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(p, q, r)-GENERATION OF THE SUZUKI GROUP Suz

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Abstract: A finite group G is called (l, m, n)-generated, if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1\rangle$.

In [17], the question of finding all triples (p, q, r) such that non-abelian finite simple group G is $(p, q, r)-$generated was posed. In this paper we answer this question for the Suzuki group Suz.

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1. Introduction

A group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $o(x) = l, o(y) = m$ and $o(xy) = n$. In this case G is the quotient of the triangle group $T(l, m, n)$ and for any permutation π of S_3, the

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group G is also $((l)\pi,(m)\pi,(n)\pi)$-generated. Therefore we may assume that $l \leq m \leq n$. By [3], if the non-abelian simple group G is (l,m,n)-generated, then either $G \cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Hence for a non-abelian finite simple group G and divisors l,m,n of the order of G such that $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$, it is natural to ask if G is a (l,m,n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [20]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations (for details see [21]).

In a series of papers, [13-18] Moori and Ganief established all possible $(p,q,r)-$ generations, p,q,r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. Also, the second author in [2] and [7-12], did the same for the sporadic groups Co_1, Ru, $O'N$, Ly and He. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Throughout this paper we use the same notation as in [1], [7], [9] and [10]. In particular, $\Delta(G)=\Delta(lX,mY,nZ)$ denotes the structure constant of G for the conjugacy classes lX,mY,nZ, whose value is the cardinality of the set $\Lambda=\{(x,y)|xy=z\}$, where $x\in lX, y\in mY$ and z is a fixed element of the conjugacy class nZ. Also, $\Delta^\star(G)=\Delta^\star_G(lX,mY,nZ)$ and $\sum(H_1\cup H_2\cup\cdots\cup H_r)$ denote the number of pairs $(x,y)\in\Lambda$ such that $G=\langle x,y\rangle$ and $\langle x,y\rangle\subseteq H_i$ (for some $1\leq i\leq r$), respectively. The number of pairs $(x,y)\in\Lambda$ generating a subgroup H of G will be given by $\sum^\star(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^\star(G)>0$, then G is (lX,mY,nZ)-generated and (lX,mY,nZ) is called a generating triple for G.

The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H(cf. [22]). In most cases, we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP [19].

Now we begin with some theorems which give us useful techniques in resolving generation type questions for finite groups.

Lemma 1.2. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action* *on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

We listed in Table 3, the value h for some conjugacy classes of the Suzuki group which we need.

Lemma 1.3. (see [3]). *Suppose a and b are permutations of N points such that a has λ_u cycles of length u (for $1 \leq u \leq l$) and b has μ_v cycles of length v (for $1 \leq v \leq m$) and their product ab is an involution having k transpositions and $N - 2k$ fixed points. If a and b generate a transitive group on these N points, then there exists a non-negative integer p such that*

$$k = 2p - 2 + \sum_{1 \leq v \leq m} \lambda_u + \sum_{1 \leq v \leq m} \mu_v.$$

Throughout this paper our notation is standard and taken mainly from [1], [13] and [15]. The main result of this paper is the following theorem.

Main Theorem. *The Suzuki group* Suz *is $(p, q, r)-$generated if and only if $(p, q, r) \neq (2, 3, 5)$.*

2. (p, q, r)-Generations for Suz

First of all, we investigate the $(2, 3, p)-$generations of the Suzuki group Suz. If this group is $(2, 3, p)-$generated, then by Conder's result [3], $G \cong A_5$, the alternating group on five letters or $\frac{1}{2} + \frac{1}{3} + \frac{1}{p} < 1$. Thus we only need to consider the cases that $p = 7, 11, 13$. On the other hand, by [4, Theorem 3.1], the group Suz is not $(2, 3, 7)-$generated, so it is enough to investigate the cases $p = 11, 13$.

Lemma 2.1 *The group* Suz *is $(2X, 3Y, 11Z)-$generated if and only if $(X, Z) = (B, A)$ and $Y \in \{B, C\}$.*

Proof. We should investigate six cases for $(2X, 3Y, 11A)$, where $X \in \{A, B\}$ and $Y \in \{A, B, C\}$.

Case $(2X, 3A, 11A)$, $X \in \{A, B\}$. In these cases, $\Delta(\text{Suz}) = 0$, so Suz is not $(2A, 3A, 11A)-$ or $(2B, 3A, 11A)-$generated.

Case $(2A, 3B, 11A)$. In this case, $M_{12}.2$ is one of the maximal subgroups of Suz which has non-empty intersection with all conjugacy classes in this triple. Our calculation give $\sum(M_{12}.2) = 11$. Hence $\Delta^*(\text{Suz}) \leq \Delta(\text{Suz}) - \sum(M_{12}.2) = 11 - 11 = 0$ and so Suz is not $(2A, 3B, 11A)-$generated.

Case $(2A, 3C, 11A)$. Consider the second maximal subgroup M_{12} of Suz. By Table 3, we have $\sum(M_{12}) = 55$ and $\Delta^*(\text{Suz}) \leq \Delta(\text{Suz}) - \sum(M_{12}) = 55 - 55 = 0$, which shows the non-generation of Suz by this triple.

Case $(2B, 3B, 11A)$. By Table 3, the only maximal subgroup of Suz which have non-empty intersection with all conjugacy classes in this triple is, up to isomorphism, $M_{12}.2$. So we have, $\Delta^*(\text{Suz}) \leq \Delta(\text{Suz}) - \sum(M_{12}.2) = 77 - 11 > 0$. Hence Suz is $(2B, 3B, 11A)-$generated.

Case $(2B, 3C, 11A)$. By Table 3, the only maximal subgroup of Suz which have non-empty intersection with all conjugacy classes in this triple is, up to isomorphism, $M_{12}.2$. Now we have, $\Delta^*(\text{Suz}) \leq \Delta(\text{Suz}) - \sum(M_{12}.2) = 715 - 11 > 0$. So Suz is $(2B, 3C, 11A)-$generated, proving the lemma. □

Lemma 2.2 *The group* Suz *is* $(2X, 3Y, 13Z)-$*generated if and only if* $(X, Y, Z) = (A, C, A)$ *or* $X = B, Y \in \{B, C\}, Z \in \{A, B\}$.

Proof. Set $R = \{(2A, 3A, 13A), (2A, 3A, 13B), (2A, 3B, 13A), (2A, 3B, 13B), (2B, 3A, 13A), (2B, 3A, 13B)\}$ and $S = \{(2A, 3C, 13A), (2A, 3C, 13B), (2B, 3B, 13A), (2B, 3B, 13B), (2B, 3C, 13A), (2B, 3C, 13B)\}$. If $(2X, 3Y, 13Z) \in R$, then we have $\Delta(\text{Suz}) = 0$ and so Suz is not $(2X, 3Y, 13Z)-$generated. Hence it is enough to investigate the cases that $(2X, 3Y, 13Z) \in S$. Our main proof consider some cases.

Case $(2A, 3C, 13A)$. By Table 3, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes in this triple are isomorphic to $G_2(4), L_3(3).2$ and $\text{Suz}\,M_{15}$. We calculate that $\Delta^*(\text{Suz}) \geq \Delta(\text{Suz}) - \sum(G_2(4)) - \sum(L_3(3).2) - \sum(\text{Suz}\,M_{15}) = 39 - 13 - 0 - 13 > 0$. So Suz is $(2A, 3C, 13A)-$generated.

Case $(2A, 3C, 13B)$. In this case we consider the action of Suz on the cosets of maximal subgroup isomorphic to $G_2(4)$. Since this action is transitive, if χ denotes the permutation character of this action, then $\chi = 1_{G_2(4)}^{\text{Suz}}$ and we have $\chi = 1a + 780a + 1001a$, in which, na denotes the first irreducible character with degree n, in the library of GAP. Now for $g \in \text{Suz}$, the value of $\chi(g)$ is the number of cosets of Suz fixed by g. Suppose $N = |\,\text{Suz} : G_2(4)|$. Then we have:

$$\lambda_3 = \frac{N - 18}{3} = 588\,, \quad \mu_{13} = \frac{N - 1}{13} = 137\,, \quad k = \frac{N - 54}{2} = 864.$$

Therefore by Lemma 1.3, $P = \frac{1591}{2}$ must be an integer, a contradiction. Thus Suz is not $(2A, 3C, 13B)-$generated.

Case $(2B, 3B, 13A)$. By Table 3, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes in this triple are, up to isomorphism $L_3(3).2$ and Suz M_{15}. We have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - \sum(L_3(3).2) - \sum(\mathrm{Suz}\, M_{15}) = 65 - 0 - 0 > 0$. Hence Suz is $(2B, 3B, 13A)-$ generated.

Case $(2B, 3B, 13B)$. In this case, we can see that $L_3(3).2$ and Suz M_{15} are, up to isomorphism, the only maximal subgroups of Suz which have non-empty intersections with all conjugacy classes in this triple. Hence we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - \sum(L_3(3).2) - \sum(\mathrm{Suz}\, M_{15}) = 65 - 0 - 0 > 0$, proving the generation of this triple.

Case $(2B, 3C, 13A)$. The only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes in this triple are, up to isomorphism, $G_2(4)$, $L_3(3).2$, Suz M_{15} and $L_2(25)$. Now we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - \sum(G_2(4)) - \sum(L_3(3).2) - \sum(\mathrm{Suz}\, M_{15}) - 3\sum(L_2(25)) = 858 - 312 - 0 - 0 - 3\times 78 > 0$, so Suz is $(2B, 3C, 13A)-$generated.

Case $(2B, 3C, 13B)$. The only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes in this triple are, up to isomorphism, $G_2(4)$, $L_3(3).2$, Suz M_{15} and $L_2(25)$. We have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - \sum(G_2(4)) - \sum(L_3(3).2) - \sum(\mathrm{Suz}\, M_{15}) - 3\sum(L_2(25)) = 858 - 312 - 0 - 0 - 3\times 78 > 0$. Thus Suz is $(2B, 3C, 13B)-$generated and the proof is completed. □

We now investigate the $(2, 5, p)-$generations of Suz, for $p \geq 7$. In the following lemma, we find all of $(2, 5, 7)-$generations of this group.

Lemma 2.3. *The group* Suz *is* $(2X, 5Y, 7Z)-$*generated if and only if* $(X, Y, Z) = (B, B, A)$.

Proof. We should investigate four cases.

Case $(2A, 5A, 7A)$. The group Suz acts transitively on the cosets of maximal subgroup $2^{1+6}.U_4q_2$. If χ denotes the permutation character of this action, then $\chi = 1_{2^{1+6}.U_4q_2}^{\mathrm{Suz}}$ and we have $\chi = 1a + 780a + 1001a$. For $g \in \mathrm{Suz}$, the value of $\chi(g)$ is the number of cosets of Suz fixed by g. Suppose $N = |\,\mathrm{Suz} : 2^{1+6}.U_4q_2|$. Then we have:

$$\begin{aligned} \lambda_5 &= \frac{135135 - 45}{5} = 27018\,, \quad \mu_7 = \frac{135135 - 0}{7} = 19305\,, \\ k &= \frac{135135 - 415}{2} = 67360. \end{aligned}$$

Therefore by Lemma 1.3, $P = \frac{k+2+\lambda_5+\mu_{13}}{2}$ must be an integer, a contradiction. Thus Suz is not $(2A, 5A, 7A)-$generated.

Case$(2A,5B,7A)$. Using similar argument as above, we consider the action of Suz on cosets of $J_2.2$. Then we have, $N = |\,\mathrm{Suz} : J_2.2|$ and

$$\begin{aligned}\lambda_5 &= \frac{370656-1}{5} = 74131\,, \quad \mu_7 = \frac{370656-6}{7} = 52950\,,\\ k &= \frac{370656-864}{2} = 184896.\end{aligned}$$

Therefore by Lemma 1.3, $P = \frac{k+2+\lambda_5+\mu_7}{2}$ must be an integer, a contradiction. Thus Suz is not $(2A,5B,7A)-$generated.

Case $(2B,5A,7A)$. For this case, we consider maximal subgroup isomorphic to $2^{1+6}U_4q_2$ and do similarity and we see;

$$\begin{aligned}\lambda_5 &= \frac{135135-45}{5} = 27018\,, \quad \mu_7 = \frac{135135-0}{7} = 19305\,,\\ k &= \frac{135135-315}{2} = 67410.\end{aligned}$$

Therefore by Lemma 1.3, $P = \frac{k+2+\lambda_5+\mu_7}{2}$ must be an integer, a contradiction. Thus Suz is not $(2B,5A,7A)-$generated.

Case $(2B,5B,7A)$. By Table 3, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes in this triple are, up to isomorphism, $G_2(4)$, $3_2.U_4(3).2_3$, $J_2.2$, $A_4 \times PSL(3,4):2$ and A_7. Now we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 4\sum(G_2(4)) -4\sum(3_2.U_4(3).2_3) - 6\sum(J_2.2) - \sum(A_4 \times PSL(3,4):2) - 24\sum(A_7) = 7812 - 4\times 336 - 0 - 6\times 14 - 0 - 24\times 28 > 0$. Hence Suz is $(2B,5B,7A)$-generated. This completes the proof. □

Lemma 2.4. *The group* Suz *is* $(2X,5Y,11A)-$*generated if and only if* $X = Y = A$ *or* $X = B$, $Y \in \{A,B\}$.

Proof. We should consider four cases.

Case $(2X,5A,11A)$, $X \in \{A,B\}$. By Table 3, there is no maximal subgroup of Suz with non-empty intersection with all conjugacy classes in triples $(2A,5A,11A)$ or $(2B,5A,11A)$. We calculate that $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) > 0$. Hence Suz is $(2X,5A,11A)-$generated for $X \in A,B$.

Case $(2A,5B,11A)$. In this case, we consider the action of Suz on the cosets of maximal subgroup $J_2.2$. Since this action is transitive, if χ denotes the permutation character of this action, then $\chi = 1_{J_2.2}^{\mathrm{Suz}}$ and we have $\chi = 1a + 780a + 1001a$. Suppose $N = |\,\mathrm{Suz} : J_2.2|$. Then we have:

$$\begin{aligned}\lambda_5 &= \frac{370656-1}{5} = 74131\,, \quad \mu_{11} = \frac{370656-0}{11} = 33696\,,\\ k &= \frac{370656-864}{2} = 184896.\end{aligned}$$

Therefore by Lemma 1.3, $P = \frac{292725}{2}$ must be an integer, which is a contradiction. Thus Suz is not $(2A, 5B, 11A)-$generated.

Case $(2B, 5B, 11A)$. The only maximal subgroup of Suz which has non-empty intersection with all conjugacy classes in this triple is, up to isomorphism, $M_{12}.2$. We have, $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) - \sum(M_{12}.2) = 9295 - 26 > 0$, so Suz is $(2B, 5B, 11A)$-generated, proving the lemma. □

Lemma 2.5. *The group* Suz *is* $(2X, 5Y, 13Z)-$*generated for all 2-, 5- and 13-conjugacy classes.*

Proof. We consider two cases that $X = A$ and $X = B$.

Case $X = A$. In these cases, the only maximal subgroup of Suz which has non-empty intersections with all conjugacy classes in the triples is, up to isomorphism, $G_2(4)$. Our calculations show that $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) - \sum(G_2(4))$. Thus, $\Delta^*(\text{Suz})(2A, 5A, 13A) = 78 - 26 > 0$, $\Delta^*(\text{Suz})(2A, 5A, 13B) = 78-26 > 0$, $\Delta^*(\text{Suz})(2A, 5B, 13A) = 390-26 > 0$ and $\Delta^*(\text{Suz})(2A, 5B, 13B) = 390 - 26 > 0$. Hence Suz is $(2A, 5Y, 13Z)-$generated, for $Y, Z \in A, B$.

Case $X = B$. The only maximal subgroups of Suz which have non-empty intersections with all conjugacy classes of the triples are, up to isomorphism, $G_2(4)$ and $L_2(25)$. Our calculations give $\Delta^*(\text{Suz}) \geq \Delta(\text{Suz}) - \sum(G_2(4)) - 3\sum(L_2(25))$, and hence $\Delta^*(\text{Suz})(2B, 5A, 13A) \geq 1547 - 416 - 3 \times 39 > 0$, $\Delta^*(\text{Suz})(2B, 5A, 13B) \geq 1547 - 416 - 3 \times 39 > 0$, $\Delta^*(\text{Suz})(2B, 5B, 13A) \geq 8996-416-3\times39 > 0$ and $\Delta^*(\text{Suz})(2B, 5B, 13B) \geq 8996-416-3\times39 > 0$. This shows that the sporadic group Suz is $(2B, 5Y, 13Z)-$generated for $Y, Z \in A, B$, which proves the lemma. □

Lemma 2.6. *The group* Suz *is* $(2X, 7A, 11A)-$*generated, for* $X = A, B$.

Proof. There are just two cases, $(2A, 7A, 11A)$ and $(2B, 7A, 11A)$. In these cases, there is no maximal subgroup of Suz which has non-empty intersection with all cojugacy classes in the triples, so in the first case we have $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) = 1474 > 0$, and for the second case, we have, $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) = 30184 > 0$. Hence Suz is $(2X, 7A, 11A)$, for $X = A, B$. □

Lemma 2.7. *The group* Suz *is* $(2X, 7A, 13Z)-$*generated, for* $X, Z \in \{A, B\}$.

Proof. We must investigate four triples $(2A, 7A, 13A)$, $(2A, 7A, 13B)$, $(2B, 7A, 13A)$ and $(2B, 7A, 13B)$. For each cases, the only maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple is, up to isomorphism, $G_2(4)$. Our calculations give $\Delta^*(\text{Suz})(2A, 7A, 13A) = 1755 - 221 > 0$, $\Delta^*(\text{Suz})(2A, 7A, 13B) = 1755 - 221 > 0$, $\Delta^*(\text{Suz})(2B, 7A,$

$13A) = 33709 - 31120 > 0$ and $\Delta^*(\mathrm{Suz})(2B, 7A, 13B) = 33709 - 31120 > 0$. Therefore the group Suz is $(2X, 7A, 13Z)-$generated, for $X, Z \in \{A, B\}$. □

Lemma 2.8. *The group* Suz *is* $(2X, 11A, 13Z)-$*generated, for* $X, Z \in \{A, B\}$.

Proof. We must consider four triples $(2A, 11A, 13A)$, $(2A, 11A, 13B)$, $(2B, 11A, 13A)$ and $(2B, 11A, 13B)$. In any case, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes and so $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) > 0$. Hence Suz is $(2X, 11A, 13Z)$-generated for $X, Z \in \{A, B\}$. □

Lemma 2.9. *The group* Suz *is* $(3X, 5Y, 7A)-$*generated if and only if* $X \in \{B, C\}, Y \in \{A, B\}$.

Proof. We must consider six triples.

Case $(3A, 5A, 7A)$. The group Suz acts transitively on the cosets of maximal subgroup $3_2.U_4(3).2_3$. If χ denotes the permutation character of this action, then $\chi = 1_{3_2.U_4(3).2_3}^{\mathrm{Suz}}$ and we have $\chi = 1a + 364a + 780a + 5940a + 15795a$. For $g \in \mathrm{Suz}$, the value of $\chi(g)$ is the number of cosets of Suz fixed by g. Suppose $N = |\,\mathrm{Suz} : 3_2.U_4(3).2_3|$. Then we have:

$$\lambda_5 = \frac{22880 - 0}{5} = 4576\,, \quad \mu_7 = \frac{22880 - 4}{7} = 3268\,,$$
$$k = \frac{22880 - 281}{3} = 7533.$$

Therefore by Lemma 1.3, $P = \frac{k+2+\lambda_5+\mu_7}{2}$ must be an integer, a contradiction. Thus Suz is not $(3A, 5A, 7A)-$generated.

Triple $(3A, 5B, 7A)$. For this case, again we consider the action of Suz on the cosets of the maximal subgroup isomorphic to $3_2.U_4(3).2_3$, and we have;

$$\lambda_5 = \frac{22880 - 10}{5} = 4574\,, \quad \mu_7 = \frac{22880 - 4}{7} = 3268\,,$$
$$k = \frac{22880 - 281}{3} = 7533.$$

Therefore by Lemma 1.3, $P = \frac{k+2+\lambda_5+\mu_7}{2}$ must be a integer, a contradiction. Thus Suz is not $(3A, 5B, 7A)-$generated.

Triple $(3B, 5A, 7A)$. In this case, there is no maximal subgroup of Suz which have non-empty intersection with all conjugacy classes of the triple. So $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) = 16128 > 0$, hence Suz is $(3B, 5A, 7A)-$generated.

Case $(3B, 5B, 7A)$. In this case, the only maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple is, up to isomorphism, $3_2.U_4(3).2_3$. So $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 4\sum(3_2.U_4(3).2_3) = 61208 - 4 \times 190 > 0$, so Suz is $(3B, 5B, 7A)-$generated.

Triple $(3C, 5A, 7A)$. In this case, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes of the triple are, up to isomorphism, $G_2(4)$ and $J_2.2$. So $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 4\sum(G_2(4)) - 6\sum(J_2.2) = 75852 - 4 \times 7938 - 6 \times 686 > 0$, so Suz is $(3C, 5A, 7A)-$generated.

Triple $(3C, 5B, 7A)$. In this case, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes of the triple are, up to isomorphism, $G_2(4)$, $3_2.U_4(3).2_3$, $J_2.2$, $A_4 \times PSL(3,4) : 2$ and A_7. So $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 4\sum(G_2(4)) - 4\sum(3_2.U_4(3).2_3) - 6\sum(J_2.2) - \sum(A_4 \times PSL(3,4) : 2) - 24\sum(A_7) = 438186 - 4 \times 9282 - 4 \times 8064 - 6 \times 112 - 882 - 24 \times 126 > 0$, hence Suz is $(3C, 5B, 7A)-$generated and the result follows. □

Lemma 2.10. *The group* Suz *is* $(3X, 5Y, 11A)-$*generated if and only if* $X \in \{A, B, C\}$ *and* $Y \in \{A, B\}$ *unless* $(X, Y) = (A, A)$.

Proof. First of all, we see that if $X = Y = A$, then $\Delta(\mathrm{Suz}) = 0$, so Suz is not $(3A, 5A, 11A)-$generated.

We must consider four Cases.

Case $(3A, 5B, 11A)$. By Table III, the only maximal subgroups of Suz which have non-empty intersections with all conjugacy classes in this triple are, up to isomorphism, $U_5(2)$ and $3^5 : M_{11}$. Now we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 2\sum(U_5(2)) - 2\sum(3^5 : M_{11}) = 132 - 2 \times 44 - 0 > 0$. So Suz is $(3A, 5B, 11A)-$generated.

Case $(3B, 5A, 11A)$ *and* $(3C, 5A, 11A)$. By Table 3, there is no maximal subgroup of Suz which have non-empty intersection with all conjugacy classes in this triple. So we have, $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) > 0$. Hence Suz is $(3X, 5A, 11A)-$generated for $X \in \{B, C\}$.

Case $(3B, 5B, 11A)$. The only maximal subgroups of Suz which have non-empty intersections with all conjugacy classes in this triple are, up to isomorphism, $U_5(2)$, $3^5 : M_{11}$ and $M_{12}.2$. Now we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 2\sum(U_5(2)) - 2\sum(3^5 : M_{11}) - \sum(M_{12}.2) = 48864 - 2 \times 7480 - 2 \times 1782 - 198 > 0$, hence Suz is $(3B, 5B, 11A)-$generated.

Case $(3C, 5B, 11A)$. The only maximal subgroups of Suz which have non-empty intersections with all conjugacy classes in this triple are, up to isomorphism, $3^5 : M_{11}$ and $M_{12}.2$. So we have, $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - 2\sum(3^5 : M_{11}) - \sum(M_{12}.2) = 481547 - 2 \times 3564 - 253 > 0$, hence Suz is $(3C, 5B, 11A)-$generated. □

Lemma 2.11. *The group* Suz *is* $(3X, 5Y, 13Z)$-*generated for* $X \in \{B, C\}$, $Y, Z \in \{A, B\}$ *or* $(X, Y) = (A, B)$,$Z \in \{A, B\}$.

Proof. The group Suz is not $(3A, 5A, 13A)$ and $(3A, 5A, 13B)$-generated because the subgroup $G_2(4)$ have non-empty intersection with all of conjugacy classes in these cases. Thus for both cases, we have, $\Delta^*(\mathrm{Suz}) \leq \Delta(\mathrm{Suz}) - \sum(G_2(4)) = 26-26 = 0$. Hence Suz is not $(3A, 5A, 13A)$ and $(3A, 5A, 13B)-$generated. We show that the group Suz is $(3X, 5Y, 13Z)-$ generated for remainder cases.

Case $(3A, 5B, 13Z), Z \in \{A, B\}$. The only maximal subgroup of Suz which has non-empty intersections with all conjugacy classes of the triple is, up to isomorphism, $G_2(4)$. So for both cases, we have $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) - \sum(G_2(4)) = 130 - 26 > 0$, hence Suz is $(3A, 5B, 13Z)-$ generated for $Z \in \{A, B\}$.

Case $(3B, 5Y, 13Z), Y, Z \in \{A, B\}$. By Table 3, Suz has no maximal subgroup which has non-empty intersection with all conjugacy classes of the triple in these four cases. Thus we have $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) > 0$. Hence Suz is $(3B, 5Y, 13Z)-$generated for $Y, Z \in \{A, B\}$.

Case $(3C, 5Y, 13Z), Y, Z \in \{A, B\}$. By Table 3, in all of these cases, the only maximal subgroups of Suz which have non-empty intersection with all conjugacy classes of the triple are, up to isomorphism, $G_2(4)$ and $L_2(25)$. Our calculations show that $\Delta^*(\mathrm{Suz}) \geq \Delta(\mathrm{Suz}) - \sum(G_2(4)) - 3\sum(L_2(25))$. Thus, $\Delta^*(\mathrm{Suz})(3C, 5A, 13A) \geq 76843 - 9282 - 3 \times 78 > 0$, $\Delta^*(\mathrm{Suz})(3C, 5A, 13B) \geq 76843 - 9282 - 3 \times 78 > 0$, $\Delta^*(\mathrm{Suz})(3C, 5B, 13A) \geq 461006 - 9282 - 3 \times 78 > 0$ and $\Delta^*(\mathrm{Suz})(3C, 5B,\ 13B) \geq 461006 - 9282 - 3 \times 78 > 0$. Therefore Suz is $(3C, 5Y, 13Z)$-generated for $Y, Z \in \{A, B\}$ which proves the Lemma. □

Lemma 2.12. *The group* Suz *is* $(3X, 7Y, 11Z)-$*generated if and only if* $X \in \{A, B, C\}$, *and* $Y = Z = A$.

Proof. We must investigate three triples $(3A, 7A, 11A)$, $(3B, 7A, 11A)$ and $(3C, 7A, 11A)$. In any case, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple and so $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) > 0$. Hence Suz is $(3X, 7A, 11A)-$generated for $X \in \{A, B, C\}$. □

Lemma 2.13. *The group* Suz *is* $(3X, 7A, 13Z)-$*generated if and only if* $X \in \{A, B, C\}$ *and* $Z \in \{A, B\}$.

Proof. It is enough to consider two following cases.

Case $X \in \{A, C\}$. In each of these cases, the only maximal subgroup of Suz which has non-empty intersection with all conjugacy classes in the triple is, up to isomorphism, $G_2(4)$ and we have $\Delta^*(\mathrm{Suz}) = \Delta(\mathrm{Suz}) - \sum(G_2(4)) > 0$. Thus $\Delta^*(\mathrm{Suz})(3A, 7A, 13A) = 598 - 25 > 0$, $\Delta^*(\mathrm{Suz})(3A, 7A, 13B) = 598 - 221 > 0$, $\Delta^*(\mathrm{Suz})(3C, 7A, 13A) = 1647685 - 66625 > 0$ and $\Delta^*(\mathrm{Suz})\ (3C, 7A, 13B) = 1647685 - 66625 > 0$. Therefore, Suz is $(3X, 7A, 13Z)-$ generated for $X \in \{A, C\}$, $Z \in \{A, B\}$.

Case $X = B$. In this case, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple and we have, $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) = 154570 > 0$. Hence Suz is $(3B, 7A, 13Z)-$generated for $Z \in \{A, B\}$ and the proof is completed. □

Lemma 2.14. *The group* Suz *is* $(3X, 11A, 13Z)-$*generated if and only if* $X \in \{A, B, C\}$ *and* $Z \in \{A, B\}$.

Proof. We must consider six triples $(3A, 11A, 13A)$, $(3A, 11A, 13B)$, $(3B, 11A, 13A)$, $(3B, 11A, 13B)$, $(3C, 11A, 13A)$ and $(3C, 11A, 13B)$. In any case, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple and so $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) > 0$. Therefore Suz is $(3X, 11A, 13Z)-$generated for $X \in \{A, B, C\}$, $Z \in \{A, B\}$. □

Lemma 2.15. *The group* Suz *is* $(5X, 7A, 11A)-$*generated if and only if* $X \in \{A, B\}$.

Proof. We must consider two triples $(5A, 7A, 11A)$ and $(5B, 7A, 11A)$. In both cases, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple and so $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) > 0$. Hence Suz is $(5X, 7A, 11A)-$generated for $X \in \{A, B\}$. □

Lemma 2.16. *The group* Suz *is* $(5X, 7A, 13Z)-$*generated if and only if* $X, Z \in \{A, B\}$.

Proof. We must consider four triples $(5A, 7A, 13A)$, $(5A, 7A, 13B)$, $(5B, 7A, 13A)$ and $(5B, 7A, 13B)$. In each cases, the only maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple is, up to isomorphism, $G_2(4)$. So $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) - \sum(G_2(4))$, and hence $\Delta^*(\text{Suz})(5A, 11A, 13A) = 2965534 - 79794 > 0$, $\Delta^*(\text{Suz})(5A, 11A, 13B) = 2965534 - 79794 > 0$, $\Delta^*(\text{Suz})\ (5B, 11A, 13A) = 17837586 - 79794 > 0$ and $\Delta^*(\text{Suz})(5B, 11A, 13B) = 17837586 - 79794 > 0$. Therefore Suz is $(5X, 7A, 13Z)-$generated for $X, Z \in \{A, B\}$. □

Lemma 2.17. *The group* Suz *is* $(5X, 11A, 13Z)-$*generated if and only if* $X, Z \in \{A, B\}$.

Proof. We must consider four triples $(5A, 11A, 13A)$, $(5A, 11A, 13B)$, $(5B, 11A, 13A)$ and $(5B, 11A, 13B)$. In any case, there is no maximal subgroup of Suz which has non-empty intersection with all conjugacy classes of the triple, so $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) > 0$. Therefore Suz is $(5X, 11A, 13Z)-$generated for $X, Z \in \{A, B\}$. □

Lemma 2.18. *The group* Suz *is* $(7A, 11A, 13Z)-$*generated if and only if* $Z \in \{A, B\}$.

Proof. We should consider two triples $(7A, 11A, 13A)$ and $(7A, 11A, 13B)$.

Group	Order	Group	Order
$G_2(4)$	$2^{12}.3^3.5^2.7.13$	$M_{12}.2$	$2^7.3^3.5.11$
$3_2.U_4(3).2_3$	$2^8.3^7.5.7$	$3^{2+4}:2(2^2\times A_4)2$	$2^6.3^7$
$U_5(2)$	$2^{10}.3^5.5.11$	$(A_6\times A_5).2$	$2^6.3^3.5^2$
$2^{1+6}.U_4(2)$	$2^{13}.3^4.5$	$(3^2:4\times A_6).2$	$2^6.3^4.5$
$3^5:M_{11}$	$2^4.3^7.5.11$	$L_3(3).2$	$2^5.3^3.13$
$J_2.2$	$2^8.3^3.5^2.7$	Suz M_{15}	$2^5.3^3.13$
$2^{4+6}:3A_6$	$2^{13}.3^3.5$	$L_2(25)$	$2^3.3.5^2.13$
$A_4\times PSL(3,4)$	$2^9.3^3.5^2.7$	A_7	$2^3.3^2.5.7$
$2^{2+8}(A_4\times S_3)$	$2^{13}.3^2.5$		

Table 1: The maximal subgroups of Suz

pX	$\Delta(2A,3A,pX)$	$\Delta(2A,3B,pX)$	$\Delta(2A,3C.pX)$	$\Delta(2B,3A,pX)$
$11A$	0	0	55	0
$13A$	0	0	39	0
$13B$	0	0	39	0
pX	$\Delta(2B,3B,pX)$	$\Delta(2B,3C,pX)$		
$11A$	77	715		
$13A$	65	858		
$13B$	65	858		

Table 2: The structure constants of the Group Suz

In each cases, there is no maximal subgroup which has non-empty intersection with all conjugacy classes of the triple, and so $\Delta^*(\text{Suz}) = \Delta(\text{Suz}) > 0$. Therefore Suz is $(7A, 11A, 13Z)-$generated for $Z \in \{A,\ B\}$. □

Therefore, by the above lemmas, we proved the main Theorem.

pX	$\Delta(2A,5A,pX)$	$\Delta(2A,5B,pX)$	$\Delta(2B,5A,pX)$	$\Delta(2B,5B,pX)$
7A	252	686	2436	7812
11A	66	671	1232	9295
13A	78	390	1547	8996
13B	78	390	1547	8996
pX	$\Delta(2A,7A,pX)$	$\Delta(2B,7A,pX)$	$\Delta(2A,11A,pX)$	$\Delta(2B,11A,pX)$
11A	1474	30184		
13A	1775	33709	12285	252785
13B	1775	33709	12285	252785
pX	$\Delta(3A,5A,pX)$	$\Delta(3A,5B,pX)$	$\Delta(3B,5A,pX)$	$\Delta(3B,5B,pX)$
7A	336	392	16128	16128
11A	0	132	7172	48862
13A	26	130	7085	41730
13B	26	130	7085	41730
pX	$\Delta(3C,5A,pX)$	$\Delta(3C,5B,pX)$	$\Delta(3A,7A,pX)$	$\Delta(3B,7A,pX)$
7A	75852	438186		
11A	70642	481547	308	154330
13A	76843	461006	598	154570
13B	76843	461006	598	154570
pX	$\Delta(3C,7A,pX)$	$\Delta(3A,11A,pX)$	$\Delta(3B,11A,pX)$	$\Delta(3C,11A,pX)$
11A	1606968			
13A	1647685	4160	11664800	12580295
13B	1647685	4160	11664800	12580295
pX	$\Delta(5A,7A,pX)$	$\Delta(5B,7A,pX)$	$\Delta(5A,11A,pX)$	$\Delta(5B,11A,pX)$
11A	2853136	17789134		
13A	2965534	17837586	22644440	135863910
13B	2965534	17837586	22644440	135863910
pX	$\Delta(7A,11A,pX)$			
13A	485216550			
13B	485216550			

Table 2: (Continuation) The maximal subgroups of Suz

$G_2(4)$-class	2a	2b	3a	3b	5a	5b	5c	5d	7a	
$\rightarrow$ Suz	2A	2B	3A	3C	5B	5B	5A	5A	7A	
h									4	
$G_2(4)$-class	13a	13b								
$\rightarrow$ Suz	13A	13B								
h	1	1								
$3_2.u_4(3).2_3$-class	2a	2b	3a	3b	3c	3d	3e	5a	7a	
$\rightarrow$ Suz	2A	2B	3A	3A	3B	3B	3C	5B	7A	
h									4	
$U_5(2)$-class	2a	2b	3a	3b	3c	3d	3e	3f	5a	
$\rightarrow$ Suz	2A	2B	3A	3A	3B	3B	3B	3B	5B	
$U_5(2)$-class	11a	11b								
$\rightarrow$ Suz	11A	11A								
h	2	2								
$3^5 : M_{11}$-class	2a	3a	3b	3c	3d	5a	11a	11b		
$\rightarrow$ Suz	2A	3A	3B	3B	3C	5B	11A	11A		
h							2	2		
$J_2.2$-class	2a	2b	2c	3a	3b	5a	5b	7a		
$\rightarrow$ Suz	2A	2B	2B	3A	3C	5B	5A	7A		
h								6		
$A_4 \times PSL_3(4) :$ 2-class	2a	2b	2c	2d	3a	3b	3c	5a	7a	
$\rightarrow$ Suz	2A	2B	2B	2B	3C	3A	3C	5B	7A	
h									1	
$M_{12}.2$-class	2a	2b	2c	3a	3b	5a	11a			
$\rightarrow$ Suz	2B	2A	2B	3B	3C	5B	11A			
h							1			
$L_3(3).2$-class	2a	2b	3a	3b	13a	13b				
$\rightarrow$ Suz	2A	2B	3B	3C	13A	13B				
h					1	1				
Suz M_{15}-class	2a	2b	3a	3b	13a	13b				
$\rightarrow$ Suz	2A	2B	3B	3C	13B	13A				
h					1	1				
$L_2(25)$-class	2a	3a	5a	5b	13a	13b	13c	13d	13e	13f
$\rightarrow$ Suz	2B	3C	5A	5B	13A	13B	13A	13B	13A	13B
h					3	3	3	3	3	3
A_7-class	2a	3a	3b	5a	7a	7b				
$\rightarrow$ Suz	2B	3C	3C	5B	7A	7A				
h					24	24				

Table 3: The partial fusion maps into Suz

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