal generator of G is a global null Killing vector field, say ξ, on M. The metric g restricted to its screen distribution space $S(TM)$ then induces a Riemannian metric, say g', on M'. Since ξ is non-vanishing on M, we can take $\xi = \frac{\partial}{\partial\theta}$ a global null coordinate vector field for some global function θ on M. Thus, θ induces a diffeomorphism on M such that $(M = M' \times C', g = \pi^\star g')$ is a global product manifold. Finally, the integrability of $S(TM)$ follows from the equivalence of (a) and (b) in above quoted Theorem of Duggal [4]. □

In [5, Example 2] it has been shown that there exists a pair of globally null hypersurfaces of a globally hyperbolic spacetime [3] manifold. Based on this result, we prove the following result.

Theorem 4. *Let* (M, g, G) *be a special globally null hypersurface of an* $(n+2)$*-dimensional* $(n > 1)$ *globally hyperbolic spacetime manifold* $(\bar{M}, \bar{g})$. *Suppose the induced Ricci tensor on* M *is symmetric. Then, one can choose a canonical integrable distribution* $S(TM)^{\star}$, *of* M, *with respect to which* M *is a screen globally conformal lightlike hypersurface.*

Proof. Since Rad TM is integrable, using the fibre bundle theory, we construct a special globally null hypersurface of $\bar{M}$ as follows: Let $(x^0 = t\,; x^a, x^{n+1})$ be coordinates on $\bar{M}$ such that $(t\,; x^a)$ are coordinates on M induced by the foliation determined by Rad TM and (x^{n+1}) is a coordinate on the fiber of Rad TM, where $a \in \{1, \ldots, n\}$. It is easy to see that this construction is invariant with respect to any allowable coordinates transformations. Assume that Rad TM is spanned by a global null vector field ξ. Consider a natural basis $\{\frac{\partial}{\partial t}\,; \frac{\partial}{\partial x^a}\,, \frac{\partial}{\partial x^{n+1}}\}$, with respect to which the global null vector field ξ is given by

$$\xi = \xi^0 \frac{\partial}{\partial t} + \sum_{\alpha=1}^{n+1} \xi^\alpha \frac{\partial}{\partial x^\alpha}, \quad \sum_{\alpha=1}^{n+1} (\xi^\alpha)^2 = (\xi^0)^2. \tag{32}$$

It is known that a globally hyperbolic spacetime may be written as a topological product $\bar{M} = \mathbf{R} \times \mathbf{S}$ where $\mathbf{S}$ is a Cauchy hypersurface. Such a spacetime is of the form $(\bar{M} = \mathbf{R} \times \mathbf{B}, \bar{g} = -dt^2 \oplus \bar{g}')$ with $(\mathbf{B}, \bar{g}')$ a Riemannian manifold. Moreover, Beem and Ehrlich [3] have shown that there exists a class of globally hyperbolic warped product spacetimes of the form

$$\bar{M} = M_1 \times {}_f\, M_2, \quad \bar{g} = g_1 \oplus f\, g_2, \tag{33}$$

where (M_1, g_1), (M_2, g_2) and f are a time oriented globally hyperbolic spacetime, a complete Riemannian manifold and a smooth function on M_1. Such product spacetimes possess at least one timelike covariant constant vector field. Thus, we may choose (without any loss of generality), along M, a timelike constant vector field $V = -\frac{\partial}{\partial t}$ which satisfies the required condition $\bar{g}(V, \xi) = \xi^0 \neq 0$, that is, V is not tangent to M. It follows that the vector bundle $\mathcal{B} = \mathrm{Span}\{\frac{\partial}{\partial t}, \xi\}$ is non-degenerate on M. Take the complementary orthogonal vector bundle $S(TM)^{\star}$ to $\mathcal{B}$ in $T\bar{M}$, which is a non-degenerate distribution on M complementary to Rad TM. This means that $S(TM)^{\star}$ is a screen distribution on M such that $\mathcal{B} = S(TM)^{\perp}$, which we call the canonical integrable (follows from Proposition 2) screen distribution on M. Using this, (3), (4) and

(32) we obtain

$$N = (\xi^0)^{-1}\,(V + \frac{1}{2\,\xi^0}\,\xi\,), \tag{34}$$

which we call the unique canonical null transversal vector bundle of M. Since by hypothesis induced Ricci tensor on M is symmetric, it follows from Proposition 3.2 of [7, p. 99] there exists a pair $\{\xi, N\}$ on $\mathcal{U}$ such that $\tau(X) = 0$, $\forall X \in \Gamma(TM)$. Now using (10) and (34), we get

$$\begin{aligned} 0 = \tau(X) &= \bar{g}(\bar{\nabla}_X N, \xi) = X(\xi^0)^{-1}\,\bar{g}(V,\,\xi) + \frac{1}{2}\,(\xi^0)^{-2}\,\bar{g}(\bar{\nabla}_X\,\xi,\,\xi) \\ &= X(\xi^0)^{-1}\,\xi^0. \end{aligned}$$

Consequently, ξ^0 is a constant. Also, the two Weingarten equations (10) and (16) reduce to

$$\bar{\nabla}_X\, N = -A_N\, X\,, \quad \nabla_X\,\xi = -\overset{\star}{A}_\xi\;X, \tag{35}$$

respectively. Now using (19) and (34) we obtain

$$\bar{\nabla}_X\, N = \frac{1}{2}(\xi^0)^{-2}\bar{\nabla}_X\,\xi = \frac{1}{2}(\xi^0)^{-2}\nabla_X\,\xi = -\,\frac{1}{2}(\xi^0)^{-2}\;\overset{\star}{A}_\xi\;X\,.$$

Finally, using the reduced Weingarten equations (35) and above we get

$$A_N = \frac{1}{2\,(\xi^0)^2}\;\overset{\star}{A}_\xi\;. \tag{36}$$

Thus, as per Definition 1, (M, g, G) is a screen globally conformal lightlike hypersurface of $\bar{M}$ with constant conformal function

$$\varphi(x) = \frac{1}{2\,(\xi^0)^2}.$$

□

Remark 4. A globally hyperbolic spacetime of the form $\mathbf{R}\times\mathbf{B}$ includes Minkowski space and the Einstein static universe, but it fails to include several other physical spacetimes such as Schwarzchild, Reissner - Nordström, and Kerr spacetimes. This is why we have used warped products (in the proof of Theorem 4) to include above and more general spacetimes. Also, in the following physical model we show that the use of warped product globally hyperbolic spacetimes is necessary for some cases.

Physical model. Consider a 4 - dimensional stationary spacetime $(\bar{M},\,\bar{g})$ which is chronological, that is, $\bar{M}$ admits no closed timelike curves. It is well-known [8] that a stationary $\bar{M}$ admits a smooth 1-parameter group, say $\bar{G}$, of isometries whose orbits are timelike curves in $\bar{M}$. A static spacetime is stationary with the additional condition that its timelike Killing vector field, say T, is hypersurface orthogonal, that is, there exists a spacelike hypersurface orthogonal to T. Our model will be applicable to both of these types. Denote by $\bar{M}'$ the Hausdorff and paracompact 3-dimensional Riemannian orbit space of the action $\bar{G}$. The projection $\bar{\pi} : \bar{M} \to \bar{M}'$ is a principal $\mathbf{R}$-bundle, with the timelike fiber $\bar{G}$. Let $T = \frac{\partial}{\partial t}$ be the non-vanishing timelike Killing vector field, where t is a global time coordinate function on $\bar{M}'$. Then, the metric $\bar{g}$ induces a Riemannian metric $g_{_{\bar{M}'}}$ on $\bar{M}'$ such that

$$\bar{M} = \mathbf{R} \times \bar{M}', \quad \bar{g} = -\,u^2\,(dt+\eta)^2 + \bar{\pi}^{\star}\, g_{_{\bar{M}'}},$$

where η is a connection 1-form for the $\mathbf{R}$-bundle $\bar{\pi}$ and

$$u^2 = -\bar{g}(T,\,T) > 0.$$

It is known that a stationary spacetime $(\bar{M},\,\bar{g})$ uniquely determines the orbit data $(\bar{M}', \bar{g}_{_{\bar{M}'}}, u, \eta)$ as described above, and conversely. Suppose the orbit space $\bar{M}'$ has a non-empty metric boundary $\partial\bar{M}' \neq \emptyset$. Consider the maximal solution data in the sense that it is not extendible to a larger domain $(\mathcal{M}', g'_{_{\mathcal{M}'}}, u', \eta') \supset (\bar{M}', g_{_{\bar{M}'}}, u, \eta)$ with $u' > 0$ on an extended spacetime $\mathcal{M}'$. Under these conditions, it is known [8] that in any neighborhood of a point $x \in \partial\bar{M}'$, either the metric $g_{_{\bar{M}'}}$ or the connection 1-form η degenerates, or $u \to 0$. The third case implies that the timelike Killing vector T becomes null and, there exists a Killing horizon $H = \{u \to 0\}$ of $\bar{M}$. This Killing horizon H is related to special globally null hypersurfaces as follows:

Let $(M,\,g)$ be a lightlike hypersurface of a stationary spacetime $(\bar{M},\,\bar{g})$. First, we state the following known general result.

Theorem. (Duggal-Bejancu [7, p. 87-88]) *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, the induced connection ∇, on M, is independent of $S(TM)$ if and only if the second fundamental form h of M vanishes. Also, the following assertions are equivalent:*

(i) *M is totally geodesic, that is, B vanishes identically on M.*

(ii) There exists a unique Levi-Civita connection ∇ on M.

(iii) Rad TM *is a parallel distribution with respect to ∇.*

(iv) Rad TM *is a Killing distribution on M.*

Suppose (M, g) satisfies above result. Then, its Rad TM is a Killing distribution. Therefore, it follows from the equivalence of (i) and (iv), of above result, that a globally null (M, g) is a totally geodesic (i.e., invariant) hypersurface of a stationary spacetime manifold $\bar{M}$. Now consider the case when the timelike vector field $T \in \bar{M}$ becoming null, that is,

$$\mathrm{Lim}(T)_{u \to 0} = \xi,$$

where $\xi \in \Gamma(T(\bar{M}))$ is a null Killing vector field. Then the spacelike hypersurface $\bar{M}'$ of $\bar{M}$ degenerates to a 3-dimensional globally null hypersurface M of $\bar{M}$ such that a leaf M' of $S(TM)$ is identified with $\partial \bar{M}'$, that is,

$$M' = \partial \bar{M}' \subset \bar{M}$$

is a common 2-dimensional submanifold of both M and $\bar{M}$. Since M is totally geodesic in $\bar{M}$, we conclude that M admits a smooth 1-parameter group G of isometries (induced from the group $\bar{G}$ of $\bar{M}$) whose orbits are global null curves in M. Obviously, for a totally geodesic M the induced Ricci tensor must be symmetric. Thus, it follows from Theorem 4 that there exists a screen globally conformal lightlike hypersurface $(M,\, g,\, G)$ of a 4-dimensional stationary spacetime manifold $(\bar{M},\, \bar{g})$ of general relativity.

Physically, it is important to find those stationary spacetimes $\bar{M}$ which are geodesically complete, chronological and their orbit space $\bar{M}'$ has a non-empty metric boundary $\partial \bar{M}'$. The last condition is necessary for the existence of a screen globally conformal hypersurface of such a spacetime $\bar{M}$ with a Killing horizon. For this purpose, we recall the following result.

Theorem. (Anderson, [2]) *Let $(\bar{M},\, \bar{g})$ be a geodesically complete, chronological, stationary vacuum spacetime. Then $\bar{M}$ is the flat Minkowski space $\mathbf{R}^4_1$, or a quotient of Minkowski space by a discrete group Γ of isometries of $\mathbf{R}^3$, commuting with $\bar{G}$. In particular, $\bar{M}$ is diffeomorphic to $\mathbf{R} \times \bar{M}'$, $d\theta = 0$ and $u =$ constant.*

Anderson result above implies that only a non-flat $\bar{M}$ will have a non-empty metric boundary of its orbit space. It turns out that Asymptotically flat spacetimes are best physical systems for the non-flat stationary spacetimes, many of them do have Killing horizons. For example, among stationary spacetimes, Schwarzchild, Reissner-Nordström, and Kerr spacetimes, all have Killing horizons [8]. As mentioned in Remark 4, for such physical cases, it is necessary to use warped product globally hyperbolic spacetimes of the form (33) which include above mentioned three stationary spacetimes and several others. Using above information one can show that, in particular, Theorem 4 will also hold if the ambient manifold is a warped product non-flat stationary spacetime having Killing horizon.

Example. Consider the case of a 4-dimensional Schwarzchild spacetime $(\bar{M}, \bar{g})$, with the metric

$$g = -A(r)\, dt^2 + A(r)^{-1}\, dr^2 + r^2\, ds^2_{s^2}, \quad A(r) = 1 - \frac{2m}{r}\,,$$

where m and r are the mass and the radius of an isolated star or black hole and represents their vacuum exterior regions. This metric is spherically symmetric on $\bar{M}' = (2m, \infty) \times S^2$, such that $\partial(\bar{M}')$ is given by a totally geodesic S^2 and $r = 2m$. Consider a new coordinate system (v, r', θ, ϕ) such that $v = t + \bar{r}$ a null coordinate, where $\bar{r} = r + 2m \log(r - 2m)$. Then above metric transforms into

$$ds^2 = -A(r)\, dv^2 + 2\, dv\, dr + r^2\, ds^2_{s^2}\,,$$

which is non-singular for any non-zero r. For $r = 2m$ we have $A(r) = 0$, then, due to the absence of the term dv^2, we get a lightlike hypersurface, say, $(M, g, r = 2m, v = \text{ constant})$ of $\bar{M}$, whose screen distribution is topologically a 2-sphere S^2. Denote by C' the 1-dimensional null manifold generated by the null vector field ξ. Then, it follows that (M, g) has a Killing horizon given by S^2 such that

$$M = S^2 \times C', \quad g = \pi^\star g_{S^2}, \quad \pi : M \to S^2.$$

Finally, as explained above for the general case, (M, g) is a special globally null (and, therefore, a screen globally conformal lightlike) hypersurface of a 4-dimensional Schwarzchild spacetime $\bar{M}$ with a Killing horizon S^2 in the sense that the null Killing vector field $\xi \in \Gamma(T(M))$ vanishes on S^2.

Remark 5. Mathematically, the existence of geodesically complete and chronological stationary spacetimes puts strong conditions on both the topology and geometry of its orbit space $\bar{M}'$ outside large compact sets. For purely geometric reasons, it is interesting to find those conditions, both with respect to flat and non-flat stationary spacetimes. However, physically, since there do exist Killing horizons of some spacetimes, our Theorem 4 and its physical model establishes a relevance of screen globally conformal lightlike hypersurfaces with some physically significant stationary spacetimes.

6. Stability with Screen Distribution Change

As mentioned before, the screen distribution is not unique, although for a chosen screen distribution, we have a unique rigging tr(TM) which is complementary vector bundle to TM in $T\bar{M}$. In this section, we are interested in the behavior of the screen conformality with respect to a change in screen distribution. That is to find how much there are which satisfy this property? Is this an intrinsic property?

In the following we denote by $\mathcal{S}^1$ the first derivative of the screen distribution, that is the distribution

$$\mathcal{S}^1(x) = \text{span}\,\{[X,Y]|_x, \quad X_x,\ Y_x\ \in S(T_xM)\}\,, \tag{37}$$

for $x \in M$. As the screen distribution is integrable, $\mathcal{S}^1$ is a subbundle of $S(TM)$. Let $S(TM)$ and $S(TM)'$ denote two screen distributions on M, h and h' their second fundamental forms with respect to $\text{tr}\,(TM)$ and $\text{tr}\,(TM)'$ respectively with respect to the same $\xi \in \Gamma(TM^\perp|_{\mathcal{U}})$. We know from [7, p. 83] that the local second fundamental form B of M is independent of the screen distribution, that is, $B = B'$.

Lemma 1. *M is screen locally (or globally) conformal if and only if the second fundamental forms C and B of the screen distribution and M respectively satisfy*

$$C(X,PY) = \varphi\, B(X,Y), \quad \forall X,Y \in \Gamma(TM|_{\mathcal{U}})\,, \tag{38}$$

for some smooth function φ on $\mathcal{U} \subset M$.

Proof. From (20), (21) and (22) we obtain

$$\begin{aligned} C(X,PY) &= g(A_N X, PY) = \varphi\, g(\overset{\star}{A}_\xi\, X, PY) \\ &= \varphi\, B(X,PY) = \varphi\, B(X,Y). \end{aligned}$$

Now consider $\{W_i\}, 1 \le i \le n$ an orthonormal basis of $\Gamma(S(TM)|_{\mathcal{U}})$. For the screen distribution $S(TM)'$, consider also the local orthonormal basis $\{W_i'\}, 1 \le i \le n$. As $\text{tr}\,(TM)$ and $\text{tr}\,(TM)'$ are rigging with respect to the same $\xi \in \Gamma(TM^\perp|_{\mathcal{U}})$, it is easy to check that [7, p. 87]

$$\begin{aligned} N' &= N - \frac{1}{2}\left\{\sum_{i=1}^{n} \varepsilon_i (c_i)^2\right\}\xi + \sum_{i=1}^{n} c_i W_i, \\ W_i' &= \sum_{j=1}^{n} A_i^j (W_j - \varepsilon_j c_j \xi)\,, \quad 1 \le i \le n, \end{aligned} \tag{39}$$

where the c_i are smooth functions on $\mathcal{U}$, $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the signature of $\{W_1, \cdots, W_n\}$ and (A_i^j) is an orthogonal matrix of $S(T_xM)$ at any $x \in \mathcal{U}$. Thus, the induced connections ∇ and ∇', with respect to $S(TM)$ and $S(TM)'$, are related by:

$$\nabla'_X Y = \nabla_X Y + B(X,Y)\left\{\frac{1}{2}(\sum_{i=1}^{n}\varepsilon_i(c_i)^2)\xi - \sum_{i=1}^{n} c_i W_i\right\}. \tag{40}$$

□

Lemma 2. *A relationship between the second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$ respectively is as follows*

$$C'(X, PY) = C(X, PY) - \frac{1}{2}||W||^2 B(X,Y) + g(\nabla_X PY, W), \tag{41}$$

where the characteristic vector field of the screen change is given by

$$W = \sum_{i=1}^{n} c_i W_i. \tag{42}$$

Proof. Using (39) and (40) we get

$$\begin{aligned} C'(X, PY) &= \bar{g}(\nabla'_X PY, N') \\ &= \bar{g}\left[\nabla_X PY + B(X,Y)\{\frac{1}{2}(\{\sum_{i=1}^{n}\varepsilon_i(c_i)^2)\xi - \sum_{i=1}^{n} c_i W_i\},\ N'\right] \\ &= \bar{g}\,(\nabla_X PY, N) + \bar{g}(\nabla_X PY, \sum_{i=1}^{n} c_i W_i) \\ &\quad + B(X,Y)\left\{\frac{1}{2}(\sum_{i=1}^{n}\varepsilon_i(c_i)^2) - \sum_{j=1}^{n}\sum_{i=1}^{n} g(c_i W_i, c_j W_j)\right\} \\ &= C(X, PY) + g(\nabla_X PY, W) - \frac{1}{2}||W||^2 B(X,Y) \end{aligned}$$

which is the desired formula. □

Let us denote ω the dual one form of the characteristic vector field W with respect to the metric tensor g, that is

$$\omega(X) = g(X, W), \quad \forall X \in \Gamma(TM). \tag{43}$$

Theorem 5. *If the screen distribution $S(TM)'$ generated by $\{W_i'\}$ as given in (39) is screen locally conformal then the one form ω in (43) vanishes identically on the first derivative space distribution $\mathcal{S}^1$. In particular, if $\mathcal{S}^1$ coincide* *with $S(TM)$ then there is a unique screen locally conformal distribution, up to an orthogonal transformation.*

Proof. If $S(TM)'$ is locally (or globally) conformal then by (38), (41) and the fact that $B = B'$ we obtain

$$g(\nabla_X PY, W) = \phi B(X,Y), \quad \forall X,\ Y \in \Gamma(TM)\,, \tag{44}$$

for some smooth function ϕ on M. As the right hand side of (44) is symmetric in X and Y we obtain

$$g(\nabla_X PY - \nabla_Y PX, W) = 0, \quad \forall X,\ Y \in \Gamma(TM).$$

Therefore, we have $g(\nabla_X Y - \nabla_Y X, W) = 0, \quad \forall X,\ Y \in \Gamma(S(TM))$, that is, $\omega([X,Y]) = g([X,Y],W) = 0,\ \forall\ X,\ Y \in \Gamma(S(TM))$ which is the announced necessary condition.

Now assume that $\mathcal{S}^1 = S(TM)$ that is ω vanishes on $S(TM)$. By (42) and (43) we obtain $W = 0$ that is the functions c_i are zero. Thus, (39) becomes $W_i' = \sum_{j=1}^n A_i^j W_j \ \ (1 \le i \le n)$, where (A_i^j) is an orthogonal matrix of $S(T_xM)$ at any $x \in M$ which completes the proof. □

Acknowledgments

The first author (Atindogbe) is thankful to the Centre Béninois de Recherches Scietifiques et Techniques (CBRST) for its research support through the program Project d'Appui *â* La Formation de Jeunes Chercheurs (PAFJC). The second author (Duggal) is grateful to the Natural Sciences and Engineering Research Council (NSERC) of Canada for an operating research grant.

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