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ON THE CONTACT LOCI OF TANGENT HYPERPLANES TO VARIETIES IN A PROJECTIVE SPACE

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Abstract: Here, following a paper by Chiantini and Ciliberto, we study integral varieties $X \subset \mathbf{P}^r$ such that for general $P_1, \dots, P_{k+1}$ many hyperplanes containing $T_{P_1}X \cup \dots \cup T_{P_{k+1}}X$ have positive-dimensional contact locus.

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1. Contact Loci

Let $X \subset \mathbf{P}^r$ be an integral n-dimensional non-degenerate variety. Fix an integer $k \geq 0$ and $k+1$ general points $P_1, \dots, P_{k+1}$. According to [1] the variety X is said to be weakly k-defective if the general hyperplane containing all the tangent spaces $T_{P_i}X$, $1 \leq i \leq k+1$, is tangent to X along a positive dimensional variety containing some P_i; obviously, we have to assume $\langle T_{P_1}X \cup \dots \cup T_{P_{k+1}} \rangle \neq \mathbf{P}^r$,

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so that there is at least one such hyperplane. In low dimension the weakly 0-defective varieties are classified and in arbitrary dimension they are "described" (use the Gauss mapping to see that they are exactly the developable scrolls (with the cones as a particular case)). In this case the general contact locus is a linear space ([2], lines 6–9 on p. 173).

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. The characteristic zero assumption is essential.

A key tool in this subject is the following infinitesimal form of Bertini Theorem proved by L. Chiantini and C. Ciliberto ([1], Theorem 2.2).

Lemma 1. (see [1], Theorem 2.2) *Let X be an integral projective variety and $\{H_y\}_{y\in Y}$ an algebraic variety of Cartier divisors parametrized by an integral variety Y. Let $y \in Y$ be a general point and $S := S_y$ the closure in X of $\mathrm{Sing}(H) \cap X_{reg}$. If Y is contained in a linear system $|V|$, then the projective tangent space to Y in $|V|$ is contained in $V(-S)$.*

We will always use the following notation.

Notation 1. Let $X \subset \mathbf{P}^r$ be an integral n-dimensional non-degenerate closed subvariety and $|V|$ the linear system of all hyperplanes of $\mathbf{P}^r$. For any zero-dimensioal $Z \subset X$ set $|V| := \{H \in |V| : Z \subset H\}$. For any $P \in X_{red}$ the first infinitesimal neighborhood of P in X. Hence $|V|(-2P)$ denotes the set of all hyperplanes containing the tangent space T_PX. Fix an integer $k \geq 0$ and $k+1$ general points of X. We will always assume $\langle T_{P_1}X \cup \cdots \cup T_{P_{k+1}}X\rangle \neq \mathbf{P}^r$, i.e. $|V|(-2P_1 - \cdots - 2P_{k+1}) \neq \emptyset$. Set $\delta_k := (n+1)(k+1) - 1 - \dim(\langle T_{P_1}X \cup \cdots \cup T_{P_{k+1}}X\rangle)$. For any $H \in |V|(-2P_1 - \cdots - 2P_{k+1})$ let $\tilde{\Sigma}_H$ be the union of all positive dimensional components of the contact locus containing at least one of the points $P_1, \dots, P_{k+1}$ and Σ_H the union of all irreducible components of $\tilde{\Sigma}_H$ with maximal dimension. Let $Y \subseteq |V|(-2P_1 - \cdots - 2P_{k+1})$ be a closed integral subvariety; we allow the case in which Y is degenerate. Let $\beta(Y) \geq 0$ be the codimension of Y in $|V|(-2P_1 - \cdots - 2P_{k+1})$. We assume $\Sigma_H \neq \emptyset$ for a general $H \in Y$. Set $\nu_k(Y) := \dim(\Sigma_H)$ for any general $H \in Y$.

Theorem 1. *Let $X \subset \mathbf{P}^r$ be an integral n-dimensional non-degenerate closed subvariety. Fix non-negative integers k and $\beta \leq r$. Assume the existence of an integral codimension β subvariety $\mathcal{Y}$ of $|V|$ such that for general $P_1, \dots, P_{k+1} \in X$ the set $Y := \mathcal{Y} \cap |V|(-2P_1 - \cdots - 2P_{k+1})$ is a non-empty integral codimension β subvariety of $|V|(-2P_1 - \cdots - 2P_{k+1})$ such that $\nu_k(Y) > 0$ and that $\mathcal{Y} \cap |V|(-P_1 - \cdots - P_{k+1})$ is an integral variety of dimension $r - \beta - k - 1$. Then for a general $H \in Y$ we have*

$$k + 1 \leq r - \dim(\langle \Sigma_H \rangle) \leq (k+1)(1 + \nu_k(Y)) - \delta_k + \beta\,, \tag{1}$$

and therefore

$$(k+1)\nu_k(Y) \geq \delta_k - \beta\,. \tag{2}$$

If $\mathcal{Y} \cap |V|(-P_1 - \cdots - P_{k+1})$ *is not linear, then we have strict inequality in the first inequality of (1) and hence strict inequality in (2).*

Proof. Just copy the proof of [1], Theorem 1.1, using $\mathcal{Y}$ instead of $|V|$ (called $\mathcal{H}$ there): the only difference is that $\mathcal{Y}$ has dimension $r - \beta$ instead of dimension r. In the quoted proof a key tool was [1], Theorem 2.2, which we quoted as Lemma 1 in the form we apply here. The condition $\mathcal{Y} \cap |V|(-P_1 - \cdots - P_{k+1})$ is again used and the proof gives the last assertion, too. □

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