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A NOTE ON LIOUVILLE THEOREM OF p-HARMONIC MAPS BETWEEN SPHERES

Xiaoli Chao [§]

Department of Mathematics
Southeast University
Nanjing 210096, P.R. CHINA
and
Department of Mathematics
Nanjing Normal University
Nanjing 210097, P.R. CHINA
e-mail: xlchao@seu.edu.cn

Abstract: In this note, we obtain a Liouville Theorem of p-harmonic maps between spheres.

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1. Introduction

The theory of harmonic maps, critical points of the energy functional defined on the space of smooth maps between two Riemannian manifolds, is one of the most important actual research themes. A natural generalization of the notion of a harmonic map is that of a p-harmonic map ($p > 1$), that is a critical point of the p-energy. The p-harmonic maps appear in many contexts in physics: non-Newtonian fluids, nonlinear elasticity, glaciology.

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[§]Correspondence address: Department of Mathematics, Southeast University, Nanjing 210096, P.R. CHINA

Let (M,g) be a Riemannian manifold (without boundary) of dimension m with metric g, and (N,h) a complete Riemannian manifold of dimension n with metric h. For a smooth map $u: M \to N$ and a number $2 \leq p \leq \infty$, we define the p-energy functional $E_p(u)$ of u by

$$E_p(u) = \frac{1}{p}\int_M |du(x)|^p dv_g,$$

where $|du(x)|$ is the norm of the differential $du(x)$ of u at $x \in M$ and dv_g is the volume element of M.

A p-harmonic map is a critical point of the p-energy functional. The Euler-Lagrange equation for p-harmonic map is

$$\triangle_p u = \delta(\|du\|^{p-2} du) = 0,$$

where δ is adjoint of exterior differential d w.r.t. the L^2-norm (see [1] for more detail).

In the past decades we have witness many development in the theory of harmonic maps ([1], [2]). It is interesting to generalize these results in case of p-harmonic maps. As for Liouville Theorem of harmonic maps, Schoen and Yau derived the following theorem.

Theorem A. (see [6]) *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and N be a Riemanian manifold with nonpositive sectional curvature. Then each harmonic map u from M to N with finite energy has to be a constant map.*

Takeuchi gave a Liouville type theorem for p-harmonic maps in a way analogous to [6].

Theorem B. (see [7]) *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and N be a Riemanian manifold with nonpositive sectional curvature. Then each p-harmonic map u from M to N with $\int_M \|du\|^{2p-2} dv_g < \infty$ has to be a constant map.*

Now, in this note, we obtain the following result.

Theorem C. *Let $u: S^m \to S^n (m \geq 3)$ be a p-harmonic map of class C^1. If $E_m(u) < \left(\min\left\{\frac{p-1}{p^2}\frac{m^2(m-2)}{m-1},\ m\right\}\right)^{m/2}\omega_m$, then u is constant.*

2. The Proof of Theorem C

Lemma. (see [4]) *Let $u : S^m \to S^n$ be a p-harmonic map of class C^1. Then we have the inequality*

$$\int_{S^m} |du|^{p-2}\big(|d|du||^2 - \frac{1}{p-1}\frac{m-1}{m}|du|^4\big) \le -\frac{m-1}{p-1}\int_{S^m}|du|^p. \tag{1}$$

Obviously, we have the identity

$$|du|^{p-2}|d|du||^2 = ||du|^{\frac{p}{2}-1}d|du||^2 = \frac{4}{p^2}|d(|du|^{\frac{p}{2}})|^2. \tag{2}$$

In order to simplify the notations, we set $f = |du|^{p/2}$. Then, from (1) and (2), we obtain

$$\begin{aligned} \frac{1}{p-1}\frac{m-1}{m}\int_{S^m} f^2 f^{4/p} &\ge \frac{4}{p^2}\int_{S^m}|df|^2 + \frac{m-1}{p-1}\int_{S^m} f^2 \\ &= \frac{4}{p^2}\|df\|_2^2 + \frac{m-1}{p-1}\|f\|_2^2\,. \end{aligned} \tag{3}$$

By Hölder inequality

$$\begin{aligned} \int_{S^m} f^2 f^{4/p} &\le \Big(\int_{S^m} f^{2m/(m-2)}\Big)^{(m-2)/m}\Big(\int_{S^m} f^{2m/p}\Big)^{2/m} \\ &= \|f\|^2_{2m/(m-2)}\|f^{2/p}\|_m^2\,, \end{aligned} \tag{4}$$

and Sobolev type inequality (see [3])

$$\|f\|^2_{2m/(m-2)} \le \frac{4}{m(m-2)}\omega_m^{-2/m}\|df\|_2^2 + \omega_m^{-2/m}\|f\|_2^2\,, \tag{5}$$

we have

$$\begin{aligned} \frac{1}{p-1}\frac{m-1}{m}\Big(\frac{4}{m(m-2)}\|df\|_2^2 + \|f\|_2^2\Big)\omega_m^{-2/m}\|f^{2/p}\|_m^2 \\ \ge \frac{4}{p^2}\|df\|_2^2 + \frac{m-1}{p-1}\|f\|_2^2\,, \end{aligned}$$

$$\begin{aligned} \Big(\frac{4}{p^2} - \frac{1}{p-1}\frac{m-1}{m}\frac{4}{m(m-2)}\omega_m^{-2/m}\|f^{2/p}\|_m^2\Big)\|df\|_2^2 \\ + \frac{m-1}{p-1}\Big(1 - \frac{1}{m}\omega_m^{-2/m}\|f^{2/p}\|_m^2\Big)\|f\|_2^2 \le 0. \end{aligned} \tag{6}$$

Since

$$\|f^{2/p}\|_m^2 = \Big(\int_{S^m} |du|^m\Big)^{2/m} = (E_m(u))^{2/m}. \tag{7}$$

If

$$E_m(u) < \Big(\min\Big\{\frac{p-1}{p^2}\frac{m^2(m-2)}{m-1},\ m\Big\}\Big)^{m/2}\omega_m,$$

we get, from (6) and (7), $|du| = 0$, i.e., u is constant.

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