International Journal of Pure and Applied Mathematics
Volume 12 No. 1 2004, 1-21

A MULTIPLICATIVE SCHWARZ ALGORITHM FOR THE GALERKIN BOUNDARY ELEMENT APPROXIMATION OF THE WEAKLY SINGULAR INTEGRAL OPERATOR IN THREE DIMENSIONS

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Abstract: We study a multiplicative Schwarz method for the h-version Galerkin boundary element method for a weakly singular integral equation of the first kind in 3 dimensions. We prove a bound for the contraction rate of the multiplicative Schwarz operator which depends on the quotient $n_H = H/h$ of the mesh sizes H and h of the coarse grid space and the fine grid space, respectively. The boundary element space consists of piecewise constant functions on surface meshes. Computational results are presented which support our theory.

AMS Subject Classification: 65N55, 65N38
Key Words: Galerkin boundary element method, weakly singular integral equations, multiplicative Schwarz, domain decomposition

Received: April 7, 2003

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1. Introduction

We study a multiplicative Schwarz method for the h-version of the Galerkin boundary element method applied to the weakly singular integral equation on open manifolds (cracks). This equation is an integral reformulation for boundary value problems with the Laplace equation and Dirichlet boundary conditions (see [4]). The use of the Galerkin method to solve this integral equation results in a linear system with a dense, symmetric and positive definite stiffness matrix. Since the condition number of this matrix grows like h^{-2} [14], the contraction rate of the conjugate gradient algorithm approaches 1 (the non-convergent status of the iterative method) like $1 - ch^{1/2}$ as $h \to 0$ with a positive constant c. We shall propose a multiplicative Schwarz algorithm which significantly reduces the contraction rate compared with the conjugate gradient method.

We use an integral mean zero basis to compute the local Galerkin matrix for the weakly singular operator instead of preconditioning with a second order difference operator as in [1]. The analysis is based on the abstract framework of [2] which requires a strengthened Cauchy-Schwarz inequality, the proof of which is the major task of this paper. A comprehensive introduction of the analysis of the method can be found in [15]. See also [3] for the design of multiplicative Schwarz algorithms for nonsymmetric and indefinite problems.

The strengthened Cauchy-Schwarz inequality that we prove does not involve the coarse-grid subspace. Instead, the contribution of this subspace to the error prolongation operator is represented in the bounds for the extremal eigenvalues of the corresponding additive Schwarz operator; see [9], see also [2, 15]. In the proof of the contraction rate, we slightly modify the analysis in [2] to obtain an estimate without a log term (see Theorem 4.3 and Remark 4.4).

In Section 2 we introduce the general setting of multiplicative Schwarz methods as in [2, 15], and present the fundamental theorem for the convergence analysis of the methods. The model problem is introduced in Section 3. In Section 4 we prove the strengthened Cauchy-Schwarz inequality and give estimates for the contraction rate of the multiplicative Schwarz method. Numerical experiments are presented in Section 5. Section 5 is devoted to the proofs of some technical results.

2. General Setting of Multiplicative Schwarz Methods

Multiplicative Schwarz methods (as additive Schwarz methods) are in general defined via a subspace decomposition of the space of test and trial functions together with projections onto these subspaces. In this section, we present the abstract form of the methods.

Consider a general problem on a finite-dimensional space V:

$$\text{Find } u \in V \text{ such that } a(u,v) = f(v)\,, \quad \forall v \in V, \tag{2.1}$$

where $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a symmetric, bounded, and coercive bilinear form, and $f : V \to \mathbb{R}$ is a bounded linear functional. The bilinear form $a(\cdot,\cdot)$ defines a norm on V:

$$\|v\|_a = a(v,v)^{1/2}.$$

To define a multiplicative Schwarz method for (2.1), we represent V as

$$V = V_0 + \ldots + V_J,$$

where V_j, $j = 0,\ldots,J$, are subspaces of V. Let $P_j : V \to V_j$, $j = 0,\ldots,J$, be projections defined by

$$a(P_j v, \phi) = a(v,\phi)\,, \quad \forall v \in V, \quad \phi \in V_j.$$

The multiplicative Schwarz operator is then defined as

$$P_{\mathrm{MS}} := I - E_J,$$

where I is the identity map on V, and

$$E_J := (I - P_0)(I - P_1)\cdots(I - P_J)$$

is the error propagation operator. The corresponding additive Schwarz operator is defined as

$$P_{\mathrm{AS}} := \sum_{j=0}^{J} P_j.$$

By defining

$$E_{-1} := I \quad \text{and} \quad E_i := (I - P_i)E_{i-1}, \quad i = 0,\ldots,J-1,$$

we obtain

$$E_{j-1} - E_j = P_j E_{j-1};$$

which in turn yields

$$I - E_i = \sum_{j=0}^{i} P_j E_{j-1}, \quad i = 0, \ldots, J. \tag{2.2}$$

Let Θ be a $J \times J$ matrix whose elements θ_{ij}, $i, j = 1, \ldots, J$ are defined by

$$\theta_{ij} := \sup_{u \in V_i, v \in V_j} \frac{a(u,v)}{a(u,u)^{1/2} a(v,v)^{1/2}}, \tag{2.3}$$

and let $\|\Theta\|_2$ denote the 2-norm of the matrix Θ. Note that the space V_0, traditionally defined as the coarse-grid space, does not involve in Θ. The following result, see e.g. [2, 9, 15], describes how the error propagation operator of the multiplicative Schwarz method depends on bounds for the minimal and maximal eigenvalues of the additive Schwarz operator and on $\|\Theta\|_2$.

Theorem 2.1. *If there exist positive constants C_0, C_2 satisfying*

$$C_0 a(v,v) \le a(P_{\mathrm{AS}} v, v) \le C_2 a(v,v)\,, \quad \forall v \in V,$$

and if C_1 is a constant defined by $C_1 := 2\max\{C_2, \|\Theta\|_2^2\}$, then there holds

$$\|E_J v\|_a^2 \le \left(1 - \frac{C_0}{C_1}\right) \|v\|_a^2.$$

As we shall see later, a direct use of this theorem results in an estimate of the form

$$\|E_J v\|_a^2 \le \left(1 - \frac{1}{c \cdot n_H^2 \log n_H}\right) \|v\|_a^2.$$

In Section 4, we shall prove that a more optimal estimate without the log term can be achieved.

3. A Model Problem

We consider the weakly singular integral equation

$$W\psi(x) := \frac{1}{2\pi} \int_\Gamma \frac{\psi(y)}{|x-y|}\, ds_y = f(x), \quad x \in \Gamma = (-1,1)^2 \times \{0\} \subset \mathbb{R}^3. \tag{3.1}$$

As shown in [4], W is continuous and invertible from $\widetilde{H}^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. Here the Sobolev space $H^{1/2}(\Gamma)$ is defined as the space of functions which are traces of functions in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, and $\widetilde{H}^{-1/2}(\Gamma)$ is its dual with respect to the L^2-inner product; see e.g. [5, 7].

It is known that there exists a constant $\gamma > 0$ such that

$$\langle Wv, v\rangle \geq \gamma \|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)},$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 duality on Γ. Hence W defines a continuous, symmetric and positive-definite bilinear form on $\widetilde{H}^{-1/2}(\Gamma)$:

$$a(v,w) := W(v,w) := \langle Wv, w\rangle\,, \quad \forall v, w \in \widetilde{H}^{-1/2}(\Gamma).$$

The norm defined by a will now be denoted by $\|\cdot\|_W$ instead of $\|\cdot\|_a$.

Let $h := 2/n$ for some positive integer n, and let

$$\begin{cases} x_i & := -1 + ih, \qquad i = 0, \ldots, n, \\ y_j & := -1 + jh, \qquad j = 0, \ldots, n. \end{cases}$$

We consider a partition of Γ into subelements Γ_{ij},

$$\Gamma = \bigcup_{i,j=1}^{n} \Gamma_{ij},$$

where $\Gamma_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and define on this mesh the space V of piecewise-constant functions. We note that V is a finite-dimensional subspace of $\widetilde{H}^{-1/2}(\Gamma)$ of dimension $N := n^2$. The h-version boundary element method for (3.1) reads as:

$$\text{Find } u \in V \text{ such that } a(u,v) = \langle f, v\rangle\,, \quad \forall v \in V. \tag{3.2}$$

The stability and convergence of the Galerkin scheme (3.2) was proved in [12]. The condition number of the Galerkin matrix grows like $\mathcal{O}(N)$ or $\mathcal{O}(h^{-2})$. We now design the multiplicative Schwarz method by defining a subspace decomposition of V.

Let n_H be a positive integer such that $J_0 := n/n_H$ is also an integer. We define

$$H := hn_H \quad \text{and} \quad J := J_0^2,$$

and consider a coarser mesh on Γ with mesh size H as follows

$$\begin{cases} \tilde{x}_i & := x_{in_H}, \qquad i = 0, \ldots, J_0, \\ \tilde{y}_j & := y_{jn_H}, \qquad j = 0, \ldots, J_0. \end{cases} \tag{3.3}$$

This coarse mesh forms J elements, denoted by $\Gamma_1, \dots, \Gamma_J$, each element being a union of n_H^2 subelements Γ_{ij}; see Figure 1. Each element Γ_p will be written as

$$\Gamma_p = [\tilde{x}_{i_p}, \tilde{x}_{i_p+1}] \times [\tilde{y}_{j_p}, \tilde{y}_{j_p+1}], \tag{3.4}$$

i.e., $(\tilde{x}_{i_p}, \tilde{y}_{i_p})$ denotes the lower left corner of Γ_p. We shall also denote this corner as $(x_{i(p)}, y_{j(p)})$, in other words (see (3.3))

$$i(p) := i_p n_H \quad \text{and} \quad j(p) := j_p n_H. \tag{3.5}$$

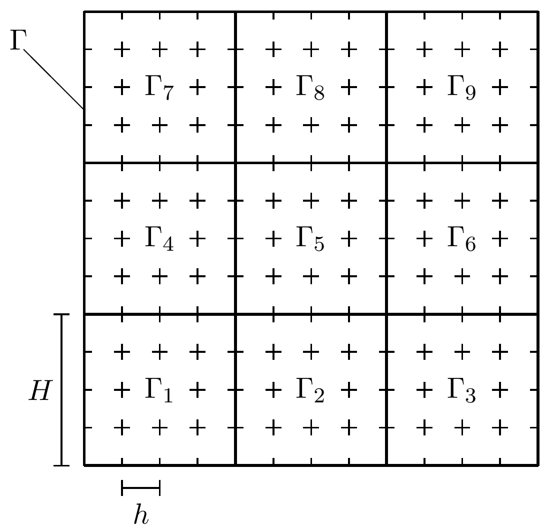

Figure 1: Subspace decomposition

By defining

$$\begin{aligned} V_0 &:= \{v \in V \mid v|_{\Gamma_p} = \text{ const}, p = 1, \dots, J\}, \\ V_p &:= \{v \in V \mid \operatorname{supp} v \subseteq \bar{\Gamma}_p, \textstyle\int_{\Gamma_p} v(x)\,dx = 0\}, \quad p = 1, \dots, J, \end{aligned} \tag{3.6}$$

we have

$$V = V_0 \oplus V_1 \oplus \dots \oplus V_J, \tag{3.7}$$

and

$$\dim V_0 = J, \quad \dim V_p = n_H^2 - 1.$$

For the additive Schwarz method defining from this subspace decomposition the following lemma has been proved.

Lemma 3.1. (see [6]) *Let $P_{\mathrm{AS}} : V \to V$ be the two-level additive Schwarz operator and let $\lambda_{\min}$ and $\lambda_{\max}$ denote its smallest and largest eigenvalue, respectively. Then there are constants $c_1, c_2 > 0$, which are independent of H and h, such that*

$$\lambda_{\min} \geq c_1(1+\log n_H)^{-1} \quad \text{and} \quad \lambda_{\max} \leq c_2(1+\log n_H).$$

Alternatively, we may write

$$c_1(1+\log n_H)^{-1}W(u,u) \leq W(P_{\mathrm{AS}}u,u) \leq c_2(1+\log n_H)W(u,u),$$

for all $u \in V$.

Remark 3.2. Suboptimal estimates for the additive Schwarz operator were firstly given in [10], where its condition number is bounded by n_H^2. For the multilevel case see [11].

Referring to Theorem 2.1, our next step is to find an upper bound for $\|\Theta\|_2$.

4. Strengthened Cauchy-Schwarz Inequality

In this section we derive a strengthened Cauchy-Schwarz inequality and give a bound for the contraction rate of the multiplicative Schwarz method.

Lemma 4.1. *For any $p, q = 1, \ldots, J$, the following holds*

$$|W(u,v)| \leq \epsilon_{pq} h \|u\|_{L^2(\Gamma_p)} \|v\|_{L^2(\Gamma_q)}, \quad \forall u \in V_p, v \in V_q, \tag{4.1}$$

with

$$\epsilon_{pq} := \begin{cases} c_1 n_H & \text{if } |i_p - i_q| \leq 1,\ |j_p - j_q| \leq 1, \\ \dfrac{6n_H}{(|i_p - i_q| - 1)^3} & \text{if } |i_p - i_q| > 1,\ |j_p - j_q| \leq 1, \\ \dfrac{6n_H}{(|j_p - j_q| - 1)^3} & \text{if } |i_p - i_q| \leq 1,\ |j_p - j_q| > 1, \\ \dfrac{6n_H}{[(|i_p - i_q| - 1)^2 + (|j_p - j_q| - 1)^2]^{3/2}} & \text{if } |i_p - i_q| > 1,\ |j_p - j_q| > 1, \end{cases} \tag{4.2}$$

where c_1 is a positive constant independent of h and H.

Proof. Using the canonical basis functions

$$\phi_{i,j}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}], \\ 0 & \text{otherwise,} \end{cases}$$

we shall represent $u \in V_p$ and $v \in V_q$ as, noting (3.5),

$$u = \sum_{l=0}^{n_H-1} \sum_{m=0}^{n_H-1} u_{l,m} \phi_{i(p)+l,j(p)+m}$$

and

$$v = \sum_{l'=0}^{n_H-1} \sum_{m'=0}^{n_H-1} v_{l',m'} \phi_{i(q)+l',j(q)+m'}.$$

Then

$$W(u,v) = \sum_{l,m=0}^{n_H-1} \sum_{l',m'=0}^{n_H-1} u_{l,m} v_{l',m'} W(\phi_{i(p)+l,j(p)+m}, \phi_{i(q)+l',j(q)+m'}). \tag{4.3}$$

Let

$$F(\delta_1,\delta_2) := \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{(\delta_1 + s_1 - s_2)^2 + (\delta_2 + t_1 - t_2)^2}} \, dt_2 \, ds_2 \, dt_1 \, ds_1.$$

Then for arbitrary $i, j, l, m \in \mathbb{N}$ we have

$$W(\phi_{i,j}, \phi_{l,m}) = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \int_{x_l}^{x_{l+1}} \int_{y_m}^{y_{m+1}} \frac{1}{\sqrt{(\tilde{s}_1 - \tilde{s}_2)^2 + (\tilde{t}_1 - \tilde{t}_2)^2}} \, d\tilde{t}_2 \, d\tilde{s}_2 \, d\tilde{t}_1 \, d\tilde{s}_1 = h^3 F(i-l, j-m).$$

With both patches Γ_p and Γ_q in mind, we define

$$F_{pq}(\delta_1,\delta_2) := F(i(p) - i(q) + \delta_1, j(p) - j(q) + \delta_2), \quad \forall \delta_1, \delta_2 \in \mathbb{R},$$

to obtain

$$W(\phi_{i(p)+l,j(p)+m}, \phi_{i(q)+l',j(q)+m'}) = h^3 F_{pq}(l - l', m - m').$$

Let

$$\begin{aligned} a_{l,m,l',m'} &:= F_{pq}(l-l', m-m'), \\ c_{l,m} &:= -F_{pq}(l,m), \\ d_{l',m'} &:= -F_{pq}(n_H-1-l', n_H-1-m') + F_{pq}(n_H-1, n_H-1), \\ b_{l,m,l',m'} &:= a_{l,m,l',m'} + c_{l,m} + d_{l',m'}. \end{aligned}$$

Then using Lemma A.2 in the appendix, we obtain from (4.3)

$$\begin{aligned} |W(u,v)| &= h^3 \left| \sum_{l,m=0}^{n_H-1} \sum_{l',m'=0}^{n_H-1} u_{l,m} v_{l',m'} a_{l,m,l',m'} \right| \\ &\le h^3 \|(u_{l,m})\|_2 \|(v_{l',m'})\|_2 \, |||\, B \,|||_2 = h\|u\|_{L^2(\Gamma_p)} \|v\|_{L^2(\Gamma_q)} \, |||\, B \,|||_2, \quad (4.4) \end{aligned}$$

where $B = (b_{l,m,l',m'})$, and where we have used $h\|(u_{l,m})\|_2 = \|u\|_{L^2(\Gamma_p)}$ and $h\|(v_{l',m'})\|_2 = \|v\|_{L^2(\Gamma_q)}$.

We now estimate $|||\, B \,|||_2$. Setting

$$f(a,b) := F_{pq}(l-a, m-b) - F_{pq}(n_H-1-a, n_H-1-b), \ \forall a,b \in \mathbb{R},$$

we obtain by applying the Mean Value Theorem twice

$$\begin{aligned} b_{l,m,l',m'} &= f(l',m') - f(0,0) = (l',m') \cdot \nabla f(\theta_1 l', \theta_1 m') \\ &= -(l',m') \cdot \big(\nabla F_{pq}(l-\theta_1 l', m-\theta_1 m') \\ &\qquad - \nabla F_{pq}(n_H-1-\theta_1 l', n_H-1-\theta_1 m') \big) \\ &= (l',m') \cdot HF_{pq}(\delta_1,\delta_2) \cdot \begin{pmatrix} n_H-1-l \\ n_H-1-m \end{pmatrix}, \end{aligned}$$

for $\theta_1, \theta_2 \in (0,1)$, where the Hessian matrix is defined as usual as

$$HF_{pq}(\delta_1,\delta_2) := \begin{pmatrix} \partial_1^2 F_{pq}(\delta_1,\delta_2) & \partial_1\partial_2 F_{pq}(\delta_1,\delta_2) \\ \partial_2\partial_1 F_{pq}(\delta_1,\delta_2) & \partial_2^2 F_{pq}(\delta_1,\delta_2) \end{pmatrix},$$

with

$$(\delta_1,\delta_2) := (l-\theta_1 l' + \theta_2(n_H-1-l), m-\theta_1 m' + \theta_2(n_H-1-m)).$$

Since $0 \le l,m,l',m' \le n_H-1$, there follows

$$|b_{l,m,l',m'}| \le n_H^2 \Big\{ |\partial_1^2 F_{pq}(\delta_1,\delta_2)| + 2|\partial_1\partial_2 F_{pq}(\delta_1,\delta_2)| + |\partial_2^2 F_{pq}(\delta_1,\delta_2)| \Big\}.$$

Let $M := M(x,y,s_1,s_2,t_1,t_2) := (x+s_1-s_2)^2+(y+t_1-t_2)^2$. The second derivatives of

$$F(x,y) = \int_0^1\int_0^1\int_0^1\int_0^1 M^{-1/2}\,dt_2\,ds_2\,dt_1\,ds_1$$

are given by

$$\partial_1^2 F(x,y) = \int_0^1\int_0^1\int_0^1\int_0^1 \left\{2(x+s_1-s_2)^2-(y+t_1-t_2)^2\right\} \times M^{-5/2}\,dt_2\,ds_2\,dt_1\,ds_1,$$

$$\partial_1\partial_2 F(x,y) = 3\int_0^1\int_0^1\int_0^1\int_0^1 (x+s_1-s_2)(y+t_1-t_2) \times M^{-5/2}\,dt_2\,ds_2\,dt_1\,ds_1,$$

$$\partial_2^2 F(x,y) = \int_0^1\int_0^1\int_0^1\int_0^1 \left\{-(x+s_1-s_2)^2+2(y+t_1-t_2)^2\right\} \times M^{-5/2}\,dt_2\,ds_2\,dt_1\,ds_1.$$

Using the estimate $|(x+s_1-s_2)(y+t_1-t_2)| \le \frac{1}{2}M$ and summing up we obtain

$$\begin{aligned} &|\partial_1^2F(x,y)| + 2|\partial_1\partial_2F(x,y)| + |\partial_2^2F(x,y)| \\ &\le 6\int_0^1\int_0^1\int_0^1\int_0^1 M^{-3/2}\,dt_2\,ds_2\,dt_1\,ds_1 \\ &\le 6\max_{s_1,s_2,t_1,t_2\in[0,1]} M^{-3/2}(x,y,s_1,s_2,t_1,t_2) \\ &\le 6\begin{cases} (|x|-1)^{-3} & \text{if } |x|>1,\ y\in\mathbb{R}, \\ (|y|-1)^{-3} & \text{if } |y|>1,\ x\in\mathbb{R}, \\ [(|x|-1)^2+(|y|-1)^2]^{-3/2} & \text{if } |x|>1,\ |y|>1. \end{cases} \end{aligned}$$

(Note that the three cases of different values of x and y do not exclude each other. For example, when $|x|>1$ and $|y|>1$, the first two bounds $(|x|-1)^{-3}$ and $(|y|-1)^{-3}$ are still valid; however, $[(|x|-1)^2+(|y|-1)^2]^{-3/2}$ is a better estimate.) Consequently we can estimate $b_{l,m,l',m'}$ by

$$|b_{l,m,l',m'}| \le 6n_H^2\begin{cases} (|\hat{x}|-1)^{-3} & \text{if } |\hat{x}|>1,\ \hat{y}\in\mathbb{R}, \\ (|\hat{y}|-1)^{-3} & \text{if } |\hat{y}|>1,\ \hat{x}\in\mathbb{R}, \\ [(|\hat{x}|-1)^2+(|\hat{y}|-1)^2]^{-3/2} & \text{if } |\hat{x}|,|\hat{y}|>1, \end{cases} \qquad (4.5)$$

where $\hat{x} = i(p) - i(q) + \delta_1$ and $\hat{y} = j(p) - j(q) + \delta_2$. Note that due to $\theta_1, \theta_2 \in (0,1)$ and $0 \le l, m, l', m' \le n_H - 1$, there hold

$$\begin{aligned} \delta_1 &= l - \theta_1 l' + \theta_2(n_H - 1 - l) \le l + (n_H - 1 - l) = n_H - 1, \\ \delta_1 &= l - \theta_1 l' + \theta_2(n_H - 1 - l) \ge -\theta_1 l' \ge -(n_H - 1), \end{aligned}$$

i.e.

$$|\delta_1| \le n_H - 1.$$

Therefore, there holds

$$|\hat{x}| - 1 \ge |i(p) - i(q)| - |\delta_1| - 1 \ge |i(p) - i(q)| - n_H = n_H(|i_p - i_q| - 1),$$

which implies $|\hat{x}| > 1$ if $|i_p - i_q| > 1$. Similarly, we have

$$|\hat{y}| - 1 \ge n_H(|j_p - j_q| - 1),$$

which implies $|\hat{y}| > 1$ if $|j_p - j_q| > 1$. It follows then from (4.5) that

$$\begin{aligned} &|b_{l,m,l',m'}| \\ &\le 6n_H^{-1} \begin{cases} (|i_p - i_q| - 1)^{-3} & \text{if } |i_p - i_q| > 1, \\ (|j_p - j_q| - 1)^{-3} & \text{if } |j_p - j_q| > 1, \\ [(|i_p - i_q| - 1)^2 \\ \quad + (|j_p - j_q| - 1)^2]^{-3/2} & \text{if } |i_p - i_q| > 1 \text{ and } |j_p - j_q| > 1. \end{cases} \end{aligned} \tag{4.6}$$

When $|i_p - i_q| \le 1$ and $|j_p - j_q| \le 1$, instead of (4.4), we estimate $|W(u,v)|$ directly as (see [13])

$$\begin{aligned} |W(u,v)| &\le c_0 \|u\|_{\tilde{H}^{-1/2}(\Gamma_p)} \|v\|_{\tilde{H}^{-1/2}(\Gamma_q)} \\ &\le c_1 H \|u\|_{L^2(\Gamma_p)} \|v\|_{L^2(\Gamma_q)} \\ &= h c_1 n_H \|u\|_{L^2(\Gamma_p)} \|v\|_{L^2(\Gamma_q)}. \end{aligned} \tag{4.7}$$

Now if ϵ_{pq} is defined by (4.2) (noting that, not as in (4.6), more conditions are imposed on the values of $|i_p - i_q|$ and $|j_p - j_q|$ so that ϵ_{pq} is well defined), then by combining (4.4), (4.6), (4.7), and using Lemma A.3 we obtain (4.1), finishing the proof of the lemma. □

The following bound for the spectral norm of the coefficient matrix in the strengthened Cauchy-Schwarz inequality is crucial for the analysis of the multiplicative Schwarz method.

Theorem 4.2. *For the matrix* Θ *defined in (2.3), there holds*

$$\|\Theta\|_2 \le C\, n_H,$$

with a constant C *independent of* h, H *and* N.

Proof. We first note from the definitions of Θ and ϵ_{pq} (see (2.3) and (4.2)), and from the symmetry of (ϵ_{pq}) that

$$\|\Theta\|_2 \le \|(\epsilon_{pq})\|_2 \le \|(\epsilon_{pq})\|_\infty. \tag{4.8}$$

In order to estimate $\|(\epsilon_{pq})\|_\infty$, we define a function $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ as

$$g(x,y) := \begin{cases} c_1 n_H & \text{if } x \le 1,\ y \le 1, \\ \dfrac{6n_H}{(x-1)^3} & \text{if } x > 1,\ y \le 1, \\ \dfrac{6n_H}{(y-1)^3} & \text{if } x \le 1,\ y > 1, \\ \dfrac{6n_H}{[(x-1)^2+(y-1)^2]^{3/2}} & \text{if } x > 1,\ y > 1, \end{cases}$$

which implies

$$\epsilon_{pq} = g(|i_p - i_q|, |j_p - j_q|).$$

Therefore (see (3.3) and (3.4))

$$\begin{aligned} \sum_{q=1}^{J} \epsilon_{pq} &= \sum_{i_q=0}^{J_0-1} \sum_{j_q=0}^{J_0-1} g(|i_p - i_q|, |j_p - j_q|) \\ &\le 4 \sum_{l=0}^{J_0-1} \sum_{m=0}^{J_0-1} g(l,m) \\ &= 4[g(0,0) + g(1,0) + g(0,1) + g(1,1)] \\ &\quad + 4 \sum_{l=2}^{J_0-1} (g(l,0) + g(l,1)) + 4 \sum_{m=2}^{J_0-1} (g(0,m) + g(1,m)) \\ &\quad + 4 \sum_{l=2}^{J_0-1} \sum_{m=2}^{J_0-1} g(l,m) \\ &=: T_1 + \cdots + T_4. \end{aligned}$$

We estimate each of the terms on the right-hand side as

$$\begin{aligned} T_1 &\leq 16cn_H, \\ T_2 &= 8\sum_{l=2}^{J_0-1}\frac{6n_H}{(l-1)^3} = 48n_H\sum_{l=1}^{J_0-2} l^{-3} \leq cn_H, \\ T_3 &= 8\sum_{m=2}^{J_0-1}\frac{6n_H}{(m-1)^3} = 48n_H\sum_{m=1}^{J_0-2} m^{-3} \leq cn_H, \end{aligned}$$

$$T_4 \leq 4\sum_{l=2}^{J_0-1}\sum_{m=2}^{J_0-1}\frac{6n_H}{2^{3/2}(l-1)^{3/2}(m-1)^{3/2}} = 6\sqrt{2}n_H\sum_{l=1}^{J_0-2}\sum_{m=1}^{J_0-2}\frac{1}{l^{3/2}m^{3/2}} \leq cn_H.$$

(Here, and in the sequel, c denotes a generic constant which may take different values at different occurrences.) Hence

$$\|(\epsilon_{pq})\|_\infty = \max_{p=1,\ldots,J}\sum_{q=1}^{J}|\epsilon_{pq}| \leq Cn_H,$$

which together with (4.8) proves the theorem. □

Theorem 4.3. *For the multiplicative Schwarz method with the subspace decomposition (3.6) there exists a constant $c > 0$ with*

$$\|E_J u\|_W \leq \sqrt{1 - \frac{1}{c \cdot n_H^2}}\|u\|_W. \tag{4.9}$$

Proof. For all $i = 0, \ldots, J$ and for all $u \in V$ there holds

$$\begin{aligned} &\|E_{i-1}u\|_W^2 - \|E_i\|_W^2 \\ =\ & W(E_{i-1}u, E_{i-1}u) - W((I-P_i)E_{i-1}u, (I-P_i)E_{i-1}u) \\ =\ & W(P_iE_{i-1}u, E_{i-1}u) + W(E_{i-1}u, P_iE_{i-1}u) - W(P_iE_{i-1}u, P_iE_{i-1}u) \\ =\ & W(P_iE_{i-1}u, E_{i-1}u). \end{aligned}$$

Summing up we obtain

$$\|E_J u\|_W^2 = \|u\|_W^2 - \sum_{i=0}^{J} W(P_iE_{i-1}u, E_{i-1}u). \tag{4.10}$$

We now prove a lower estimate for $\sum_{i=0}^{J} W(P_i E_{i-1}u, E_{i-1}u)$. Due to (3.7), there holds

$$u = \sum_{i=0}^{J} u_i\,, \quad \forall u \in V,$$

where $u_i \in V_i$ is the L^2 projection of u onto V_i, i.e.,

$$\langle u_i, v_i \rangle = \langle u, v_i \rangle\ , \quad \forall v_i \in V_i.$$

On the other hand, due to (2.2) we can write

$$u = P_0 u + E_{i-1}u + \sum_{j=1}^{i-1} P_j E_{i-1}u, \quad i = 1, \ldots, J.$$

Therefore,

$$\begin{aligned} & W(u,u) \\ = \;& W(u,u_0) + \sum_{i=1}^{J} W(u,u_i) \\ = \;& W(P_0 u, u_0) \\ & + \sum_{i=1}^{J} \left(W(P_0 u, u_i) + W(E_{i-1}u, u_i) + \sum_{j=1}^{i-1} W(P_j E_{j-1}u, u_i) \right) \\ = \;& W(P_0 u, u) + \sum_{i=1}^{J} W(P_i E_{i-1}u, u_i) \\ & + \sum_{i=1}^{J} \sum_{j=1}^{i-1} W(P_j E_{j-1}u, u_i). \end{aligned} \tag{4.11}$$

Using the estimate

$$\sum_{i=1}^{J} \|u_i\|_W^2 \le c\, n_H \|u\|_W^2,$$

which was proved in [10, Lemma 2.6], we can estimate the second term in the right-hand side of (4.11) as

$$\begin{aligned}
&\sum_{i=1}^{J} W(P_i E_{i-1} u, u_i) \\
\leq\ & \left(\sum_{i=1}^{J} W(P_i E_{i-1} u, P_i E_{i-1} u) \right)^{1/2} \left(\sum_{i=1}^{J} W(u_i, u_i) \right)^{1/2} \\
\leq\ & c \left(\sum_{i=1}^{J} W(P_i E_{i-1} u, E_{i-1} u) \right)^{1/2} \sqrt{n_H} \|u\|_W .
\end{aligned}$$

Using Theorem 4.2 we can estimate the third term in the right-hand side of (4.11) as

$$\begin{aligned}
&\sum_{i=1}^{J} \sum_{j=1}^{i-1} W(P_j E_{j-1} u, u_i) \\
\leq\ & cH \left(\sum_{i=1}^{J} \|P_i E_{i-1} u\|_{L^2(\Gamma)}^2 \right)^{1/2} \left(\sum_{i=1}^{J} \|u_i\|_{L^2(\Gamma)}^2 \right)^{1/2} \\
\leq\ & cH \left(\sum_{i=1}^{J} \|P_i E_{i-1} u\|_{L^2(\Gamma)}^2 \right)^{1/2} \|u\|_{L^2(\Gamma)} \\
\leq\ & cHh^{-1} \left(\sum_{i=1}^{J} \|P_i E_{i-1} u\|_W^2 \right)^{1/2} \|u\|_W \\
=\ & cn_H \left(\sum_{i=1}^{J} W(P_i E_{i-1} u, E_{i-1} u) \right)^{1/2} \|u\|_W .
\end{aligned}$$

Therefore, (4.11) implies

$$\begin{aligned}
W(u,u) &\leq W(P_0 u, u) + cn_H \left(\sum_{i=1}^{J} W(P_i E_{i-1} u, E_{i-1} u) \right)^{1/2} \|u\|_W \\
&\leq cn_H \left(\sum_{i=0}^{J} W(P_i E_{i-1} u, E_{i-1} u) \right)^{1/2} \|u\|_W ,
\end{aligned}$$

which, together with (4.10), yields (4.9) □

Remark 4.4. A less optimal estimate than (4.9) for the multiplicative Schwarz method, namely

$$\|E_J u\|_W \leq \sqrt{1 - \frac{1}{c \cdot n_H^2 \log n_H}} \|u\|_W$$

follows directly from Theorem 2.1, Lemma 3.1 and Theorem 4.2.

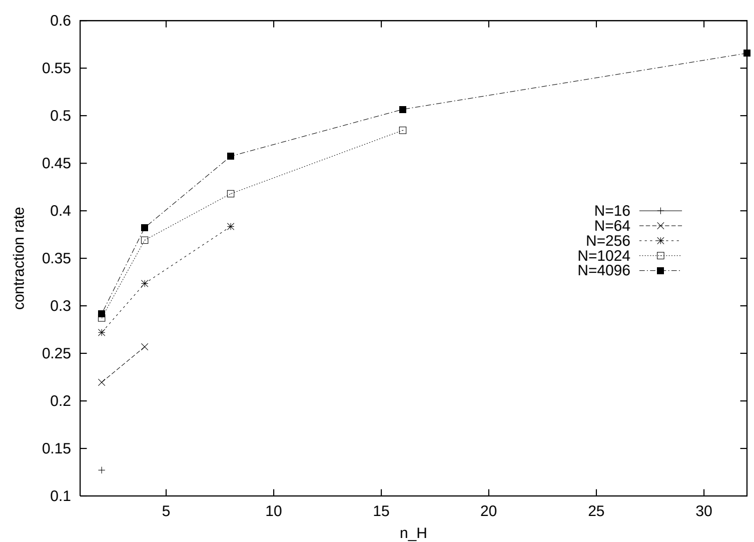

Figure 2: Contraction rates

5. Numerical Experiments

We consider the weakly singular integral equation (3.1) with the right hand side $f(x) \equiv 1$. We solve the Galerkin equation (3.2) by the multiplicative Schwarz algorithm and observe the expected behavior of the contraction rate with respect to N and n_H (see Figure 2 and Table 1), i.e. the contraction rate is independent of N and increases with n_H. Table 2 shows the computing times and iteration numbers. We note that the iteration numbers do not depend on N.

The subspace V_0 consists of brick functions defined on a mesh with mesh size H. The subspaces V_p consists of Haar basis functions on a mesh with mesh size h without global constant to take into account the integral mean zero condition onto V_p. The elements of the Galerkin matrix have been calculated numerically. The Galerkin projections for $V_0, V_1, \ldots, V_p$ have been computed by explicitly computing the inverse of the corresponding Galerkin matrices in advance. Then the multiplicative Schwarz method is implemented as a block Jacobi method.

The contraction rates are evaluated by computing the error reduction operator E_J in matrix form explicitly and determining the largest eigenvalue using the *LAPACK*-package.

For the computation of the iteration numbers and CPU-times we have always used the stopping criterion

$$\frac{\|x^k - x^{k-1}\|_2}{\|x^k\|_2} \leq 10^{-10}, \tag{5.1}$$

where $\|\cdot\|_2$ is the Euclidean norm and x^{k-1}, x^k are successive iterates. The numerical experiments were performed on a *Sun-E450* (480MHz) at the University of Hannover using the program system *maiprogs* [8].

$N \backslash n_H$	2	4	8	16	32
16	0.1271424				
64	0.2194417	0.2568840			
256	0.2719622	0.3234754	0.3834260		
1024	0.2872697	0.3690567	0.4179214	0.4846657	
4096	0.2914583	0.3822296	0.4573208	0.5066855	0.5657064

Table 1: Contraction rates

$N \backslash n_H$	2	4	8	16	32
16	0.0037 (12)				
64	0.0095 (17)	0.0069 (18)			
256	0.0537 (17)	0.0462 (21)	0.0659 (23)		
1024	1.2524 (17)	1.3704 (21)	1.7388 (26)	2.5067 (30)	
4096	35.753 (17)	40.636 (21)	50.724 (26)	64.708 (32)	88.853 (36)

Table 2: Computing times in seconds and iteration numbers

Acknowledgments

The first two authors would like to thank the Australian National University for supporting their research visit during the CTAC99 Conference. They would also like to thank the German Research Foundation (DFG) and the Australian Research Council (ARC) for support through the DFG-ARC German-Australian Collaboration Workshop in Numerical Analysis of Boundary Integral Methods and Applications in Sydney in September 1999, where this work was initialised. The second author was partially supported by the DFG under grant Ste 573/3. The third author was supported by Graduiertenkolleg GRK 615 sponsored by the DFG during his visit to the University of Hannover, where this work was finalised.

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A. Appendix

Definition A.1. We extend the usual notation of vector- and matrix-norms to multi-indexed objects and define:

$$\forall u = (u_{ij}) \in \mathbb{R}^{n\times n} : \qquad \|u\|_2 = \left(\sum_{i,j=1}^{n} |u_{ij}|^2 \right)^{1/2} ,,$$

and obtain for all $B := (b_{ij,lm}) \in \mathbb{R}^{n\times n\times n\times n}$

$$|||\, B\, |||_2 := \sup_{x\in\mathbb{R}^{n\times n}} \frac{\|Bx\|_2}{\|x\|_2} ,$$

with $(Bx)_{ij} = \sum_{l,m=1}^{n} b_{ij,lm} x_{lm}$.

Lemma A.2. *Let* $u = (u_{ij}), v = (v_{lm}) \in \mathbb{R}^{n\times n}$ *with* $\sum_{i,j=1}^n u_{ij} = \sum_{l,m=1}^n v_{lm} = 0$. *Let* $A = (a_{ij,lm}) \in \mathbb{R}^{n\times n\times n\times n}$. *Then there holds*

$$|uAv| \leq \|(u_{ij})\|_2 \cdot \|(v_{lm})\|_2 \cdot |||\, B \,|||_2 \tag{A.1}$$

with $B = (b_{ij,lm}) = (a_{ij,lm} + d_{lm} + c_{ij})$, *where* $(c_{ij}), (d_{lm}) \in \mathbb{R}^{n\times n}$ *are arbitrary.*

Proof. Due to $\sum_{i,j=1}^n u_{ij} = \sum_{l,m=1}^n v_{lm} = 0$ we obtain

$$\begin{aligned}
|uAv| = |\sum_{l,m=1}^n v_{lm} \sum_{i,j=1}^n u_{ij} a_{ij,lm}| &= |\sum_{l,m=1}^n v_{lm} \sum_{i,j=1}^n u_{ij}(a_{ij,lm} + d_{lm})| \\
&= |\sum_{i,j=1}^n u_{ij} \sum_{l,m=1}^n v_{lm}(a_{ij,lm} + d_{lm})| \\
&= |\sum_{i,j=1}^n u_{ij} \sum_{l,m=1}^n v_{lm}(a_{ij,lm} + d_{lm} + c_{ij})| \\
&= |\sum_{i,j=1}^n u_{ij} \sum_{l,m=1}^n v_{lm} b_{ij,lm}| = |uBv| \leq \|(u_{ij})\|_2 \|(v_{lm})\|_2 \,|||\, B \,|||_2 \,. \quad \square
\end{aligned}$$

Lemma A.3. *Let* $B = (b_{ij,lm}) \in \mathbb{R}^{n\times n\times n\times n}$ *with* $|b_{ij,lm}| \leq c$ *for all* $i, j, l, m = 1, \ldots, n$. *Then there holds*

$$|||\, B \,|||_2 \leq cn^2 \tag{A.2}$$

Proof. We have

$$
\begin{aligned}
|||\, B \,|||_2 &= \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{\|Bx\|_2}{\|x\|_2} \\
&= \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{\|(\sum_{l,m=1}^n b_{ij,lm}x_{lm})_{ij}\|_2}{\|x\|_2} \\
&\le \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{\|(\sum_{l,m=1}^n c|x_{lm}|)_{ij}\|_2}{\|x\|_2} \\
&= c \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{\|(1)_{ij}\|_2 \sum_{l,m=1}^n |x_{lm}|}{\|x\|_2} \\
&= c \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{n \sum_{l,m=1}^n |x_{lm}|}{\|x\|_2} \\
&\le c \sup_{x=(x_{ij})\in\mathbb{R}^{n\times n}} \frac{n\sqrt{\sum_{l,m=1}^n 1 \sum_{l,m=1}^n x_{lm}^2}}{\|x\|_2} \\
&= cn^2 \,.
\end{aligned}
$$

□