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NON-EXTENDIBILITY OF CERTAIN HOLOMORPHIC VECTOR BUNDLES ON INFINITE-DIMENSIONAL DOMAINS

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Abstract: As a sample we give the following result. Let V be a complex locally convex, Hausdorff and Baire topological vector space such that its dual V' is equipped with a Hausdorff and sequentially complete locally convex topology for which the natural pairing $V \times V' \to \mathbf{C}$ is continuous. Let H be any closed hyperplane of V'. Fix any $P \in V$. There is a holomorphic vector bundle E on $V \backslash \{P\}$ with fibers isomorphic to H which is not the restriction of a holomorphic vector bundle on V and such that for every open neighborhood Ω of P in V the holomorphic vector bundle $E|\Omega \cap U$ is not trivial.

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1. Introduction

As an appetizer we will give the following extension of a result of S. Dineen concerning the first cohomology group of analytic sheaves on pseudoconvex domains of complex topological vector spaces equipped with the finite open topology (see [4], Proposition 1).

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Theorem 1. *Let U be a pseudoconvex open subset of a complex topological vector space equipped with the finite open topology and A a holomorphic vector bundle on U with fibers isomorphic to a Banach space. Then $H^1(U, A) = 0$.*

Now we describe a few non-extendibility results for holomorphic vector bundles.

Theorem 2. *Let V be a complex locally convex, Hausdorff and Baire topological vector space such that its dual V' is equipped with a Hausdorff and sequentially complete locally convex topology for which the natural pairing $V \times V' \to \mathbf{C}$ is continuous. Let $U \subsetneqq V$ be an open subset and $P \in \partial U$ such that U is C^2 at P and 2-concave at P, i.e. such that the following condition is satified:*

(α) There is an open neighborhood M of P in V and a C^2-function $f : M \to \mathbf{R}$ such that $M \cap U = \{Q \in M : f(Q) < 0\}$, $df(P) \neq 0$ and there is a two-dimensional complex vector space $N \subset V$ transversal to the real hyperplane $T_P(\partial(U)$ of V and such that the restriction to N of the Levi form of f at P is strictly definite negative.

Then there is a holomorphic vector bundle E on U with fibers isomomorphic to V' with the following property: there is no neighborhood Ω of P in V such that the holomorphic vector bundle $E|U \cap \Omega$ is the restriction of a holomorphic vector bundle on Ω.

For instance in Theorem 2 we may take as V any Banach space. As an immediate corollary we obtain the following result.

Corollary 1. *Let V be a complex locally convex, Hausdorff and Baire topological vector space such that its dual V' is equipped with a Hausdorff and sequentially complete locally convex topology for which the natural pairing $V \times V' \to \mathbf{C}$ is continuous. Let H be any closed hyperplane of V'. Fix any $P \in V$. There is a holomorphic vector bundle E on $V \backslash \{P\}$ with fibers isomorphic to H which is not the restriction of a holomorphic vector bundle on V and such that for every open neighborhood Ω of P in V the holomorphic vector bundle $E|\Omega \cap U$ is not trivial.*

Theorem 3. *Let V be a complex locally convex and Hausdorff topological vector space, U an open subset of V and $P \in \partial(U)$ such that U is C^2 at P and 2-concave at P. Fix an integer $k \geq 2$ and assume the existence of holomorphic functions f_i, $1 \leq i \leq k$, on V such that $P \in Z$ and $Z \cap U = \emptyset$, where $Z := \{Q \in V : f_1(Q) = \cdots = f_k(Q) = 0\}$. Then there is a rank $k-1$*

holomorphic vector bundle E on U with the following property: there is no neighborhood Ω of P in V such that the holomorphic vector bundle $E|U \cap \Omega$ is the restriction of a holomorphic vector bundle on Ω.

Proof of Theorem 1. The case $A \cong \mathcal{O}_U$ is just [4], Proposition 1. Since A is locally free, the tensor product with A is an exact functor in the category of $\mathcal{O}_U$-sheaves. Hence tensoring with A diagram (1) at p. 338, one can copy verbatim the proof of [4], pp. 339–339, if one knowns in advance the case in which U has finite-dimensional. If A has finite-rank, the case U finite-dimensional is true by Theorem B of Cartan-Serre. For the extension of this theorem in which A is an arbitrary Banach bundle, see [3], 3.5, or [1]. □

Proof of Theorem 2. Without losing generality we may assume $P = 0$. Let $\mathcal{O}_{V,0}$ be the sheaf of germs of holomorphic functions on V vanishing at 0. The evaluation map induces a surjection $f : \mathcal{O}_V^{V'} \to \mathcal{I}_{V,0}$. Set $G := \mathrm{Ker}(f)$, $F := G|V\backslash\{0\}$ and $E := F|U$. G is an analytic sheaf on V. □

First Claim. *F is a holomorphic vector bundle on $V\backslash\{0\}$ with fibers isomorphic to H.*

Proof of First Claim. We need to check the local freeness of F. Fix $Q \in V\backslash\{0\}$. By Hahn-Banach there is $h \in V'$ such that $h(Q) \neq 0$. Set $M := \mathrm{Ker}(h)$. By the Open Mapping Theorem $M \cong H$. We will see that $F|V\backslash M \cong (V\backslash M) \times H$ (the trivial bundle) in the following way. Define the map $u : (V\backslash M) \times H \to \mathrm{Ker}(f)|(V\backslash M)$ sending any $(Q', m) \in (V\backslash M) \times H$ to $m - (m(Q')/h(Q'))h$. By construnction $\mathrm{Im}(u) \subseteq \mathrm{Ker}(f)|(V\backslash M)$ and it is easy to check that we have equality because at each $Q' \in (V\backslash M)$ the linear space $\mathrm{Im}(u)|\{Q'\}$ is a hyperplane of V. □

By First Claim E is a holomorphic vector bundle on U with fibers isomorphic to H. We only need to check that E satifies the thesis of Theorem 2.

Second Claim. *For every open neighborhood Ω of P in V and every holomorphic bundle A on $\Omega\backslash\{0\}$ the restriction map $H^0(\Omega\backslash\{P\}, A) \to H^0(\Omega \cap U, A|\Omega \cap U)$ is surjective.*

Proof of Second Claim. If V is finite-dimensional and A has finite rank, then Second Claim is a classical property of 2-concavity. If V is finite-dimensional, but A not finite rank, see [2]. The case "V infinite-dimensional" follows from the case finite-dimensional for Gâteaux analytic sections of A. Since a Gâteaux analytic section of A whose restriction to $A \cap U$ is Fréchet analytic (see [5], part (b) of Theorem 2.4.4), we conclude the proof of Second Claim. □

Since sections of a trivial bundle induce a trivialization of that bundle, it is sufficient to prove that F is not trivial and that for every open neighborhood

Ω of P in V the restriction map $\rho : H^0(\Omega\backslash\{P\}, F|\Omega) \to H^0(\Omega \cap U, E|\Omega)$ is surjective. The surjectivity of ρ is Second Claim. Assume that F is trivial. By the definition of f we have the following exact sequence on V:

$$0 \to \mathrm{Ker}(f) \to \mathcal{O}_V^{V'} \to \mathcal{I}_{V,0} \to 0\,. \tag{1}$$

Since $\dim(V) \geq 2$, the restriction map

$$\eta : H^0(V, \mathcal{O}_V^{V'}) \to H^0(V\backslash\{0\}, \mathcal{O}_{V\backslash\{0\}}^{V'})$$

is surjective. Since $\mathcal{I}_{V,0}$ has no torsion, a diagram chasing using (1) gives the surjectivity of the restriction map $H^0(V, \mathrm{Ker}(f)) \to H^0(V\backslash\{0\}, F)$ and that this is true after restricting to any neighborhood Ω of P. Hence any trivialization of F would induce a trivialization of $\mathrm{Ker}(f)$. The sheaf $\mathrm{Ker}(f)$ is not trivial because its restriction to any finite-dimensional proper linear subspace of V has torsion, concluding the proof. □

Proof of Theorem 3. Set $f := (f_1, \ldots, f_k)$. Let W be the scheme-theoretic zero-locus of $f_1, \ldots, f_k$. Thus $Z = W_{red}$ and we have a surjection $f : \mathcal{O}_V^{\oplus k} \to \mathcal{I}_W$. Set $E := \mathrm{Ker}(f)|U$ and copy the proof of Theorem 2. □

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