International Journal of Pure and Applied Mathematics
Volume 12 No. 2 2004, 125-132

DYNAMIC ASSIGNMENTS ON CONSECUTIVE-2 SYSTEMS

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Abstract: A consecutive-2 system is a line of n components such that the system is failed if and only if there exist two consecutive failed components. In a dynamic assignment, the order of n positions in the system to receive assignment can be arbitrary and the state of a component (working or failed) becomes known once it is assigned to the position. We prove the nonexistence of an invariant dynamic assignment on the consecutive-2 system, but reduce the number of candidate asymptotically optimal assignments to $\lfloor n/2 \rfloor$.

AMS Subject Classification: 90B25, 05A05
Key Words: reliability, consecutive-2 system, dynamic assignment

1. Introduction

Assume that a system S has n components each of which can either work or be failed. A *consecutive*-2 *system* is a linear arrangement of the n components such

Received: October 23, 2003

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that the system fails if and only if there exist two consecutive failed components. The system reliability $R(S)$ is the probability that the system works.

Derman, Lieberman and Ross (DLR) [1] introduced the optimal permutation problem to consecutive-2 systems. Assume that the n components are functionally interchangeable and have different reliabilities (probabilities of working), which permutation yields the largest system reliability. Alternatively, we may view components as objects which exist independently of the system. Then a permutation becomes an assignment of the n components to the n positions, called *nodes*, of the consecutive-2 system, thus justifying the title of this paper.

Actually DLR studied two different assignment problems, which we will call *nonadaptive* and *sequential*, respectively. In a nonadaptive assignment, the mapping from components to nodes is specified all at once; while in a sequential assignment, the mapping is given one node at a time in the order of the line. The difference is that in the sequential assignment, the state of a component (working or failed) becomes known as soon as it is assigned to a node, and this information can be used to determine the next assignment. Clearly, an optimal sequential assignment cannot be worse than an optimal nonadaptive assignment.

Let p_i denotes the reliability of component i. Without loss of generality, we assume $p_1 \leq p_2 \leq \ldots \leq p_n$. In practical situations we may know only the ranking of the p_i's, but not their exact values. Therefore it would be nice if an optimal assignment depends only on the ranking. We call such an optimal assignment an *invariant assignment*.

DLR conjectured the invariant nonadaptive assignment which was later proved by Malon [4] and Du and Hwang [2]. In fact, the latter proved the more general case, where the linear assignment is replaced by the circular assignment. DLR also gave the invariant sequential arrangement. Recently, Hwang and Pai [3] proved the nonexistence of an invariant sequential assignment on the circular consecutive-2 system, but reduced the number of candidate optimal assignments to $\lfloor n/2 \rfloor$.

In this paper we generalize the sequential assignment to the *dynamic assignment* in which the order of nodes to receive assignment can be arbitrary. For example, we can assign to node 5 first and then to node 2. The sequential assignment is then a special case when the ordering is $1, 2, \ldots$. We prove that the invariant sequential assignment is no longer invariant in the larger class of dynamic assignments. In fact no invariant dynamic assignment exists for $n \geq 4$. Under the assumption that all p_i's are close to 1, we reduce the number of candidate asymptotically optimal dynamic assignments to $\lfloor n/2 \rfloor$.

2. Dynamic Assignments and 1-1 Mappings

Let $s = (s_1, \ldots, s_n)$, where s_i is the state of component i. Let S be the set of 2^n distinct s. For a dynamic ordering D, let $D(s) = (t_1, \ldots, t_n)$ be the output of the line given the input s, where t_j is the state of node j. Let $T = \{D(s) : s \in S\}$.

Proposition 1. $T = S$.

Proof. It suffices to prove that $s \neq s'$ implies $D(s) \neq D(s')$. Let $\pi = (\pi_1, \ldots, \pi_n)$ denote the ordering of nodes in D. Since $s \neq s'$, there exists a smallest i such that $s_i \neq s'_i$. Since $s_j = s'_j$ for $1 \leq j \leq i-1$, $\pi_j(s) = \pi_j(s')$ for $1 \leq j \leq i-1$. Therefore $D(s)$ and $D(s')$ assign the same component, say component c, to node i. But $s_i \neq s'_i$, hence $\pi_i(s) \neq \pi_i(s')$. Namely, $D(s) \neq D(s')$. □

Remark. Were $\pi_{i-1}(s) \neq \pi_{i-1}(s')$, then $D(s)$ could assign component c, and $D(s')$ component c', to node i. Then s_c could equal $s_{c'}$, and hence $\pi_i(s) = \pi_i(s')$ even though $s \neq s'$.

Corollary 2. *The number of input sequences inducing a system failure is independent of D and equal to $2^n - f_{n+1}$, where f_i is the Fibonacci number defined by $f_0 = f_1 = 1$, and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 2$.*

Proof. From Proposition 1 we note that D is a 1-1 mapping from the set N of 2^n binary n-sequences to itself. Let g_n denote the number of n-sequences in N containing no 2 consecutive failed components. It is easily verified that g_n satisfies the recurrence relation: $g_1 = 2$, $g_2 = 3$, and $g_n = g_{n-1} + g_{n-2}$ for $n \geq 3$. Therefore, $g_n = f_{n+1}$ and there are $2^n - f_{n+1}$ input sequences inducing a system failure. □

In fact, a nonadaptive assignment also defines a 1-1 mapping from N to N, and therefore, has the same number of failed sequences. However, the nonadaptive assignment, the sequential assignment and the dynamic assignment each induces different constraints on the mapping. To study these differences, we need to make some simplifying assumptions which cut down the details but preserve the main points.

3. The Nonexistence of Invariant Dynamic Assignment

Santha and Zhang [5] introduced an asymptotic analysis by assuming all p_i's are close to 1. Therefore the system failure is dominated by the exactly-two-failed-components (ETFC) events. Namely, if one assignment is better than another with respect to those ETFC events, then even it is worse with respect to other events, the probability of the latter can never make up the difference in the former. Santha and Zhang called invariance under the ETFC restriction *first-order invariance.*

For a nonadaptive assignment, the two components at the two ends appear once, while other components appear twice in the system-failure set F. For example, for $n = 9$, the optimal nonadaptive assignment is:

$$1\ 9\ 3\ 7\ 5\ 6\ 4\ 8\ 2,$$

which induces $F^n = \{(19), (93), (37), (75), (56), (64), (48), (82)\}$.

For a sequential assignment, each component must appear in F. This is because component i must be assigned during the process. Consider the case that all components assigned before i, except the next to last if i is last, are working. Then the failure of component i and its succeeding assignment (preceding if i is last) constitute a pair in F. Under this constraint, the best a sequential assignment can do is to have

$$F^s = \{(1n), (2n), \ldots, (n-1, n)\}.$$

F^s is achieved by the invariant sequential assignment $A(n)$ of DLR [1].

Assignment $A(n)$**.**

$A1$. Assign component 1 to node 1.

$A2$. Let c_i denote the component assigned to node i. For $i = 2, \ldots, n$, if c_{i-1} works, then c_i is chosen to be the one that has the lowest reliability of all remaining components; else c_i is chosen to be the one that has the highest reliability of all remaining components.

Intuitively, one feels that the requirement that all components appear in F should hold for all assignments. Indeed, it holds for all nonadaptive assignments and hence F^s is better than F^n. But surprisingly, this is not the case for dynamic assignments. For example, for $n = 5$, suppose we first assign component 5 to node 2. If component 5 works, assign component 1 to node 1. If 5 fails, assign 4 and 3 to nodes 1 and 3. Then regardless of the input state, component 1 does not appear in F.

To prove the nonexistence of an invariant dynamic assignment, we consider the following assignment.

Assignment $H(n)$**.**

$H1$. Assign component $\lceil n/2 \rceil + 1$ to node 2.

$H2$. If it works, assign component 1 to node 1 and use $H(n-2)$ on the rest components.

$H3$. If it does not work, assign the two best components to nodes 1 and 3 in arbitrary order. Apply $H(n-3)$ to the rest components.

It is easily verified that $H(n)$ yields

$$F^H = \{(n, n-1), (n, n-2), \ldots, (n, \lfloor n/2 \rfloor), \\ (n-1, n-2), (n-1, n-3), \ldots, (n-1, \lceil n/2 \rceil)\}.$$

For example, for $n = 9$,

$$F^H = \{(98), (97), (96), (95), (94), (87), (86), (85)\}.$$

Let P be the partial order on $\binom{n}{2}$ ordered pairs such that $(i_1, i_2) \leq (j_1, j_2)$ if $i_1 \leq j_1$ and $i_2 \leq j_2$. Then F' would be better than or equal to F, denoted by $F \leq F'$, if there exists a 1-1 onto mapping M from F to F' such that $(i, j) \leq M(i, j)$ for all $(i, j) \in F$. It can be easily verified that $F^n \leq F^H$, but F^H and F^s are not comparable since both are top-sets ($(i, j) \in F$ and $(i, j) \leq (i', j') \Rightarrow (i', j') \in F$). In fact when $p_n > p_{n-1} = \cdots = p_1$, then F^s is the unique optimal assignment, and when $p_n = p_{n-1} > p_{n-2} = \cdots = p_{\lfloor n/2 \rfloor} > p_{\lfloor n/2 \rfloor - 1} \geq \cdots \geq p_1$, then F^H is the unique optimal assignment. Since $F^H \neq F^s$ for $n \geq 4$, we have the following theorem.

Theorem 3. *There is no first-order invariant dynamic assignment for $n \geq 4$.*

Corollary 4. *There is no invariant dynamic assignment for $n \geq 4$.*

4. Asymptotically Optimal Dynamic Assignments

In this section, we show that it suffices to search a set of $\lfloor n/2 \rfloor$ candidates for an asymptotically optimal dynamic assignment.

When all components are working, a dynamic assignment assigns component c_1 to node n_1 first, then c_2 to n_2, ..., c_n to n_n. We use $\begin{pmatrix} c_1 & \ldots & c_n \\ n_1 & \ldots & n_n \end{pmatrix}$ to denote this procedure and let e_i denote the number of nodes adjacent to n_i but not assigned yet at the stage immediately after n_i is assigned. For example, for the assignment $H(n)$ with even n,

$$\begin{pmatrix} c_1 & \ldots & c_n \\ n_1 & \ldots & n_n \end{pmatrix} = \begin{pmatrix} n/2+1 & 1 & n/2+2 & 2 & \ldots & n & n/2 \\ 2 & 1 & 4 & 3 & \ldots & n & n-1 \end{pmatrix},$$

and $(e_1, \dots, e_n) = (2020 \dots 2010)$.

We note that for any dynamic assignment, $\sum_{i=1}^{n} e_i = n-1$ is the number of pairs in the system-failure set F. Further, if $e_i = 1$ then the component c_i contributes a pair (c_i, x) to F for some $x \in \{c_{i+1}, \dots, c_n\}$; if $e_i = 2$ then c_i contributes two pairs (c_i, x) and (c_i, y) to F for some $x, y \in \{c_{i+1}, \dots, c_n\}$. To maximize F, we choose x and y to be the largest and the second largest components in $\{c_{i+1}, \dots, c_n\}$. That is, when $e_i \neq 0$ and the components $c_1, c_2, \dots, c_{i-1}$ all work but c_i does not work, we assign the e_i currently most reliable components to the nodes adjacent to n_i. Note that such a policy is adopted in assignments $A(n)$ and $H(n)$. We will refer to it as the *maximal choice policy.*

Now we give a set of $\lfloor n/2 \rfloor$ dynamic assignments and prove that it suffices to search this set for an asymptotically optimal dynamic assignment.

Assignment $O(n,i)$, $i = 1, 2, \dots, \lfloor n/2 \rfloor$.

$O1$. When all components are working,

$$\begin{pmatrix} c_1 & \dots & c_n \\ n_1 & \dots & n_n \end{pmatrix}$$
$$= \begin{pmatrix} i & i+1 & \dots & n-i-1 & n-i & n-i+1 & \dots \\ 1 & 2 & \dots & n-2i & n-2i+2 & n-2i+4 & \dots \end{pmatrix.$$
$$\begin{matrix} \dots & n-1 & 1 & 2 & \dots & i-1 & n \\ \dots & n & n-2i+1 & n-2i+3 & \dots & n-3 & n-1 \end{pmatrix} .$$

$O2$. If $c_1, \dots, c_{j-1}$ are all working and c_j is failed, the assignment then adopts the maximal choice policy for the unassigned neighbor(s) of n_j and uses an arbitrary rule for the remaining assignment (since it will not affect the system-failure set F).

Note that for $O(n,i)$, we have $(e_1, \dots, e_n) = (1 \dots 12 \dots 210 \dots 0)$ where the first $n-i$ digits are not zeroes. Thus $O(n,i)$ generates the system-failure set

$$F^{O(n,i)} = \{(n, n-1), (n, n-2), \dots, (n, i), \\ (n-1, n-2), (n-1, n-3), \dots, (n-1, n-i)\},$$

which is a top set with $n-i$ pairs involving n, and i pairs involving $n-1$ (the pair $(n, n-1)$ is counted in both). In particular, $O(n,1)$ generates the same F as $A(n)$ and $O(n, \lfloor n/2 \rfloor)$ generates the same F as $H(n)$.

Theorem 5. *Let* $I = \begin{pmatrix} c_1 & \dots & c_n \\ n_1 & \dots & n_n \end{pmatrix}$ *be a dynamic assignment on* n *components with* $|\{i : e_i \neq 0\}| = k$ *and* $\max\{i : e_i \neq 0\} = l$.

(1) If $e = 1$ *then* $F^I \leq F^{O(n,n-k)}$.

(2) If $e = 2$ *then* $F^I \leq F^{O(n,n-k-1)}$.

Proof. Without loss of generality, we assume that each dynamic assignment adopts the maximal choice policy. Note that if $e_i = 0$ and $e_{i+1} \neq 0$ for some i, then the interchange of the two columns (c_i, n_i) and (c_{i+1}, n_{i+1}) in I yields a new assignment I' with e_i and e_{i+1} interchanged. Further, $F^I \leq F^{I'}$. So we may assume that $e_{k+1} = e_{k+2} = \cdots = e_n = 0$ in assignment I. Then the ordering of the components c_{k+1}, c_{k+2}, ..., and c_n does not affect F. So we may assume further that $c_{k+1} < c_{k+2} < \cdots < c_n$.

Case 1. $e_k = 1$: Since each dynamic assignment adopts the maximal choice policy, if $c_k < c_{n-1}$ then the interchange of c_k and c_{n-1} yields a new assignment with a better or equal F. So we may assume c_k and c_n are the two largest components in the set $\{c_k, c_{k+1}, \ldots, c_n\}$.

Whenever c_k and c_n are the two largest components in the set $S_i = \{c_{k-i+1}$, c_{k-i+2}, ..., $c_n\}$ and $c_{k-i} > \min\{c_k, c_n\}$ for some $i \geq 1$, we can interchange c_{k-i} and $\min\{c_k, c_n\}$ to obtain a new assignment in which c_k and c_n are the two largest components in the set $S_i \cup \{c_{k-i}\}$. We note that the new assignment has a better or equal F, since each dynamic assignment adopts the maximal choice policy.

By repeatedly applying such interchange of components, finally we have that c_k and c_n are the two largest components in the set $\{c_1, \ldots, c_n\}$, *i.e.*, $\{c_k, c_n\} = \{n, n-1\}$. Then the system-failure set of this assignment contains k pairs involving n, and $n-k$ pairs involving $n-1$ (the pair $(n, n-1)$ is counted in both). Therefore $F^I \leq F^{O(n,n-k)}$.

Case 2. $e_k = 2$: We consider a new assignment J that assigns c_1 to n_1, ..., c_{k-1} to n_{k-1}, c_n to $n_k - 1$, c_k to n_k, then assigns the remaining components using an arbitrary rule. By the maximal choice policy, it is easily verified that $F^I \leq F^J$. Also we note that for assignment J, $|\{i : e_i \neq 0\}| = k+1$ and $e_{k+1} = 1$. Then it follows from Case 1 that $F^J \leq F^{O(n,n-k-1)}$. Therefore, $F^I \leq F^{O(n,n-k-1)}$. □

Corollary 6. *It suffices to search the set* $\{F^{O(n,n-i)} : \lceil n/2 \rceil \leq i \leq n-1\}$ *for an asymptotically optimal dynamic assignment.*

Proof. Note that in Theorem 5, k is at least $\lceil n/2 \rceil$ and at most $n-1$. In particular, if $k = n-1$ then $(e_1, \ldots, e_n) = (1, 1, \ldots, 1, 0)$, and hence only Case 1 applies. □

Acknowledgements

First author is supported by NSC Grant 88-2811-M-007-0054. Second author is supported by NSC Grant 88-2115-M-009-016.

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