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ON THE JOINS OF OSCULATING LINEAR SPACES TO PROJECTIVE VARIETIES

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Abstract: Let $X \subset \mathbf{P}^N$ be an integral variety. For a general $P \in X$ let $T_P X(m)$ the osculating space of order m of X in $\mathbf{P}^N$. Here we study $\dim(\langle T_{P_1} X(m) \cup \cdots \cup T_{P_k} X(m) \rangle)$ for general $P_i \in X$.

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Let X be a quasi-projective n-dimensional integral variety. For any integer $m > 0$ let mP be the infinitesimal neighborhood of order $m-1$ of P in X, i.e. the closed subscheme of X with $(\mathcal{I}_{P,X})^m$ as its ideal sheaf. Thus $(mP)_{red} = \{P\}$ and $\text{length}(mP) = \binom{n+m-1}{n}$. Fix $L \in \text{Pic}(X)$ and a finite-dimensional linear subspace V of $H^0(X, L)$. For any zero-dimensional scheme $Z \subset X$, set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes L)$. The function $\dim(V(-mP))$ is semicontinuous on X_{reg} and we will call $\alpha(m, X, V)$ its minimum. Hence $X(m, V, *) := \{P \in X_{reg} : \dim(V(-mP)) = \alpha(m, V, X)\}$ is a non-empty Zariski open subset of X. If $\alpha(m, X, V) > 0$ the integer $\delta(m, X, V) := \binom{n+m-1}{n} + \alpha(m, X, V) -$

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dim(V) is called the m-defect of V. We will say that the pair (X, V) (or just the linear system V) is m-defective if $\alpha(m, X, V) > 0$ and $\delta(m, X, V) > 0$. See each $V(-mP)$, $P \in X(m, V, ast)$ as a linear subspace of V and set $W_m := \cap_{P \in X(m,V,ast)} V(-mP)$ and $\gamma(m, V, * := \dim(W_m)$. To obtain W_m it is sufficient to take the intersections for all P in any non-empty open subset of $X(m, V, *)$. If $W_m \neq \{0\}$ then we will say that V is m-strange and call W_m or its projectivization $|W_m|$ the m-vertex of V or of its projectivization $|V|$. Here we prove the following result and hence the two following immediate corollaries.

Theorem 1. *Let X be a quasi-projective n-dimensional integral variety, $L \in \mathrm{Pic}(X)$, V a finite-dimensional linear subspace of $H^0(X, L)$ and m a positive integer. Fix a general $(P_1, P_2, P_3) \in X \times X \times X$ and assume $\delta(m, V, X) > 0$, $V(-mP_1)$ not m-strange and $V(-mP_1 - mP_2 - mP_3) \neq \{0\}$. Then $\dim(V(-mP_1-mP_2))-\dim(V(-mP_1-mP_2-mP_3)) < \dim(V(-mP_1))-\dim(V(-mP_1 - mP_2))$*

Corollary 1. *Let X be a quasi-projective n-dimensional integral variety, $L \in \mathrm{Pic}(X)$, V a finite-dimensional linear subspace of $H^0(X, L)$ and m a positive integer. Fix general $P_i \in X$, $i \geq 1$. Assume V not m-strange and that for every $i \geq 1$ either $V(-mP_1 - \cdots - mP_i) \neq \{0\}$ or $V(-mP_1 - \cdots - mP_i)$ is not m-strange. Set $v(0) := \dim(V)$, $v(i) := \dim(V(-mP_1 - \cdots - mP_i))$, $i \geq 1$, and $w(j) := v(j-1) - v(j)$, $j \geq 1$. Assume $\delta(m, V, X) > 0$. Then $w(j) < w(j-1)$ for every integer $j \geq 1$ such that $V(-mP_1 - \cdots - mP_j) \neq \{0\}$*

Corollary 2. *Let X be a quasi-projective n-dimensional integral variety, $L \in \mathrm{Pic}(X)$, V a finite-dimensional linear subspace of $H^0(X, L)$ and m a positive integer. Fix general $P_i \in X$, $i \geq 1$. Assume V not m-strange and that for every $i \geq 1$ either $V(-mP_1 - \cdots - mP_i) = \{0\}$ or $V(-mP_1 - \cdots - mP_i)$ is not m-strange. Assume $\delta(m, V, X) > 0$. Then $\dim(V) \leq \alpha(m, V, X) + (\alpha(m, V, X) - \delta(m, V, X))(\alpha(m, V, X) - \delta(m, V, X) + 1)/2$. In particular we have the inequality $\dim(V) \leq \binom{n+m-1}{n}(\binom{n+m-1}{n} + 1)/2$.*

We work over an algebraically closed field $\mathbf{K}$. For a proof and a geometric interpretation of these results when $m = 2$ and $\mathrm{char}(\mathbf{K}) = 0$, see [1], §1, and [2], 2.1, 2.2 and 2.3.

Remark 1. The inequality $\dim(V(-mP_1)) - \dim(V(-mP_1 - mP_2)) \leq \dim(V) - \dim(V(-mP_1))$ for general $(P_1, P_2) \in X \times X$ is quite obvious even when V is m-strange (see the proof of Theorem 1).

Proof of Theorem 1. The linear subspaces $V(-mP_1)$ and $V(-mP_2)$ of V have the same codimension. We have $V(-mP_x - mP_y) = V(-mP_x) \cap V(-mP_y)$ if $x \neq y$ and $V(-mP_1 - mP_2 - mP_3) = V(-mP_1) \cap V(-mP_2) \cap V(-mP_3)$. Hence the codimension of $V(-P_1 - mP_2 - mP_3)$ in $V(-mP_1 - mP_2)$ is strictly smaller than the codimension of $V(-mP_1 - mP_2)$ in $V(-mP_1)$ unless $V(-mP_x) \cap V(-mP_y)$ is the same linear subspace of V for all $x \neq y$. Since $V(-mP_1)$ is not m-strange, this is not the case for general P_1, P_2, P_3. □

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References

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