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ANALYSIS OF PERIODIC PREDATOR-PREY SYSTEM WITH DISPERSAL

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Abstract: This paper consider the permanence of a periodic predator-prey system, where the prey disperse in two-patch environment. We assume the Holling type predator functional response within-patch dynamics and provide a sufficient and necessary condition to guarantee the predator and prey species to be permanent.

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1. Introduction

Dispersal predator-prey systems described by autonomous ordinary differential equations have long played an important role in population biology [1-4, 6-14, 18, 19], and the references cited therein. Recently, Lou and Ma [14] studied the following predator-prey system in two-patch environments

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$$\begin{aligned} \dot{x}_1 &= x_1(b_1 - a_1x_1 - c_1y) + D(x_2 - x_1), \\ \dot{x}_2 &= x_2(b_2 - a_2x_2) + D(x_1 - x_2), \\ \dot{y} &= y(-d + c_2x_1 - qy), \end{aligned} \tag{1.1}$$

where $x_i(t)$ represents the prey population in the i-th patch, $i = 1, 2$, at time $t \geq 0, y(t)$ stands for the predator population in patch 1 at time $t \geq 0$; coefficients $a_i, b_i, c_i (i = 1, 2), d, q, D$ are all positive constants. They proved that

$$-d + c_2x_1^*(D) > 0 \tag{1.2}$$

is necessary and sufficient condition of the strong persistence of the system (1.1), where $(x_1^*(D), x_2^*(D))$ is the globally asymptotically stable equilibrium of the following system

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 - a_1x_1) + D(x_2 - x_1), \\ \dot{x}_2 &= x_2(b_2 - a_2x_2) + D(x_1 - x_2). \end{aligned}$$

Considering that the realistic models often require the effects of the changing environment, Cui [5] studied the corresponding periodic predator-prey system

$$\begin{aligned} \dot{x}_1 &= x_1[b_1(t) - a_1(t)x_1 - c_1(t)y] + D(t)(x_2 - x_1), \\ \dot{x}_2 &= x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2), \\ \dot{y} &= y[-d(t) + c_2(t)x_1 - q(t)y], \end{aligned} \tag{1.3}$$

under the assumptions that the functions $a_i(t), b_i(t), c_i(t)$ $(i = 1, 2)$, $D(t)$, $d(t)$ and $q(t)$ are all positive, $\omega-$periodic and continuous for $t \geq 0$. He obtained the permanence of (1.3) and improved the main results in [14] and [17].

This paper consider the predator-prey system

$$\begin{aligned} \dot{x}_1 &= x_1[b_1(t) - a_1(t)x_1 - \tfrac{c_1(t)y}{x_1+e(t)}] + D(t)(x_2 - x_1), \\ \dot{x}_2 &= x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2), \\ \dot{y} &= y[-d(t) + \tfrac{c_2(t)x_1}{x_1+e(t)} - q(t)y], \end{aligned} \tag{1.4}$$

where the functions $a_i(t), b_i(t), c_i(t)(i = 1, 2), D(t), d(t), e(t)$ and $q(t)$ are all positive, $\omega-$ periodic and continuous for $t \geq 0$. We will study the permanence and extinction of (1.4).

The organization of this paper is as follows. In the next section, we agree on some notations, give some definitions and state three lemmas which will be essential to our proofs. In Section 3 we obtain the necessary and sufficient condition which guarantee the system (1.4) to be permanent.

2. Notations, Definitions and Preliminaries

In this section, we introduce some definitions and notations and state some results which will be useful in the subsequent sections. Let C denotes the space of all bounded continuous functions $f : R \to R$,C_+^0 the set of nonnegative $f \in C$, and C_+ the set of all $f \in C$ such that f is bounded below by a positive constant. Given $f \in C$, we denote

$$f^M = \sup_{t \geq 0} f(t), \quad f^L = \inf_{t \geq 0} f(t),$$

and define the lower average $A_L(f)$ and upper average $A_M(f)$ of f by

$$A_L(f) = \lim_{r \to \infty} \inf_{t-s \geq r} (t-s)^{-1} \int_s^t f(\tau) d\tau \,,$$

and

$$A_M(f) = \lim_{r \to \infty} \sup_{t-s \geq r} (t-s)^{-1} \int_s^t f(\tau) d\tau \,,$$

respectively. If $f \in C$ is $\omega-$periodic, we define the average $A_\omega(f)$ of f on the time interval $[0, \omega]$ by

$$A_\omega(f) = \omega^{-1} \int_0^\omega f(t) dt.$$

Definition 2.1. The system of differential equations

$$\dot{x} = F(t, x), \quad x \in R^n,$$

is said to be permanent if there exists a compact set K in the interior of $R_+^n = \{(x_1, x_2, \cdots, x_n) \in R^n : x_i \geq 0, i = 1, 2, \cdots, n\}$, such that all solutions starting in the interior of R_+^n ultimately enter K. The system is said to be strong persistent if

$$\lim_{t \to \infty} \inf x_i(t) > 0, \quad i = 1, 2, \cdots, n$$

hold for all solutions $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ starting in the interior of R_+^n.

Definition 2.2. The system of differential equations

$$\dot{x} = F(t, x), \quad x \in R^n,$$

is said to be cooperative if the off-diagonal elements of $D_x F(t, x)$ are nonnegative and competitive if the off-diagonal elements are nonpositive, where $D_x F(t, x)$ is the $n \times n$ matrix derivative of F with respect to x.

Lemma 2.1. (see [16]) *Let $x(t)$ and $y(t)$ be solution of*

$$\dot{x} = F(t, x)$$

and

$$\dot{y} = G(t, y)\,,$$

respectively, where both systems are assumed to have the uniqueness property for initial value problems. Assume both $x(t)$ and $y(t)$ belong to a domain $D \subset R^n$ for $[t_0, t_1]$, in which one of the two systems is cooperative and

$$F(t, z) \leq G(t, z), \quad (t, z) \in [t_0, t_1] \times D.$$

If $x(t_0) \leq y(t_0)$ then $x(t_1) \leq y(t_1)$. If $F = G$ and $x(t_0) < y(t_0)$ then $x(t_1) < y(t_1)$.

To prove the permanence of the species in (1.3), we need some information on the periodic logistic models with and without dispersal.

Lemma 2.2. (see [20]) *The problem*

$$\dot{x} = x[b(t) - a(t)x], \quad x \in C_+, \tag{2.1}$$

has exactly one canonical solution U if $a \in C_+, b \in C$ and $A_L(b) > 0$. Moreover, we have the following properties :

(a) U is $\omega-$periodic (almost periodic) if a, b are $\omega-$periodic (almost periodic).

(b) U is constant if $\frac{b}{a}$ is constant. In this case, $U = \frac{b}{a}$.

(c) $u(t) - U(t) \to 0$ as $t \to \infty$, for any positive solution $u(t)$ of equation (2.1).

(d) $(\frac{b}{a})^L \leq U \leq (\frac{b}{a})^M$.

For the following dispersal logistic equations

$$\begin{aligned} \dot{x}_1 &= x_1[b_1(t) - a_1(t)x_1] + D(t)(x_2 - x_1), \\ \dot{x}_2 &= x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2), \end{aligned} \tag{2.2}$$

we have the following result.

Lemma 2.3. (see [15]) *Suppose that $b_i(t), a_i(t)(i = 1, 2)$ and $D(t)$ are all positive and ω periodic functions,then (2.2) has a positive and ω periodic solution $(x_1^*(t), x_2^*(t))$, which is globally asymptotically stable.*

3. Necessary and Sufficient Condition of Permanence in (1.4)

Theorem 3.1. *The system (1.4) is permanent if and only if*

$$A_\omega\left(-d(t)+\frac{c_2(t)x_1^*(t)}{x_1^*(t)+e(t)}\right)>0, \tag{3.1}$$

where $(x_1^*(t),x_2^*(t))$ *be the globally asymptotically stable periodic solution of (2.2).*

To prove this theorem, we need several propositions. In the rest of this paper we denote $(x_1(t),x_2(t),y(t))$ be any solution of (1.4) with positive initial condition.

Proposition 3.1. *Suppose (3.1) holds, then there exist positive constants* M_x *and* M_y *, such that*

$$\limsup_{t\to\infty} x_i(t)\le M_x,\quad \limsup_{t\to\infty} y(t)\le M_y,\quad i=1,2. \tag{3.2}$$

Proof. Obviously, R_+^3 is a positively invariant set of (1.4). Given any positive solution $(x_1(t),x_2(t),y(t))$ of (1.4), we have

$$\dot{x}_i\le x_i[b_i(t)-a_i(t)x_i]+D(t)(x_j-x_i),\quad i=1,2,\ j\ne i\,.$$

On the other hand, the following auxiliary equations

$$\dot{u}_i=u_i[b_i(t)-a_i(t)u_i]+D(t)(u_j-u_i),\quad i=1,2,\ j\ne i\,. \tag{3.3}$$

has a unique globally asymptotically stable positive $\omega-$periodic solution $(x_1^*(t), x_2^*(t))$. Let $(u_1(t),u_2(t))$ be the solution of (3.3) with $u_i(0)=x_i(0)$, by Lemma 2.1 we have

$$x_i(t)\le u_i(t),\quad i=1,2,\quad \text{for}\quad t\ge 0.$$

Moreover, from the global stability of $(x_1^*(t),x_2^*(t))$, for every given $\varepsilon>0$, there exists $T_0>0$, such that

$$u_i(t)<x_i^*(t)+\varepsilon,\quad \text{for}\quad t>T_0,$$

hence

$$x_i(t)<x_i^*(t)+\varepsilon, i=1,2,\quad \text{for}\quad t>T_0.$$

In addition we have

$$\dot{y}\le y[-d(t)+c_2(t)\frac{x_1^*(t)+\varepsilon}{x_1^*(t)+\varepsilon+e(t)}-q(t)y],\quad t\ge T_0\,,$$

because of $\frac{\partial}{\partial x_1}\left(\frac{x_1}{x_1+e(t)}\right) > 0$. By (3.1), Lemma 2.1 and Lemma 2.2, there exists $T_1 > T_0$, such that

$$y(t) < y^*(t) + \varepsilon, \quad \text{for} \quad t > T_1,$$

where $y^*(t)$ is the positive and globally asymptotically stable $\omega-$periodic solution of the following auxiliary logistic equation

$$\dot{v} = v[-d(t) + c_2(t)\frac{x_1^*(t) + \varepsilon}{x_1^*(t) + \varepsilon + e(t)} - q(t)v].$$

Denote $M_x = \max\limits_{0\le t\le\omega} \{x_i^*(t) + \varepsilon : i = 1, 2\}$ and $M_y = \max\limits_{0\le t\le\omega} \{y^*(t) + \varepsilon\}$, then (3.2) holds for system (1.4). □

Proposition 3.2. *Suppose (3.1) holds, then there exists a positive constant η_x such that*

$$\lim_{t\to\infty} \sup x_1(t) \ge \eta_x. \tag{3.4}$$

Proof. Suppose that (3.4) is not true,then there is a sequence $\{z_m\} \subset R_+^3$, such that

$$\lim_{t\to\infty} \sup x_1(t, z_m) < \frac{1}{m}, \quad m = 1, 2, \dots , \tag{3.5}$$

where $(x_1(t, z_m), x_2(t, z_m), y(t, z_m))$ is the solution of (1.4) with initial values $(x_1(t, 0),\ x_2(t, 0),\ y(t, 0)) = z_m$. Choosing sufficiently small positive constants ε_x and ε_y such that $\varepsilon_x < 1, \varepsilon_y < 1$ and

$$A_\omega\left(-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)}\right) < 0, \tag{3.6}$$

and

$$A_\omega(\phi_\varepsilon(t)) > 0, \tag{3.7}$$

where

$$\begin{aligned} \phi_\varepsilon(t) &= \min\{b_1(t) - c_1(t)\varepsilon_y \exp(\alpha\omega)/e(t) - a_1(t)\varepsilon_x, b_2(t) - a_2(t)\varepsilon_x\}, \alpha \\ &= \max_{0\le t\le\omega} \{d(t) + c_2(t) + q(t)\}. \end{aligned}$$

By (3.5), for the given $\varepsilon_x > 0$, there exists a positive integer N_0, such that

$$\lim_{t\to\infty} \sup x_1(t, z_m) < \frac{1}{m} < \varepsilon_x,$$

for $m > N_0$. In the rest of this proof we always assume that $m > N_0$.

The inequality above, implies that there exists $\tau_1^{(m)} > 0$, such that

$$x_1(t, z_m) < \varepsilon_x,$$

for $t \geq \tau_1^{(m)}$, and further

$$\dot{y}(t, z_m) \leq y(t, z_m)[-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)y(t, z_m)],$$

for $t \geq \tau_1^{(m)}$. By (3.6), any solution $v(t)$ of the following equation

$$\dot{v} = v[-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)v]\,,$$

with positive initial condition satisfies

$$\lim_{t\to\infty} v(t) = 0.$$

By Lemma 2.1, we have

$$\lim_{t\to\infty} y(t, z_m) = 0.$$

Therefore, there is a $\tau_2^{(m)} > \tau_1^{(m)}$ such that

$$y(t, z_m) < \varepsilon_y, \quad \text{for} \quad t \geq \tau_2^{(m)}. \tag{3.8}$$

This leads to

$$\begin{aligned} \dot{x}_1(t, z_m) \;&\geq\; x_1(t, z_m)[b_1(t) - c_1(t)\varepsilon_y/e(t) - a_1(t)x_1(t, z_m)] \\ &\quad +D(t)(x_2(t, z_m) - x_1(t, z_m)), \\ \dot{x}_2(t, z_m) \;&=\; x_2(t, z_m)[b_2(t) - a_2(t)x_2(t, z_m)] \\ &\quad +D(t)(x_1(t, z_m) - x_2(t, z_m)), \end{aligned}$$

for $t \geq \tau_2^{(m)}$. Let $(u_1(t), u_2(t))$ be any positive solution of the following auxiliary equations

$$\begin{aligned} \dot{u}_1 &= u_1[b_1(t) - a_1(t)u_1 - c_1(t)\varepsilon_y/e(t)] + D(t)(u_2 - u_1)\,, \\ \dot{u}_2 &= u_2[b_2(t) - a_2(t)u_2] + D(t)(u_1 - u_2)\,. \end{aligned} \tag{3.9}$$

By (3.7) and Lemma 2.3, (3.9) has a unique positive and ω periodic solution $(u_1^*(t), u_2^*(t))$, which is globally asymptotically stable. So we have

$$x_i(t, z_m) > \frac{u_i^*(t)}{2}, \quad i = 1, 2,$$

for sufficiently large $t > 0$ and $m > N_0$, which is a contradiction with (3.5). This completes the proof. □

Proposition 3.3. *Suppose (3.1) holds, then there exists positive constants* γ_x *such that*

$$\lim_{t\to\infty} \inf \rho_x(t) \geq \gamma_x, \tag{3.10}$$

where $\rho_x(t) = x_1(t) + x_2(t)$.

Proof. Suppose that (3.10) is not true, then there exists a sequence $\{z_m\} \subset R^3_+$, such that

$$\lim_{t\to\infty} \inf \rho_x(t, z_m) < \frac{\eta_x}{2m^2}, \quad m = 1, 2, \ldots.$$

On the other hand, by Proposition 3.2, we have

$$\lim_{t\to\infty} \sup \rho_x(t, z_m) \geq \lim_{t\to\infty} \sup x_1(t, z_m) \geq \eta_x, \quad m = 1, 2, \ldots .$$

Hence there are two time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions

$$0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots ,$$
$$s_q^{(m)} \to \infty, \quad t_q^{(m)} \to \infty \quad \text{as} \quad q \to \infty,$$

and

$$\begin{aligned} &\rho_x(s_q^{(m)}, z_m) = \frac{\eta_x}{m}, \quad \rho_x(t_q^{(m)}, z_m) = \frac{\eta_x}{m^2}, \\ &\frac{\eta_x}{m^2} < \rho_x(t, z_m) < \frac{\eta_x}{m}, \quad t \in (s_q^{(m)}, t_q^{(m)}). \end{aligned} \tag{3.11}$$

By Proposition 3.1, for a given integer $m > 0$,there is a $T_1^{(m)} > 0$, such that

$$x_i(t, z_m) \leq M_x, y(t, z_m) \leq M_y, \quad \text{for} \quad t \geq T_1^{(m)} \quad \text{and} \quad i = 1, 2.$$

Because of $s_q^{(m)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \geq K^{(m)}$, hence

$$\begin{aligned} \dot{x}_1(t, z_m) \geq\ & x_1(t, z_m)[b_1(t) - a_1(t)M_x - c_1(t)M_y/e(t)] \\ & + D(t)(x_2(t, z_m) - x_1(t, z_m)), \\ \dot{x}_2(t, z_m) \geq\ & x_2(t, z_m)[b_2(t) - a_2(t)M_x] + D(t)(x_1(t, z_m) - x_2(t, z_m)), \end{aligned}$$

for $q \geq K^{(m)}$, so

$$\dot{\rho}_x(t, z_m) \geq \zeta(t)\rho_x(t, z_m), \tag{3.12}$$

for $q \geq K^{(m)}, t \in [s_q^{(m)}, t_q^{(m)}]$, where $\zeta(t) = \min\{b_1(t) - a_1(t)M_x - c_1(t)M_y/e(t), b_2(t) - a_2(t)M_x\}$. Integrating (3.12) from $s_q^{(m)}$ to $t_q^{(m)}$ yields

$$\rho_x(t_q^{(m)}, z_m) \geq \rho_x(s_q^{(m)}, z_m) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t)dt,$$

or

$$-\int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t)dt \geq \ln m \quad \text{for} \quad q \geq K^{(m)}.$$

If $A_\omega(\zeta(t)) \geq 0$, this leads to a contradiction; otherwise $A_\omega(\zeta(t)) < 0$, we have

$$t_q^{(m)} - s_q^{(m)} \to \infty \quad \text{as} \quad m \to \infty, q \geq K^{(m)},$$

according to the boundedness of $\zeta(t)$. By (3.6) and (3.7),there are constants $P > 0$ and $N_0 > 0$ such that

$$\frac{\eta_x}{m} < \varepsilon_x, t_q^{(m)} - s_q^{(m)} > 2P, \tag{3.13}$$

and

$$M_y \exp \int_0^P [-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)\varepsilon_y]dt < \varepsilon_y, \quad \int_0^a \phi_\varepsilon(t)dt > 0, \tag{3.14}$$

for $m \geq N_0, q \geq K^{(m)}$ and $a \geq P$. (3.13) implies

$$x_i(t, z_m) < \varepsilon_x, \quad i = 1, 2, \quad t \in [s_q^{(m)}, t_q^{(m)}], \tag{3.15}$$

for $m \geq N_0, q \geq K^{(m)}$. For the positive ε_y satisfying (3.7) and (3.14), we have the following two circumstances:

(i) $y(t, z_m) \geq \varepsilon_y$ for all $t \in [s_q^{(m)}, s_q^{(m)} + P]$;

(ii) there exists $\tau_{q1}^{(m)} \in [s_q^{(m)}, s_q^{(m)} + P]$, such that $y(\tau_{q1}^{(m)}, z_m) < \varepsilon_y$.

If (i) holds, by (3.15) we have

$$\begin{aligned}
\varepsilon_y &\leq y(s_q^{(m)} + P, z_m) \\
&\leq y(s_q^{(m)}, z_m) \exp \int_{s_q^{(m)}}^{s_q^{(m)}+P} [-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)\varepsilon_y]dt \\
&\leq M_y \exp \int_0^P [-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)\varepsilon_y]dt < \varepsilon_y,
\end{aligned}$$

which is a contradiction.

If (ii) holds, we now claim that

$$y(t, z_m) \leq \varepsilon_y \exp(\alpha\omega), \quad t \in (\tau_{q1}^{(m)}, t_q^{(m)}]. \tag{3.16}$$

Otherwise, there exists $\tau_{q2}^{(m)} \in (\tau_{q1}^{(m)}, t_q^{(m)}]$ such that

$$y(\tau_{q2}^{(m)}, z_m) > \varepsilon_y \exp(\alpha\omega).$$

By the continuity of $y(t, z_m)$, there must exist $\tau_{q3}^{(m)} \in (\tau_{q1}^{(m)}, \tau_{q2}^{(m)})$ such that

$$y(\tau_{q3}^{(m)}, z_m) = \varepsilon_y,$$

and

$$y(t, z_m) > \varepsilon_y \quad \text{for} \quad t \in (\tau_{q3}^{(m)}, \tau_{q2}^{(m)}).$$

Denote $P^{(m)}$ the nonnegative integer such that $\tau_{q2}^{(m)} \in (\tau_{q3}^{(m)} + P^{(m)}\omega,\ \tau_{q3}^{(m)} + (P^{(m)} + 1)\omega]$, by (3.6) we obtain

$$\begin{aligned}
\varepsilon_y \exp(\alpha\omega) &< y(\tau_{q2}^{(m)}, z_m) \\
&< y(\tau_{q3}^{(m)}, z_m) \exp \int_{\tau_{q3}^{(m)}}^{\tau_{q2}^{(m)}} [-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)\varepsilon_y] dt \\
&= \varepsilon_y \exp \left\{ \int_{\tau_{q3}^{(m)}}^{\tau_{q3}^{(m)} + P^{(m)}\omega} + \int_{\tau_{q3}^{(m)} + P^{(m)}\omega}^{\tau_{q2}^{(m)}} \right\} [-d(t) + \frac{c_2(t)\varepsilon_x}{\varepsilon_x + e(t)} - q(t)\varepsilon_y] dt \\
&< \varepsilon_y \exp(\alpha\omega).
\end{aligned}$$

This contradiction establishes that (3.16) is true, particularly (3.16) holds for $t \in [s_q^{(m)} + P, t_q^{(m)}]$. By (3.11) and (3.7), we have

$$\frac{\eta_x}{m^2} = \rho_x(t_q^{(m)}, z_m) \geq \rho_x(s_q^{(m)} + P, z_m) \exp \int_{s_q^{(m)} + P}^{t_q^{(m)}} \phi_\varepsilon(t) dt > \frac{\eta_x}{m^2},$$

which is also a contradiction. This completes the proof. □

Proposition 3.4. *Suppose (3.1) holds, then there exists positive constants* $\gamma_{xi} (i = 1, 2,)$ *such that*

$$\liminf_{t \to \infty} x_i(t) \geq \gamma_{xi} \quad (i = 1, 2). \tag{3.17}$$

Proof. (3.10) implies that there exists $T_2 \geq T_1$ such that

$$\rho_x(t) = x_1(t) + x_2(t) \geq \gamma_x \quad \text{for} \quad t \geq T_2.$$

Hence,

$$\begin{aligned}\dot{x}_1 &= x_1[b_1(t) - 2D(t) - a_1(t)x_1 - \frac{c_1(t)y}{x_1 + e(t)}] + D(t)\rho_x(t) \\ &\geq -a_1^M x_1^2 + (b_1^L - 2D^M - c_1^M M_y/e^L)x_1 + D^L\gamma_x := F_1(x_1)\,,\end{aligned}$$

and

$$\dot{x}_2 \geq -a_2^M x_2^2 + (b_2^L - 2D^M)x_2 + D^L\gamma_x := F_2(x_2),$$

for $t \geq T_2$. The algebraic equation $F_1(x_1) = 0$ gives us one positive root

$$\tilde{x}_1 = \frac{b_1^L - 2D^M - c_1^M M_y/e^L + \sqrt{(b_1^L - 2D^M - c_1^M M_y/e^L)^2 + 4D^L a_1^M \gamma_x}}{2a_1^M}.$$

Clearly, $F_1(x_1) > 0$ for every positive number $x_1 (0 < x_1 < \tilde{x}_1)$. Choose $\gamma_{x1} (0 < \gamma_{x1} < \tilde{x}_1), \dot{x}_1 \mid_{x_1 = \gamma_{x1}} \geq F_1(\gamma_{x1}) > 0$. If $x_1(T_2) \geq \gamma_{x1}$ then it also holds for $t \geq T_2$; if $x_1(T_2) < \gamma_{x1}$, then

$$\dot{x}_1(T_2) \geq \inf\{F_1(x_1) \mid 0 \leq x_1 < \gamma_{x1}\} > 0,$$

there must exist $T_3 (\geq T_2)$, such that $x_1(t) > \gamma_{x1}$ for $t \geq T_3$.

Similarly, there exists $\gamma_{x2} > 0$ and $T_4 (\geq T_3)$, such that $x_2(t) > \gamma_{x2}$ for $t \geq T_4$. This completes the proof. □

Proposition 3.5. *Suppose that (3.1) holds, then there exists a positive constant η_y such that*

$$\limsup_{t\to\infty} y(t) > \eta_y. \tag{3.18}$$

Proof. By (3.1), we can choose constant $\varepsilon_0 > 0$ such that

$$A_\omega(\psi_{\varepsilon_0}(t)) > 0, \tag{3.19}$$

where

$$\psi_{\varepsilon_0}(t) = -d(t) + \frac{c_2(t)(x_1^*(t) - \varepsilon_0)}{x_1^*(t) - \varepsilon_0 + e(t)} - q(t)\varepsilon_0.$$

Consider the following equations with parameter α $(0 < \alpha < \frac{b_1^L e^L}{2c_1^M})$

$$\begin{aligned} \dot{x}_1 &= x_1[b_1(t) - 2\alpha c_1(t)/e(t) - a_1(t)x_1] + D(t)(x_2 - x_1), \\ \dot{x}_2 &= x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2). \end{aligned} \tag{3.20}$$

By Lemma 2.3, (3.20) has a unique positive $\omega-$ periodic solution $(x_{1\alpha}(t), x_{2\alpha}(t))$, which is globally asymptotically stable. Let $(\bar{x}_{1\alpha}(t), \bar{x}_{2\alpha}(t))$ be the solution of (3.20) with initial condition $\bar{x}_{i\alpha}(0) = x_i^*(0), i = 1, 2$, where $(x_1^*(t), x_2^*(t))$ be the positive and ω periodic solution of (2.2), then for the given ε_0, there exists $T_5 \geq T_4$, such that

$$| \bar{x}_{1\alpha}(t) - x_{1\alpha}(t) | < \frac{\varepsilon_0}{4} \quad \text{for} \quad t \geq T_5.$$

By the continuity of solution to parameter, we have $(\bar{x}_{1\alpha}(t), \bar{x}_{2\alpha}(t)) \to (x_1^*(t), x_2^*(t))$ uniformly in $[T_5, T_5 + \omega]$ as $\alpha \to 0$. Hence for $\varepsilon_0 > 0$, there exists positive $\alpha_0 = \alpha_0(\varepsilon_0) < \frac{b_1^L e^L}{2c_1^M}$ such that

$$| \bar{x}_{1\alpha}(t) - x_1^*(t) | < \frac{\varepsilon_0}{4} \quad \text{for} \quad t \in [T_5, T_5 + \omega], \quad 0 < \alpha < \alpha_0.$$

So we have

$$| x_{1\alpha}(t) - x_1^*(t) | \leq | \bar{x}_{1\alpha}(t) - x_{1\alpha}(t) | + | \bar{x}_{1\alpha}(t) - x_1^*(t) | < \frac{\varepsilon_0}{2},$$

for $t \in [T_5, T_5 + \omega]$. Since $x_{1\alpha}(t)$ and $x_1^*(t)$ are all $\omega-$ periodic, we have

$$| x_{1\alpha}(t) - x_1^*(t) | < \frac{\varepsilon_0}{2}, \quad \text{for} \quad t \geq 0, \quad 0 < \alpha < \alpha_0.$$

Choosing constant $\alpha_1 (0 < \alpha_1 < \alpha_0, 2\alpha_1 < \varepsilon_0)$, then

$$x_{1\alpha_1}(t) \geq x_1^*(t) - \frac{\varepsilon_0}{2}, t \geq 0. \tag{3.21}$$

Suppose that the conclusion (3.18) is not true, then there exists $Z \in R_+^3$, for the positive solution $(x_1(t), x_2(t), y(t))$ of (1.4) with initial condition $(x_1(0), x_2(0), y(0)) = Z$, we have

$$\lim_{t\to\infty} \sup y(t) < \alpha_1.$$

So there exists $T_6 \geq T_5$ such that

$$y(t) < 2\alpha_1 \quad \text{for} \quad t \geq T_6, \tag{3.22}$$

and hence

$$\dot{x}_1 \geq x_1[b_1(t) - 2\alpha_1 c_1(t)/e(t) - a_1(t)x_1] + D(t)(x_2 - x_1)\,,$$
$$\dot{x}_2 = x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2).$$

Let $(u_1(t), u_2(t))$ be the solution of (3.20) with $\alpha = \alpha_1$ and $u_i(T_6) = x_i(T_6), i = 1, 2$, by Lemma 2.1 we know that

$$x_i(t) \geq u_i(t), \quad t \geq T_6, \quad i = 1, 2.$$

By the globally asymptotically stability of $(x_{1\alpha_1}(t), x_{2\alpha_1}(t))$, for given $\varepsilon = \varepsilon_0/2$, there exists $T_7 \geq T_6$ such that

$$| u_1(t) - x_{1\alpha_1}(t) | < \frac{\varepsilon_0}{2} \quad \text{for} \quad t \geq T_7.$$

So we have

$$x_1(t) \geq u_1(t) > x_{1\alpha_1}(t) - \frac{\varepsilon_0}{2}, t \geq T_7,$$

and hence

$$x_1(t) > x_1^*(t) - \varepsilon_0, t \geq T_7.$$

This implies

$$\dot{y}(t) \geq \psi_{\varepsilon_0}(t)y(t), \quad \text{for} \quad t \geq T_7,$$

integrating above inequality from T_7 to t yields

$$y(t) \geq y(T_7) \exp \int_{T_7}^{t} \psi_{\varepsilon_0}(t)dt.$$

By (3.19) we know that $y(t) \to \infty$ as $t \to \infty$, which is a contradiction. This completes the proof. □

Proposition 3.6. *Under the assumption (3.1), there exists a positive constant γ_y such that*

$$\liminf_{t\to\infty} y(t) \geq \gamma_y. \tag{3.23}$$

Proof. Otherwise,there must exists a sequence $\{z_m\} \subset R^3_+$, such that

$$\liminf_{t\to\infty} y(t, z_m) < \frac{\eta_y}{(m+1)^2}, \quad m = 1, 2, \dots .$$

But

$$\limsup_{t\to\infty} y(t, z_m) > \eta_y, \quad m = 1, 2, \dots ,$$

according to Proposition 3.5. Hence there are two time sequence $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions

$$0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \dots ,$$

$$s_q^{(m)} \to \infty, \quad t_q^{(m)} \to \infty \quad \text{as} \quad q \to \infty,$$

and

$$\begin{aligned} y(s_q^{(m)}, z_m) &= \frac{\eta_y}{m+1}, \quad y(t_q^{(m)}, z_m) = \frac{\eta_y}{(m+1)^2}, \\ \frac{\eta_y}{(m+1)^2} &< y(t, z_m) < \frac{\eta_y}{m+1}, \quad t \in (s_q^{(m)}, t_q^{(m)}). \end{aligned} \tag{3.24}$$

By Proposition 3.1, for a given integer $m > 0$, there is a $T_1^{(m)} > 0$, such that

$$y(t, z_m) \le M_y \quad \text{for} \quad t \ge T_1^{(m)}.$$

Because of $s_q^{(m)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \ge K^{(m)}$, hence

$$\dot{y}(t, z_m) \ge y(t, z_m)[-d(t) - q(t)M_y]\,,$$

for $q \ge K^{(m)}, t \in [s_q^{(m)}, t_q^{(m)}]$. Integrating above inequality from $s_q^{(m)}$ to $t_q^{(m)}$, we get

$$y(t_q^{(m)}, z_m) \ge y(s_q^{(m)}, z_m) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} [-d(t) - q(t)M_y]dt.$$

So we have

$$\int_{s_q^{(m)}}^{t_q^{(m)}} [d(t) + q(t)M_y]dt \ge \ln(m+1) \quad \text{for} \quad q \ge K^{(m)}\,.$$

According to the boundedness of the function $d(t) + q(t)M_y$, we know that

$$t_q^{(m)} - s_q^{(m)} \to \infty \quad \text{as} \quad m \to \infty, q \ge K^{(m)}. \tag{3.25}$$

By (3.19), there exist $P > 0$ and an integer $N_0 > 0$ such that

$$\frac{\eta_y}{m+1} < \alpha_1 < \varepsilon_0, \quad t_q^{(m)} - s_q^{(m)} > 2P, \tag{3.26}$$

and

$$\int_0^a \psi_{\varepsilon_0}(t)dt > 0, \tag{3.27}$$

for $m \geq N_0, q \geq K^{(m)}$ and $a \geq P$. Further we have

$$y(t, z_m) < \alpha_1, \quad t \in [s_q^{(m)}, t_q^{(m)}],$$

for $m \geq N_0, q \geq K^{(m)}$. In addition, for $t \in [s_q^{(m)}, t_q^{(m)}]$, we have

$$\begin{array}{rcl} \dot{x}_1(t, z_m) & \geq & x_1(t, z_m)[b_1(t) - 2\alpha_1 c_1(t)/e(t) - a_1(t)x_1(t, z_m)] \\ & & +D(t)(x_2(t, z_m) - x_1(t, z_m)) , \\ \dot{x}_2(t, z_m) & = & x_2(t, z_m)[b_2(t) - a_2(t)x_2(t, z_m)] \\ & & +D(t)(x_1(t, z_m) - x_2(t, z_m)). \end{array}$$

Let $(u_1(t), u_2(t))$ be the solution of (3.20) with $\alpha = \alpha_1$ and $u_i(s_q^{(m)}) = x_i(s_q^{(m)}, z_m)$, by Lemma 2.1 we have

$$x_i(t, z_m) \geq u_i(t), \quad t \in [s_q^{(m)}, t_q^{(m)}].$$

Further, by Proposition 3.1, Proposition 3.4 and $s_q^{(m)} \to \infty$ as $q \to \infty$, we can choose $K_1^{(m)} > K^{(m)}$, such that

$$\gamma_{xi} \leq x_i(s_q^{(m)}, z_m) \leq M_x, \quad i = 1, 2\,,$$

holds for $q \geq K_1^{(m)}$. For $\alpha = \alpha_1$, (3.20) has a unique positive $\omega-$ periodic solution $(x_{1\alpha_1}(t), x_{2\alpha_1}(t))$ which is globally asymptotically stable. In addition, by the periodicity of (3.20), the periodic solution $(x_{1\alpha_1}(t), x_{2\alpha_1}(t))$ is uniformly asymptotically stable with respect to the compact set $\Omega = \{(x_1, x_2) : \gamma_{xi} \leq x_i \leq M_x, i = 1, 2\}$. Hence, for the given ε_0 in Proposition 3.5, there exists $T_0(> P)$ which is independent on m and q, such that

$$u_1(t) \geq x_{1\alpha_1}(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_0 + s_q^{(m)}.$$

Combining (3.21) we have

$$u_1(t) \geq x_1^*(t) - \varepsilon_0, \quad \text{for} \quad t \geq T_0 + s_q^{(m)}.$$

From (3.25), there exists a positive integer $N_1 \geq N_0$, such that $t_q^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2P$ for $m \geq N_1$ and $q \geq K_1^{(m)}$. So we have

$$x_1(t, z_m) \geq x_1^*(t) - \varepsilon_0, t \in [s_q^{(m)} + T_0, t_q^{(m)}]\,,$$

as $m \geq N_1$ and $q \geq K_1^{(m)}$. Hence

$$\dot{y}(t, z_m) \geq \psi_{\varepsilon_0}(t) y(t, z_m),$$

for $t \in [s_q^{(m)} + T_0, t_q^{(m)}]$. Integrating the inequality above from $s_q^{(m)} + T_0$ to $t_q^{(m)}$ yields

$$y(t_q^{(m)}, z_m) \geq y(s_q^{(m)} + T_0, z_m) \exp \int_{s_q^{(m)}+T_0}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) dt,$$

that is to say

$$\frac{\eta_y}{(m+1)^2} \geq \frac{\eta_y}{(m+1)^2} \exp \int_{s_q^{(m)}+T_0}^{t_q^{(m)}} \psi_{\varepsilon_0}(t) dt > \frac{\eta_y}{(m+1)^2},$$

which is a contradiction. This completes the proof. □

Combining the Proposition 3.1 to Proposition 3.5, we complete the proof of the sufficiency of this Theorem 3.1.

To prove the necessity of Theorem 3.1, we will show that

$$\lim_{t \to \infty} y(t) = 0\,,$$

under the following condition

$$A_\omega[-d(t) + \frac{c_2(t) x_1^*(t)}{x_1^*(t) + e(t)}] \leq 0. \tag{3.28}$$

In fact, by (3.28) we know that for every given $\varepsilon (0 < \varepsilon < 1)$, there exists $\varepsilon_1 > 0$ and $\varepsilon_0 > 0$ such that

$$A_\omega[-d(t) + \frac{c_2(t)(x_1^*(t) + \varepsilon_1)}{x_1^*(t) + \varepsilon_1 + e(t)} - q(t)\varepsilon] \leq -\frac{\varepsilon}{2} A_\omega(q(t)) \leq -\varepsilon_0. \tag{3.29}$$

Since

$$\begin{aligned} \dot{x}_1 &\leq x_1[b_1(t) - a_1(t) x_1] + D(t)(x_2 - x_1), \\ \dot{x}_2 &= x_2[b_2(t) - a_2(t) x_2] + D(t)(x_1 - x_1), \end{aligned}$$

we know that for the given ε_1 there exists $T^{(1)} > 0$ such that

$$x_1(t) \leq x_1^*(t) + \varepsilon_1 \quad \text{for} \quad t \geq T^{(1)}.$$

By (3.29) we have

$$A_\omega[-d(t) + \frac{c_2(t) x_1(t)}{x_1(t) + e(t)} - q(t)\varepsilon] \leq -\varepsilon_0\,, \tag{3.30}$$

for $t \geq T^{(1)}$.

Firstly, there must exist $T^{(2)}$ such that $y(T^{(2)}) < \varepsilon$. Otherwise, we have

$$\varepsilon \leq y(t) \leq y(T^{(1)}) \exp \int_{T^{(1)}}^{t} [-d(s) + \frac{c_2(s)x_1(s)}{x_1(s)+e(s)} - q(s)\varepsilon] ds \to 0\,,$$

as $t \to \infty$. This implies $\varepsilon \leq 0$, which is a contradiction. Let

$$M(\varepsilon) = \max_{0 \leq t \leq \omega} \left\{ d(t) + \frac{c_2(t)x_1(t)}{x_1(t)+e(t)} + q(t)\varepsilon \right\}.$$

By Proposition 3.1, we know that $x_1(t)$ is bounded. So $M(\varepsilon)$ is also bounded for $\varepsilon \in [0,1]$.

Secondly, we have

$$y(t) \leq \varepsilon \exp(M(\varepsilon)\omega) \quad \text{for} \quad t \geq T^{(2)}. \tag{3.31}$$

Otherwise,there exists $T^{(3)} > T^{(2)}$ such that

$$y(T^{(3)}) > \varepsilon \exp(M(\varepsilon)\omega).$$

By the continuity of $y(t)$, there must exist $T^{(4)} \in (T^{(2)}, T^{(3)})$ such that $y(T^{(4)}) = \varepsilon$ and $y(t) > \varepsilon$ for $t \in (T^{(4)}, T^{(3)}]$. Let P_1 be the nonnegative integer such that $T^{(3)} \in (T^{(4)} + P_1\omega, T^{(4)} + (P_1+1)\omega]$, by (3.30) we have

$$\begin{aligned}
\varepsilon \exp(M(\varepsilon)\omega) &< y(T^{(3)}) \\
&< y(T^{(4)}) \exp \int_{T^{(4)}}^{T^{(3)}} [d(t) + \frac{c_2(t)x_1(t)}{x_1(t)+e(t)} + q(t)\varepsilon] dt \\
&= \varepsilon \exp \left\{ \int_{T^{(4)}}^{T^{(4)}+P_1\omega} + \int_{T^{(4)}+P_1\omega}^{T^{(3)}} \right\} [d(t) + \frac{c_2(t)x_1(t)}{x_1(t)+e(t)} + q(t)\varepsilon] dt \\
&< \varepsilon \exp(M(\varepsilon)\omega)\,,
\end{aligned}$$

which is a contradiction. This implies (3.31) holds. Further by the arbitrariness of ε we know that $y(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 3.1.

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