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CHARACTERISTIC PROPERTIES FOR INEQUALITIES IN HILBERT SPACES

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Dedicated to Professor Mao Sheng Chang
on his retirement.

Abstract: In this article we present some new inequalities in a Hilbert space which are equivalent to the Cauchy-Schwarz inequality. Altered forms of inequalities are given, and we provide necessary and sufficient conditions for equalities. In applications these inequalities will be used to obtain some inequalities of bounded linear operators (some of them are known) on a Hilbert space with considerable simplifications in the proofs.

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1. Introduction and Basic Lemma

In what follows H is an infinite dimensional Hilbert space over the field $\mathbf{C}$ of complex numbers and $0 \in H$ is the zero vector. In this article we present some new inequalities in H which are equivalent to the Cauchy-Schwarz inequality, i.e., $|(x,y)| \leq \| x \| \| y \|$ holds for all $x, y \in H$, and the equality holds if and only if $x = \alpha y$, $\alpha \in \mathbf{C}$. In other words, of prime importance in our discussion

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is characterizations of this inequality. Some altered forms of inequalities are given, and necessary and sufficient conditions for equalities are provided (there are many inequalities in the literature, but equality conditions and altered forms are usually neglected). Applications in the last section are about inequalities of bounded linear operators on H with considerable simplfications in the proofs; those include generalization of the Heinz inequality, characterization of the Furuta inequality, and equivalence of the Löwner-Heinz formula, etc. The proofs of some well-known operator inequalities are simplified.

We shall require the following well-known two results; the first one is trivial, but the second one is not immediately obvious.

Lemma. (1) *For any nonzero vector $x \in H$, there exists a sequence of unit vectors $\{e_i\}_1^n \subseteq H$ such that x and $\{e_i\}_1^n$ are orthogonal.*

(2) If M is a proper closed linear subspace of H, then there exists a nonzero vector $z \in H$ such that z and M are orthogonal.

The proof of (2) in Lemma can be found in [10, Theorem B, §53, Chapter 10] and we shall omit. Lemma will be used without further mentioning each time when we prove a necessary condition that an inequality is equivalent to the Cauchy-Schwarz inequality.

2. Inequalities Equivalent to Cauchy-Schwarz Enequality

Theorem 1. *Let $x, y, u, v, e \in H$ with $\| e \| = 1$. Then the following are equivalent:*

(i) $| (x, y) | \leq \| x \| \| y \|$ *(Cauchy-Schwarz inequality).*

Equality holds if and only if $x = \alpha y$, $\alpha \in \mathbf{C}$.

(ii) $$| (x-u, y-v) - (x-u, e)(e, y-v) |^2 \leq [\| x-u \|^2 - | (x-u, e) |^2][\| y-v \|^2 - | (y-v, e) |^2];$$

or, $$| (x-u, y-v) - (x-u, e)(e, y-v) | \leq \| x-u-(x-u,e)e \| \| y-v-(y-v,e)e \|;$$

or, $$| (x-u, y-v) - (x-u, e)(e, y-v) |^2 \leq \{\| x \|^2 - | (x,e) |^2 - [2Re((x,u) - (x,e)(e,u)) - (\| u \|^2 - | (u,e) |^2)]\} \times \{\| y \|^2 - | (y,e) |^2 - [2Re((y,v) - (y,e)(e,v)) - (\| v \|^2 - | (v,e) |^2)]\}.$$

Equality holds if and only if $x - u = \alpha(y - v) + \beta e$, $\alpha, \beta \in \mathbf{C}$.

Proof. (i) implies (ii). Due to (i) the required inequalities can be easily obtained by simplifying the following inequality,

$$| (x-u-(x-u,e)e, y-v-(y-v,e)e) |^2 \\ \leq \| x-u-(x-u,e)e \|^2 \| y-v-(y-v,e)e \|^2,$$

and notice that

$$(x-u-(x-u,e)e, y-v-(y-v,e)e) = (x-u, y-v) - (x-u,e)(e, y-v)$$

and

$$\| x-u-(x-u,e)e \|^2 = \| x-u \|^2 - | (x-u,e) |^2 \\ = \| x \|^2 - | (x,e) |^2 - [2\,\mathrm{Re}\,((x,u)-(x,e)(e,u)) - (\| u \|^2 - | (u,e) |^2)].$$

Equality holds if and only if $x-u-(x-u,e)e$ and $y-v-(y-v,e)e$ are proportional, i.e., $x-u=\alpha(y-v)+\beta e$, $\alpha,\beta \in \mathbf{C}$.

(ii) implies (i). Let $u=v=0$, and we may choose a unit vector e such that $(x,e)=(y,e)=0$. Then we have (i), and $x=\alpha y+\beta e$ for equality condition. But $0=(x,e)=(\alpha y+\beta e,e)=\alpha(y,e)+\beta=\beta$. Hence $x=\alpha y$, and we are done. □

We remark that Theorem 1 makes it possible to establish a large part of inequalities in this paper. Corollary 1 next is a simple consequence of Theorem 1 (i.e., let $u=v=0$ in Theorem 1), and the proof should be omitted. It will be generalized later in Theorem 3 below.

Corollary 1. *Let $x,y,e \in H$ with $\| e \|=1$. Then the following are equivalent:*

(i) $| (x,y) | \leq \| x \| \| y \|$ (*Cauchy-Schwarz inequality).*
Equality holds if and only if $x=\alpha y$, $\alpha \in \mathbf{C}$.
(ii) $| (x,y)-(x,e)(e,y) |^2 \leq [\| x \|^2 - | (x,e) |^2][\| y \|^2 - | (y,e) |^2]$;
or, $| (x,y)-(x,e)(e,y) | \leq \| x-(x,e)e \| \| y-(y,e)e \|$.
Equality holds if and only if $x=\alpha y+\beta e$, $\alpha,\beta \in \mathbf{C}$.

In the next three results and elsewhere we shall make significant use of the formula that for any real numbers a, b, c, and d the following relation holds.

$$(a^2-c^2)(b^2-d^2) \leq (ab-cd)^2. \qquad (\#)$$

Corollary 2. *Let $x,y,u,v \in H$. If*

$$\| u \|^2 - | (u,e) |^2 \leq 2\,\mathrm{Re}\,((x,u)-(x,e)(e,u))$$

and

$$\| v \|^2 - | (v,e) |^2 \le \operatorname{Re}((y,v) - (y,e)(e,v)),$$

for any unit vector $e \in H$*, then*

$$[2\operatorname{Re}((x,u) - (x,e)(e,u)) - (\| u \|^2 - | (u,e) |^2)]^{1/2} \cdot [2\operatorname{Re}((y,v) - (y,e)(e,v)) - (\| v \|^2 - | (v,e) |^2)]^{1/2} + | (x-u, y-v) - (x-u,e)(e,y-v) | \le [\| x \|^2 - | (x,e) |^2]^{1/2} [\| y \|^2 - | (y,e) |^2]^{1/2}.$$

Proof. Let

$$a = [\| x \|^2 - | (x,e) |^2]^{1/2},$$
$$b = [\| y \|^2 - | (y,e) |^2]^{1/2},$$
$$c = [2Re((x,u) - (x,e)(e,u)) - (\| u \|^2 - | (u,e) |^2)]^{1/2},$$

and

$$d = [2\operatorname{Re}((y,v) - (y,e)(e,v)) - (\| v \|^2 - | (v,e) |^2)]^{1/2}.$$

Then, by the third inequality in (ii) of Theorem 1 we get

$$\begin{aligned} | (x-u, y-v) - (x-u,e)(e,y-v) |^2 &\le (a^2 - c^2)(b^2 - d^2) \le (ab - cd)^2 \quad \text{by } (\#) \\ &= \{[\| x \|^2 - | (x,e) |^2]^{1/2} [\| y \|^2 - | (y,e) |^2]^{1/2} \\ &\quad - [2\operatorname{Re}((x,u) - (x,e)(e,u)) - (\| u \|^2 - | (u,e) |^2)]^{1/2} \\ &\quad \cdot [2\operatorname{Re}((y,v) - (y,e)(e,v)) - (\| v \|^2 - | (v,e) |^2)]^{1/2}\}^2, \end{aligned}$$

and the required relation follows. □

Notice that the assumption in Corollary 2 is merely to make sure that we do not get complex numbers, since inequalities do not apply to them.

Corollary 3. *Let* $x, y, u, v \in X$*. If* $\| u \|^2 \le 2\text{Re } (x,u)$ *and* $\| v \|^2 \le 2\text{Re } (y,v)$*, then:*

(1) $[2\text{Re } (x,u) - \| u \|^2]^{1/2} [2\text{Re } (y,v) - \| v \|^2]^{1/2} + | (x-u, y-v) - (x,e)(e,y) | \le [\| x \|^2 - | (x,e) |^2]^{1/2} [\| y \|^2 - | (y,e) |^2]^{1/2},$

for some unit vector $e \in H$*.*

(2) $[2\text{Re } (x,u) - \| u \|^2]^{1/2} [2\text{Re } (y,v) - \| v \|^2]^{1/2} + | (x-u, y-v) | \le \| x \| \| y \| .$

Proof. (1) In Corollary 2 choose a unit vector e such that $(u,e)=(v,e)=0$.

(2) In (1) above choose a unit vector e such that $(x,e)=(y,e)=0$. □

Corollary 4. *Let $x,y,u,v\in H$. Then:*

(1) $|\,(x-u,y-v)-(x-u,e)(e,y-v)\,|$
$$\leq \| x-u \| \| y-v \| - |\,(x-u,e)(e,y-v)\,|,$$
for some unit vector e.

(2) $|\,(x-u,y-v)-(x,e)(e,y)\,|$
$$\leq \| x-u \| \| y-v \| - |\,(x,e)(e,y)\,|$$
for some unit vector e.

(3) $|\,(x-u,y-v)-(x,e)(e,y)\,|$
$$\leq \| x-u-(x,e)e \| \| y-v-(y,e)e \|$$
for some unit vector e. Equality holds if and only if $x-u=\alpha(y-v)+\beta e$, $\alpha,\beta\in\mathbf{C}$.

(4) $|\,(x,y)\,|\leq|\,(x,y)-(x,e)(e,y)\,|+|\,(x,e)(e,y)\,|\leq\| x \| \| y \|$.

Proof. (1) Let $a=\| x-u \|$, $b=\| y-v \|$, $c=|\,(x-u,e)\,|$, and $d=|\,(e,y-v)\,|$. Then (1) follows by the first inequality in (ii) of Theorem 1. Precisely,

$$\begin{aligned}|\,(x-u,y-v)-(x-u,e)(e,y-v)\,|^2 &\leq (a^2-c^2)(b^2-d^2)\leq(ab-cd)^2 \quad \text{by } (\#)\\ &= [\| x-u \| \| y-v \| - |\,(x-u,e)(e,y-v)\,|]^2.\end{aligned}$$

(2) In (1) above choose a unit vector e such that $(u,e)=(v,e)=0$.

(3) In the second inequality in (ii) of Theorem 1 choose a unit vector e such that $(u,e)=(v,e)=0$

(4) The first inequality is trivial. Let $u=v=0$ in (1) above. Then the second inequality follows. □

Corollary 5. *Let $x,y,z\in H$. Then:*

$$\big|\| z \|^2 (x,y)-(x,z)(z,y)\,\big|^2 \leq [\| z \|^2\| x \|^2-|\,(x,z)\,|^2][\| z \|^2\| y \|^2-|\,(y,z)\,|^2]\,,$$

or

$$\big|\| z \|^2 (x,y)-(x,z)(z,y)\,\big|\| z \|^2 \leq \big\|\| z \|^2 x-(x,z)z \big\|\big\|\| z \|^2 y-(y,z)z \big\| .$$

Equality holds if and only if $x = \alpha y + \beta z$, $\alpha, \beta \in \mathbf{C}$.

Proof. Let $e = z/ \parallel z \parallel$ in (ii) of Corollary 1. □

Corollary 6. *Let* $x, y, z, e \in H$ *with* $\parallel e \parallel = 1$. *Then:*

$$\begin{aligned} \parallel y \parallel^2 \mid 2(z,x)(x,e) - \parallel x \parallel^2 (z,e) \mid^2 + \mid 2(z,x)(x,y) - \parallel x \parallel^2 (z,y) \mid^2 \\ + \parallel z \parallel^2 \parallel x \parallel^4 \mid (y,e) \mid^2 \\ \leq \parallel y \parallel^2 \parallel z \parallel^2 \parallel x \parallel^4 + 2\mathrm{Re}\, [2(z,x)(x,e) - \parallel x \parallel^2 (z,e)] \\ \cdot [2(x,z)(y,x) - \parallel x \parallel^2 (y,z)](e,y). \end{aligned}$$

Equality holds if and only if $2(z,x)x - \parallel x \parallel^2 z = \alpha e + \beta y$, $\alpha, \beta \in \mathbf{C}$.

Proof. Replace x by $2(z,x)x - \parallel x \parallel^2 z$, y by e, and z by y in the first inequality of Corollary 5, then

$$\begin{aligned} \mid \parallel y \parallel^2 (2(z,x)x - \parallel x \parallel^2 z, e) - (2(z,x)x - \parallel x \parallel^2 z, y)(y,e) \mid^2 \\ \leq [\parallel y \parallel^2 \parallel 2(z,x)x - \parallel x \parallel^2 z \parallel^2 - \mid (2(z,x)x - \parallel x \parallel^2 z, y) \mid^2] \\ \cdot [\parallel y \parallel^2 - \mid (e,y) \mid^2], \end{aligned}$$

and note that

$$\parallel 2(z,x)x - \parallel x \parallel^2 z \parallel = \parallel z \parallel \parallel x \parallel^2 .$$

Now, let $A = 2(z,x)(x,e) - \parallel x \parallel^2 (z,e)$, $B = 2(z,x)(x,y) - \parallel x \parallel^2 (z,y)$, $C = (y,e)$, $D = \parallel z \parallel^2 \parallel x \parallel^4$, and $E = \parallel y \parallel^2$. Then, basically we have inequality

$$\mid EA - BC \mid^2 \leq [ED - \mid B \mid^2][E - \mid C \mid^2]$$

due to the inequality above. It follows that

$$E \mid A \mid^2 + \mid B \mid^2 + D \mid C \mid^2 \leq ED + 2\,\mathrm{Re}\, A\overline{BC}.$$

The desired inequality follows by a straightforward substitution and we shall omit. Equality condition is immediate. □

Remark that the purpose of Corollary 6 is actually for the next interesting result; the inequality (4) in particular in Corollary 7 below was proved in [1] by a bit complicated method, and we may call it the Cauchy-Schwarz inequality in three variables.

Corollary 7. *Let* $x, y, z \in H$.

(1) *There exists a unit vector* $e \in H$ *such that*

$$\begin{aligned} \parallel y \parallel^2 \mid 2(z,x)(x,e) - \parallel x \parallel^2 (z,e) \mid^2 + \mid 2(z,x)(x,y) - \parallel x \parallel^2 (z,y) \mid^2 \\ \leq \parallel y \parallel^2 \parallel z \parallel^2 \parallel x \parallel^4 . \end{aligned}$$

Equality holds if and only if $2(z,x)x- \parallel x \parallel^2 z = [2(z,x)(x,e)- \parallel x \parallel^2 (z,e)]e + \beta y$, $\beta \in \mathbf{C}$.

(2) *There exists a unit vector* $e \in H$ *such that*

$$\parallel y \parallel^2 \parallel x \parallel^4 | (z,e) |^2 + | 2(z,x)(x,y) - \parallel x \parallel^2 (z,y) |^2 \leq \parallel y \parallel^2 \parallel z \parallel^2 \parallel x \parallel^4 .$$

Equality holds if and only if $2(z,x)x- \parallel x \parallel^2 z = - \parallel x \parallel^2 (z,e)e + \beta y$, $\beta \in \mathbf{C}$.

(3) $| 2(z,x)(x,y)- \parallel x \parallel^2 (z,y) | \leq \parallel y \parallel \parallel z \parallel \parallel x \parallel^2$.

Equality holds if and only if $2(z,x)x- \parallel x \parallel^2 z = \beta y$, $\beta \in \mathbf{C}$.

(4) $| (z,x)(x,y) | \leq \dfrac{\parallel z \parallel \parallel y \parallel + | (z,y) |}{2} \parallel x \parallel^2$.

Proof. (1) In Corollary 6 choose a unit vector e such that $(y,e) = 0$. As for equality condition, $(y,e) = 0$ implies $\alpha = 2(z,x)(x,e)- \parallel x \parallel^2 (z,e)$.

(2) Choose a unit vector e in (1) above such that $(x,e) = 0$, and we have (2).

(3) Choose a unit vector e in (2) above such that $(z,e) = 0$.

(4) Since

$$2 | (z,x)(x,y) | - \parallel x \parallel^2 | (z,y) | \leq | 2(z,x)(x,y)- \parallel x \parallel^2 (z,y) | \leq \parallel y \parallel \parallel z \parallel \parallel x \parallel^2 .$$

The first inequality is trivial, and the second one is due to (3) above. □

The next result is motivated by the Bessel inequality which states that if $\{e_i\}_1^n \subseteq H$ is an orthonormal set and $x \in H$, then $\sum_i | (x,e_i) |^2 \leq \parallel x \parallel^2$. Remark that the definition of the set $\{u_i\}_1^n$ in Theorem 2 below comes originally from [6], and that (iv) in Theorem 2 is a generalization of the Bessel inequality.

Theorem 2. *Let* $x, y \in H$, $\{e_i\}_1^n$ *be a set of unit vectors, and let the vectors* $\{u_i\}_1^n$ *be defined as follows:* $u_0 = x$, *and* $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, $i = 1, 2, ..., n$. *Then the following are equivalent:*

(i) $| (x,y) | \leq \parallel x \parallel \parallel y \parallel$ (*Cauchy-Schwarz inequality*).

Equality holds if and only if $x = \alpha y$, $\alpha \in \mathbf{C}$.

(ii) $| (x,y) - \sum_i (u_{i-1}, e_i)(e_i, y) |^2 + \parallel y \parallel^2 \sum_i | (u_{i-1}, e_i) |^2 \leq \parallel x \parallel^2 \parallel y \parallel^2$.

Equality holds if and only if $u_n = \alpha y$, *or* $x = \sum_i (u_{i-1}, e_i)e_i + \alpha y$, $\alpha \in \mathbf{C}$.

(iii) *If* $(y, e_i) = 0$, $i = 1, 2..., n$, *then*

$$| (x,y) |^2 + \parallel y \parallel^2 \sum_i | (u_{i-1}, e_i) |^2 \leq \parallel x \parallel^2 \parallel y \parallel^2 .$$

Equality holds if and only if $x=\sum_i(u_{i-1},e_i)e_i+\alpha y$ *such that*

$$\sum_i (u_{i-1},e_i)(e_i,e_j)=(x,e_j),\quad j=1,2,...,n,\ \alpha\in\mathbf{C}.$$

(iv) *If* $\{e_i\}_1^n$ *is an orthonormal set, then*

$$|\ (x,y)-\sum_i (x,e_i)(e_i,y)\ |^2+\|\ y\ \|^2\sum_i |\ (x,e_i)\ |^2\le\|\ x\ \|^2\|\ y\ \|^2,$$

or,

$$|\ (x,y)-\sum_i (x,e_i)(e_i,y)\ |\le\|\ y\ \|\|\ x-\sum_i (x,e_i)e_i\ \|\ .$$

Equality holds if and only if $x=\sum_i(x,e_i)e_i+\alpha y$, $\alpha\in\mathbf{C}$.

(v) *If* $(y,e_i)=0$, $i=1,2...,n$, *and* $\{e_i\}_1^n$ *is an orthonormal set, then*

$$|\ (x,y)\ |^2+\|\ y\ \|^2\sum_i |\ (x,e_i)\ |^2\le\|\ x\ \|^2\|\ y\ \|^2,$$

or,

$$|\ (x,y)\ |\le\|\ y\ \|\|\ x-\sum_i (x,e_i)e_i\ \|\ .$$

Equality holds if and only if $x=\sum_i(x,e_i)e_i+\alpha y$ *such that*

$$\sum_i (x,e_i)(e_i,e_j)=(x,e_j),\quad j=1,2,...,n,\ \alpha\in\mathbf{C}.$$

Proof. (i) implies (ii). By definition of the set $\{u_i\}_1^n$, a straightforward verification shows that

$$u_n=x-\sum_i (u_{i-1},e_i)e_i$$

and

$$\|\ u_i\ \|^2=\|\ u_{i-1}\ \|^2-|\ (u_{i-1},e_i)\ |^2\ .$$

The second equality above yields

$$\|\ u_n\ \|^2=\|\ x\ \|^2-\sum_i |\ (u_{i-1},e_i)\ |^2\ .$$

The required inequality follows easily by substituting these relations into the Cauchy-Schwarz inequality $|\ (u_n,y)\ |^2\le\|\ u_n\ \|^2\|\ y\ \|^2$. Equality holds if and only if $u_n=\alpha y$, $\alpha\in\mathbf{C}$, by (i).

(ii) implies (iii). In (ii) pick up a suitable set $\{e_i\}_1^n$ of unit vectors such that $(y, e_i) = 0$ for all i, then (iii) follows. Equality condition is due to (ii) and the relation $0 = (\alpha y, e_j) = (u_n, e_j) = (x, e_j) - \sum_i (u_{i-1}, e_i)(e_i, e_j)$, $j = 1, 2, ..., n$.

(iii) implies (v). Because $(u_{i-1}, e_i) = (x, e_i)$, $i = 1, 2, ..., n$, if $\{e_i\}_1^n$ is an orthonormal set. The altered form is due to the fact that $\| x - \sum_i (x, e_i) e_i \|^2 = \| x \|^2 - \sum_i | (x, e_i) |^2$.

(v) implies (i). Let $n = 1$ in (v) and choose e_1 such that $(x, e_1) = 0$.

Implications (ii)⇒ (iv)⇒(v) are trivial now. □

A straightforward generalization of Corollary 1, using $\{e_i\}_1^n$ instead of e, is as follows.

Theorem 3. *Let $x, y \in H$, and let $\{e_i\}_1^n$ be an orthonormal set of vectors. Then the following are equivalent.*

(i) $| (x, y) | \leq \| x \| \| y \|$ (Cauchy-Schwarz inequality).

Equality holds if and only if $x = \alpha y$, $\alpha \in \mathbf{C}$.

(ii) $| (x, y) - \sum_i (x, e_i)(e_i, y) |^2$

$$\leq [\| x \|^2 - \sum_i | (x, e_i) |^2][\| y \|^2 - \sum_i | (y, e_i) |^2];$$

or, $| (x, y) - \sum_i (x, e_i)(e_i, y) |$

$$\leq \| x - \sum_i (x, e_i) e_i \| \| y - \sum_i (y, e_i) e_i \| .$$

Equality holds if and only if $x = \alpha y + \sum_i \beta_i e_i$, $\alpha, \beta_i \in \mathbf{C}$, $i = 1, 2, ..., n$.

Proof. (i) implies (ii). Consider the Cauchy-Schwarz inequality

$$| (x - \sum_i (x, e_i) e_i, y - \sum_i (y, e_i) e_i) |^2 \leq \| x - \sum_i (x, e_i) e_i \|^2 \| y - \sum_i (y, e_i) e_i \|^2 .$$

Since $\{e_i\}_1^n$ is an orthonormal set, we have

$$(x - \sum_i (x, e_i) e_i, y - \sum_i (y, e_i) e_i) = (x, y) - \sum_i (x, e_i)(e_i, y)$$

and

$$\| x - \sum_i (x, e_i) e_i \|^2 = \| x \|^2 - \sum_i | (x, e_i) |^2 .$$

Hence, inequality (ii) and the altered form follow.

Equality holds if and only if $x - \sum_i (x, e_i) e_i$ and $y - \sum_i (y, e_i) e_i$ are proportional, i.e., $x = \alpha y + \sum_i \beta_i e_i$, $\alpha, \beta_i \in \mathbf{C}$, $i = 1, 2, ..., n$.

(ii) implies (i). Let $n = 1$ in (ii). Then we have precisely (ii) of Corollary 1, which in turn implies the Cauchy-Schwarz inequality. □

Notice that Corollary 8 below appeared in [2, Lemma 2.3]. Ours follows directly from Theorem 3.

Corollary 8. *Let $x, y \in H$, and let $\{e_i\}_1^n$ be an orthonormal set of vectors. Then:*

(1) $\mid (x,y) - \sum_i (x,e_i)(e_i,y) \mid + \sum_i \mid (x,e_i)(e_i,y) \mid \leq \| x \| \| y \|$.

(2) $\mid (x,y) \mid^2 + \| y \|^2 \sum_i \mid (x,e_i) \mid^2 \leq \| x \|^2 \| y \|^2$;

or, $\mid (x,y) \mid \leq \| x - \sum_i (x,e_i)e_i \| \| y \|$.

Equality holds if and only if $x = \alpha y + \sum_i \beta_i e_i$, $\alpha, \beta_i \in \mathbf{C}$, $i = 1, 2, ..., n$.

Proof. Use the first inequality in (ii) of Theorem 3 and let $a = \| x \|$, $b = \| y \|$, $c = (\sum_i \mid (x,e_i) \mid^2)^{1/2}$, and $d = (\sum_i \mid (y,e_i) \mid^2)^{1/2}$. Then

$$
\begin{aligned}
\mid (x,y) - \sum_i (x,e_i)(e_i,y) \mid &\leq (a^2 - c^2)^{1/2}(b^2 - d^2)^{1/2} \\
&\leq (ab - cd) \quad \text{by } (\#) \\
&= \| x \| \| y \| - [\sum_i \mid (x,e_i) \mid^2]^{1/2} [\sum_i \mid (y,e_i) \mid^2]^{1/2} \\
&\leq \| x \| \| y \| - \sum_i \mid (x,e_i)(e_i,y) \mid,
\end{aligned}
$$

and we have (1).

(2) In (ii) of Theorem 3 choose an orthonormal set of vectors $\{e_i\}_1^n$ such that $(y, e_i) = 0$, $i = 1, 2, ..., n$. □

In Theorem 3 above suppose that $\{e_i\}_1^n$ is just a set of nonzero vectors, then clearly inequality (ii) is necessarily more complecated. To make inequality more reasonably expressible we might add a condition that y and $\{e_i\}_1^n$ are orthogonal; as the next result shows. Note that (ii) in Theorem 4 below is exactly (2) in Corollary 7 with a different proof.

Theorem 4. *Let $x, y \in H$, and let $\{e_i\}_1^n$ be a set of nonzero vectors such that $(y, e_i) = 0$, $i = 1, 2, ..., n$, and $j = 1, 2, ..., n$. Then the following are equivalent.*

(i) $\mid (x,y) \mid \leq \| x \| \| y \|$ *(Cauchy-Schwarz inequality).*

Equality holds if and only if $x = \alpha y$, $\alpha \in \mathbf{C}$.

(ii) $\mid (x,y) \mid^2$

$$\leq \{ \| x \|^2 - 2 \sum_i \mid (x,e_i) \mid^2 + \sum_i [\sum_j (x,e_i)(e_j,x)(e_i,e_j)] \} \| y \|^2;$$

or, $\mid (x,y) \mid \leq \| x - \sum_i (x,e_i)e_i \| \| y \|$.

Equality holds if and only if $x = \alpha y + \sum_i \beta_i e_i$, $\alpha, \beta_i \in \mathbf{C}$, $i = 1, 2, ..., n$.

Proof. (i) implies (ii). Consider the Cauchy-Schwarz inequality

$$\mid (x - \sum_i (x, e_i)e_i, y) \mid^2 \leq \parallel x - \sum_i (x, e_i)e_i \parallel^2 \parallel y \parallel^2 .$$

Since $\mid (x - \sum_i (x, e_i)e_i, y) \mid^2 = \mid (x, y) \mid^2$ and

$$\parallel x - \sum_i (x, e_i)e_i \parallel^2$$
$$= \parallel x \parallel^2 - 2\sum_i \mid (x, e_i) \mid^2 + \sum_i [\sum_j (x, e_i)(e_j, x)(e_i, e_j)],$$

and we are done.

(ii) implies (i). Let $n = 1$ in the second inequality of (ii). Then

$$\mid (x, y) \mid \leq \parallel x - (x, e_1)e_1 \parallel \parallel y \parallel,$$

for any $e_1 \in H$. Without loss of generality we may choose e_1 a unit vector such that $(x, e_1) = (y, e_1) = 0$. Then we have (i).

Since $0 = (x, e_1) = \alpha(y, e_1) + \beta_1 = \beta_1$ by equality condition in (ii). Hence, $x = \alpha y$ for equality condition in (i). □

We would like to mention at this point that besides Theorems 1, 2, 3, and 4 each inequality in every corollary is also equivalent to the Cauchy-Schwarz inequality. Because each inequality in every corollary follows by the Cauchy-Schwarz inequality. Conversely, it is easily checked that the Cauchy-Schwarz inequality is a special case of each inequality in every corollary. This is significant in our investigation of inequalities in a Hilbert space. Moreover, in our discussion of inequalities the space could be relaxed to a pre-Hilbert space without loosing any of the essential conclusions; as long as Lemma is not used in proofs.

3. Applications

In this final section we want to present some applications to illustrate the importance of inequalities obtained in Section 2. Throughout the remainder of the paper it is to be understood that the catital letters mean bounded linear operators on H. $T = U \mid T \mid$ is the polar decomposition of T with U the partial isometry and $\mid T \mid$ is the positive square root of the positive operator T^*T such that $N(U) = N(\mid T \mid)$, where $N(A)$ denotes the kernel of A. Let us recall the following well-known basic relations as we shall use them. Let $T = U \mid T \mid$ be the polar decomposition as in above. Then $U^*U = I$, the identity operator; and for $m > 0$, $\mid T^* \mid^m = U \mid T \mid^m U^*$ holds in general [3, p. 752].

Corollary 9. *Let $x, y, z \in H$ with $\mid T \mid^\alpha z \neq o$, and $\alpha \in [0, 1]$. Then*

(1) $$\frac{\mid (\mid T \mid^{2\alpha} z, z)(Tx, y) - (\mid T \mid^{2\alpha} x, z)(Tz, y) \mid^2}{(\mid T \mid^{2\alpha} z, z)^2}$$
$$+ \frac{(\mid T \mid^{2\alpha} x, x) \mid (Tz, y) \mid^2 + (\mid T^* \mid^{2(1-\alpha)} y, y) \mid (\mid T \mid^{2\alpha} x, z) \mid^2}{(\mid T \mid^{2\alpha} z, z)}$$
$$\leq (\mid T \mid^{2\alpha} x, x)(\mid T^* \mid^{2(1-\alpha)} y, y) + \frac{\mid (\mid T \mid^{2\alpha} x, z) \mid^2 \mid (Tz, y) \mid^2}{(\mid T \mid^{2\alpha} z, z)^2}.$$

or,

$$\mid (\mid T \mid^{2\alpha} z, z)(Tx, y) - (\mid T \mid^{2\alpha} x, z)(Tz, y) \mid (\mid T \mid^{2\alpha} z, z)$$
$$\leq \| (\mid T \mid^{2\alpha} z, z) U \mid T \mid^\alpha x - (\mid T \mid^{2\alpha} x, z) U \mid T \mid^\alpha z \|$$
$$\cdot \| (\mid T \mid^{2\alpha} z, z) \mid T^* \mid^{1-\alpha} y - (y, Tz) U \mid T \mid^\alpha z \|.$$

Equality holds if and only if $\mid T \mid^\alpha x = \beta \mid T \mid^{1-\alpha} U^* y + \gamma \mid T \mid^\alpha z$, $\beta, \gamma \in \mathbf{C}$.

(2) *If* $(Tz, y) = 0$, *then*

$$\mid (Tx, y) \mid^2 + \frac{(\mid T^* \mid^{2(1-\alpha)} y, y) \mid (\mid T \mid^{2\alpha} x, z) \mid^2}{(\mid T \mid^{2\alpha} z, z)}$$
$$\leq (\mid T \mid^{2\alpha} x, x)(\mid T^* \mid^{2(1-\alpha)} y, y)$$

or,

$$\mid (\mid T \mid^{2\alpha} z, z)(Tx, y) \mid (\mid T \mid^{2\alpha} z, z)$$
$$\leq \| (\mid T \mid^{2\alpha} z, z) U \mid T \mid^\alpha x - (\mid T \mid^{2\alpha} x, z) U \mid T \mid^\alpha z \|$$
$$\cdot \| (\mid T \mid^{2\alpha} z, z) \mid T^* \mid^{1-\alpha} y \|.$$

Equality holds if and only if $\mid T \mid^\alpha x = \beta \mid T \mid^{1-\alpha} U^* y + \frac{(|T|^{2\alpha} x, z)}{(|T|^{2\alpha} z, z)} \mid T \mid^\alpha z$, $\beta \in \mathbf{C}$.

Proof. First of all rewrite the first inequality in Corollary 5 as follows:

$$\frac{|\| z \|^2 (x, y) - (x, z)(z, y) \mid^2}{\| z \|^4} + \frac{\| x \|^2 \mid (y, z) \mid^2 + \| y \|^2 \mid (x, z) \mid^2}{\| z \|^2}$$
$$\leq \| x \|^2 \| y \|^2 + \frac{\mid (x, z) \mid^2 \mid (z, y) \mid^2}{\| z \|^4}.$$

Equality holds if and only if $x = \beta y + \gamma z$, $\beta, \gamma \in \mathbf{C}$.

(1) Replace x by $U \mid T \mid^{\alpha} x$, y by $\mid T^* \mid^{1-\alpha} y$, and z by $U \mid T \mid^{\alpha} z$ in the inequality above. Note that $\parallel z \parallel^2 = (U \mid T \mid^{\alpha} z, U \mid T \mid^{\alpha} z) = (\mid T \mid^{2\alpha} z, z)$; $(x, y) = (U \mid T \mid^{\alpha} x, U \mid T \mid^{1-\alpha} U^* y) = (U \mid T \mid x, y) = (Tx, y)$; $(x, z) = (\mid T \mid^{2\alpha} x, z)$; $\parallel y \parallel^2 = (\mid T^* \mid^{2(1-\alpha)} y, y)$; and $(z, y) = (U \mid T \mid^{\alpha} z, U \mid T \mid^{1-\alpha} U^* y) = (Tz, y)$. Thus we shall get (1) after simplifications, and the details should be omitted.

Equality holds if and only if $U \mid T \mid^{\alpha} x = \beta \mid T^* \mid^{1-\alpha} y + \gamma U \mid T \mid^{\alpha} z$, or, $\mid T \mid^{\alpha} x = \beta \mid T \mid^{1-\alpha} U^* y + \gamma \mid T \mid^{\alpha} z$, $\beta, \gamma \in \mathbf{C}$.

(2) Clearly this is a special case of (1). As for equality condition, since $(Tz, y) = 0$ yields

$$0 = (U \mid T \mid z, y) = (U \mid T \mid^{1-\alpha} \mid T \mid^{\alpha} z, y) = (\mid T \mid^{\alpha} z, \mid T \mid^{1-\alpha} U^* y).$$

It follows from equality condition in (1) above that

$$(\mid T \mid^{\alpha} x, \mid T \mid^{\alpha} z) = \beta(\mid T \mid^{1-\alpha} U^* y, \mid T \mid^{\alpha} z) + \gamma(\mid T \mid^{\alpha} z, \mid T \mid^{\alpha} z).$$

So, $\gamma = \frac{(|T|^{2\alpha} x, z)}{(|T|^{2\alpha} z, z)}$ and the proof is completed. □

The well-known Heinz inequality [3] is as follows: The relation

$$\mid (Tx, y) \mid^2 \leq (\mid T \mid^{2\alpha} x, x)(\mid T^* \mid^{2(1-\alpha)} y, y)$$

holds for any operator T, $x, y \in H$, and a real $\alpha \in [0, 1]$. Its slight generalization is the inequality (2) in Corollary 9 which also appeared in [6, Theorem 1] with a long proof. If we use the same replacement as in the proof of Corollary 9 into the Cauchy-Schwarz inequality, then the proof of the Heinz inequality becomes trivial. In other words, the Heinz inequality follows by the Cauchy-Schwarz inequality. Moreover, equality holds if and only if $\mid T \mid^{\alpha} x = \beta \mid T \mid^{1-\alpha} U^* y$, $\beta \in \mathbf{C}$. Remark that we may produce the Heinz-type inequalities by different replacement for x, y and z in Corollary 5, and we leave the details to the reader.

In order to have differet applications let us recall basic notations and facts as we shall use them. T is a positive operator (written $T \geq O$) in case $(Tx, x) \geq 0$ for all $x \in H$. If S and T are Hermitian, we write $T \geq S$ in case $T - S \geq O$. Remark that (i), the Furuta inequality [4], in the next result is an excellent extension of the Löwner-Heinz formula, i.e., if $A \geq B \geq O$, then $A^{\alpha} \geq B^{\alpha}$ for $\alpha \in [0, 1]$; but the inequality does not hold in general for $\alpha > 1$.

Corollary 10. *Let* $A \geq B \geq O$, $p, q, r, s \geq 0$, *and* $u, v \geq 1$ *with* $(1 + r)u \geq p + r$ *and* $(1 + s)v \geq q + s$. *If* $y \in H$, $B^{\frac{p+r}{2u}} y \neq o$ *and* $\{e_i\}_1^n$ *is an orthonormal* *set of vectors such that* $(B^{\frac{p+r}{2u}}y, e_i) = 0$, $i = 1, 2, ..., n$, *then the following are equivalent.*

(i) $B^{\frac{p+r}{u}} \leq (B^{r/2}A^pB^{r/2})^{\frac{1}{u}}$ *(Furuta inequality* [4]*);*

(ii) $\mid (B^{\frac{p+r}{2u}+\frac{q+s}{2v}}x, y) \mid^2 + \parallel B^{\frac{p+r}{2u}}y \parallel^2 \sum_i \mid (B^{\frac{q+s}{2v}}x, e_i) \mid^2$
$$\leq ((B^{s/2}A^qB^{s/2})^{\frac{1}{v}}x, x)((B^{r/2}A^pB^{r/2})^{\frac{1}{u}}y, y);$$

or, $\mid (B^{\frac{p+r}{2u}+\frac{q+s}{2v}}x, y) \mid \leq \parallel B^{\frac{q+s}{2v}}x - \sum_i (B^{\frac{q+s}{2v}}x, e_i)e_i \parallel \parallel B^{\frac{p+r}{2u}}y \parallel$

for every $x \in H$*. Equality holds if and only if* $B^{\frac{q+s}{2v}}x = \alpha B^{\frac{p+r}{2u}}y + \sum_i (B^{\frac{q+s}{2v}}x, e_i)e_i$, $\alpha \in \mathbf{C}$.

Proof. (i) implies (ii). In (v) of Theorem 2 replace x by $B^{\frac{q+s}{2v}}x$ and y by $B^{\frac{p+r}{2u}}y$, and use (i) to get (ii), its altered form and equality condition.

(ii) implies (i). Let $y = x$, $q = p$, $s = r$, and $v = u$ in (ii), and neglect the second term on the left side of (ii). Then

$$\mid (B^{\frac{p+r}{u}}x, x) \mid^2 \leq ((B^{r/2}A^pB^{r/2})^{\frac{1}{u}}x, x)^2,$$

which implies (i), and the proof is finished. □

A special case of Corollary 10 is a characterization of the Löwner-Heinz formula. Let $p = \alpha$, $q = \beta$, $r = s = 0$, and $u = v = 1$ in Corollary 10. Then $\alpha, \beta \in [0, 1]$, and immediately we have the next result without proof.

Corollary 11. *Let* $A \geq B \geq O$ *and* $\alpha, \beta \in [0, 1]$*. If* $y \in H$, $B^{\alpha/2}y \neq o$ *and* $\{e_i\}_1^n$ *is an orthonormal set of vectors such that* $(B^{\alpha/2}y, e_i) = 0$, $i = 1, 2, ..., n$, *then the following are equivalent.*

(i) $B^\alpha \leq A^\alpha$ *(Löwner-Heinz formula);*

(ii) $\mid (B^{\frac{\alpha+\beta}{2}}x, y) \mid^2 + \parallel B^{\alpha/2}y \parallel^2 \sum_i \mid (B^{\beta/2}x, e_i) \mid^2$
$$\leq (A^\beta x, x)(A^\alpha y, y);$$

or, $\mid (B^{\frac{\alpha+\beta}{2}}x, y) \mid \leq \parallel B^{\beta/2}x - \sum_i (B^{\beta/2}x, e_i)e_i \parallel \parallel B^{\alpha/2}y \parallel$,

for every $x \in H$*. Equality holds if and only if* $B^{\beta/2}x = \gamma B^{\alpha/2}y + \sum_i (B^{\beta/2}x, e_i)e_i$, $\gamma \in \mathbf{C}$.

The main result in [5] is the next corollary. Our proof is much simpler and direct.

Corollary 12. (see [5, Theorem 1]) *Let* $A, B \geq O$ *such that* $\parallel Tx \parallel \leq \parallel Ax \parallel$ *and* $\parallel T^*x \parallel \leq \parallel By \parallel$ *for all* $x, y \in H$*. Then for each* $r \geq 0$ *and* $s \geq 0$

$$\mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2$$
$$\leq ((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y),$$

for any $p, q \geq 1$, *and* $\alpha, \beta \in [0, 1]$ *such that* $(1+2r)\alpha + (1+2s)\beta \geq 1$.

Proof. In the Cauchy-Schwarz inequality let x be replaced by $U \mid T \mid^{(1+2r)\alpha} x$, and y by $\mid T^* \mid^{(1+2s)\beta} y$. Then

$$\mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2 \leq (\mid T \mid^{2(1+2r)\alpha} x, x)(\mid T^* \mid^{2(1+2s)\beta} y, y),$$

and the equality holds if and only if $U \mid T \mid^{(1+2r)\alpha} x = \alpha \mid T^* \mid^{(1+2s)\beta} y$, $\alpha \in \mathbf{C}$ (in fact, the inequality above is better than the inequality in question). But $(\mid T \mid^{2(1+2r)\alpha} x, x) \leq ((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)$ and $(\mid T^* \mid^{2(1+2s)\beta} y, y) \leq ((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$ by the Furuta inequality, so we have the required inequality. □

Corollary 13. *Let* $r \geq 0$ *and* $s \geq 0$. *Then, for* $x, y, z \in H$,

$$\mid (\mid T \mid^{2(1+2r)\alpha} z, x)(T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid \leq \frac{1}{2} \| \mid T \mid^{(1+2r)\alpha} x \|^2 \left[\| \mid T \mid^{(1+2r)\alpha} z \| \| \mid T^* \mid^{(1+2s)\beta} y \| + \mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} z, y) \mid \right],$$

for any $p, q \geq 1$, *and* $\alpha, \beta \in [0, 1]$ *such that* $(1+2r)\alpha + (1+2s)\beta \geq 1$.

Proof. All we have to do is replacing x by $U \mid T \mid^{(1+2r)\alpha} x$, y by $\mid T^* \mid^{(1+2s)\beta} y$, and z by $U \mid T \mid^{(1+2r)\alpha} z$ in (4) of Corollary 7, and straightforword simplifications. □

Remark that if $z = x$ in Corollary 13 in particular, then we obtain the inequality in the proof of Corollary 12. This is not surprising, since (4) in Corollary 7 becomes the Cauchy-Schwarz inequality if $z = x$.

In conclusion, we remark that incidental to applications above, we may obtain various other operator inequalities along the way by applying other corollaries and theorems as well.

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