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A NOTE ON MODIFIED MAXIMAL FUNCTIONS ASSOCIATED WITH GENERAL MEASURES

Yasuo Komori

School of High Technology for Human Welfare
Tokai University
Nishino 317, Numazu-City, Shizuoka 410-0395, JAPAN
e-mail: komori@wing.ncc.u-tokai.ac.jp

Abstract: Grafakos and Kinnunen showed the sharp weak type $(1,1)$ estimates for the centered maximal functions associated with general measures. In this paper we consider the similar results for the modified uncentered maximal functions. To prove our theorem we use new covering lemma due to Sawano.

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1. Introduction

Let μ be a non-negative Radon measure on R^n and f be a μ-measurable non-negative locally integrable function. It is well-known that the centered maximal function of $Mf = M_\mu f$ is weak type $(1,1)$ and following theorem.

Theorem A.

$$\mu(\{x \in R^n; Mf(x) > \lambda\}) \leq \frac{C_n}{\lambda} \int_{\{Mf(x)>\lambda\}} f(x) d\mu\,, \tag{1}$$

where C_n is the Besicovitch constant (for the definition, see lemmas in Section 2).

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Grafakos and Kinnunen [1] obtained the following sharp inequality.

Theorem B.

$$\begin{aligned}\mu(\{x \in R^n; Mf(x) > \lambda\}) + (C_n - 1)\mu(\{x \in R^n; f(x) > \lambda\}) \\ \leq \frac{1}{\lambda}\int_{\{Mf(x)>\lambda\}} f(x)d\mu + (C_n - 1)\frac{1}{\lambda}\int_{\{f(x)>\lambda\}} f(x)d\mu. \qquad (2)\end{aligned}$$

We consider the uncentered maximal function $\widetilde{M}f = \widetilde{M}_\mu f$ (for the precise definition, see Section 2). As Grafakos and Kinnunen [1] pointed out $\widetilde{M}f$ is not weak type $(1,1)$ in general (see also Journé [2], p. 10). Therefore in this paper we consider the modified uncentered maximal function

$$\widetilde{M}_k f(x) = \sup_{x \in B} \frac{1}{\mu(kB)} \int_B f(y)d\mu,$$

where kB is k times ball of B, and we shall show the following result (see Section 3):

$$\begin{aligned}\mu(\{x \in R^n; \widetilde{M}_k f(x) > \lambda\}) + (S_{n,k} - 1)\mu(\{x \in R^n; f(x) > \lambda\}) \\ \frac{1}{\lambda}\int_{\{\widetilde{M}_k f(x)>\lambda\}} f(x)d\mu + (S_{n,k} - 1)\frac{1}{\lambda}\int_{\{f(x)>\lambda\}} f(x)d\mu, \qquad (3)\end{aligned}$$

where $k > 1$ and $S_{n,k}$ is the Sawano constant. To prove our theorem we use Sawano Covering Lemma instead of Besicovitch Covering Lemma (see lemmas in Section 2).

2. Definitions and Results

We fix a non-negative Radon measure μ on R^n. The following notation is used: We write a ball of radius r centered at x_0 by $B(x_0, r) = \{x \in R^n; |x - x_0| < r\}$ and write $kB = B(x_0, kr)$, where $k \geq 1$.

Definition 1. Let $k \geq 1$ and let f be a μ-measurable non-negative locally integrable function. We define modified uncentered maximal function as follows:

$$\widetilde{M}_k f(x) = \sup_{x \in B} \frac{1}{\mu(kB)} \int_B f(y)d\mu \quad \text{if} \quad x \in \operatorname{supp}(\mu),$$

where the supremum is taken over all closed balls containing x.

Besicovitch Covering Lemma is the following (see for example [3]):

Lemma. (Besicovitch) *There exists an integer C_n, depending only on n which satisfies the following:*

Let E be a bounded subset of R^n and let $\mathcal{B}$ be a family of closed balls such that each point of E is the <u>*center*</u> *of some ball of $\mathcal{B}$. Then we can take disjoint subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{C_n}$ such that*

$$E \subset \bigcup_{i=1}^{C_n} \bigcup_{B \in \mathcal{B}_i} B.$$

C_n is called the Besicovitch constant.

In our theory, the following covering lemma due to Sawano [4] is essential.

Lemma. (Sawano) *For any $k > 1$, there exists an integer $S_{n,k}$, depending only on n and k which satisfies the following:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$ be a family of balls. Suppose $\sup_{\lambda \in L} r_\lambda < \infty$. Then we can take disjoint subfamilies

$$\{B(x_\rho, r_\rho)\}_{\rho \in L_1}, \{B(x_\rho, r_\rho)\}_{\rho \in L_2}, \dots, \{B(x_\rho, r_\rho)\}_{\rho \in L_{S_{n,k}}},$$

such that

$$\bigcup_{\lambda \in L} B(x_\lambda, r_\lambda) \subset \bigcup_{i=1}^{S_{n,k}} \bigcup_{\rho \in L_i} B(x_\rho, k r_\rho),$$

where L_i's are countable subsets of L and balls in $\{B(x_\rho, r_\rho)\}_{\rho \in L_i}$ are disjoint for each i. And we call $S_{n,k}$ the Sawano constant.

Our result is the following theorem.

Theorem.

$$\begin{aligned}\mu(\{x \in R^n; \widetilde{M}_k f(x) > \lambda\}) &+ (S_{n,k} - 1)\mu(\{x \in R^n; f(x) > \lambda\}) \\ &\le \frac{1}{\lambda} \int_{\{\widetilde{M}_k f(x) > \lambda\}} f(x) d\mu + (S_{n,k} - 1) \frac{1}{\lambda} \int_{\{f(x) > \lambda\}} f(x) d\mu,\end{aligned}$$

where $k > 1$ and $S_{n,k}$ is the Sawano constant.

Remarks. If $n = 1$, Theorem is still true for $k = 1$, but when $n \ge 2$ Theorem is not true for $k = 1$ in general (see [1] and [2], p. 10). Note that we do not assume the following doubling condition for μ:

$$\mu(B(x, 2r)) \le C \mu(B(x, r)).$$

3. Proof of Theorem

In the following, for the simplicity of notation, we write $S_{n,k} = S$. To prove our theorem we need the following lemmas.

Lemma 1.

$$\mu(\{x \in R^n; \widetilde{M}_k f(x) > \lambda\}) \leq \frac{S}{\lambda} \int_{\{\widetilde{M}_k f(x) > \lambda\}} f(x) d\mu, \tag{4}$$

where $k > 1$.

The proof of this lemma is same as that of Theorem A. Use the Sawano Covering Lemma instead of Besicovitch Covering Lemma. We shall use the same argument in the proof of our theorem.

The next two lemmas are trivial and stated in [1].

Lemma 2. *Let* $f \geq 0$ *and let* E *be a* μ*-measurable set. Then*

$$\frac{1}{\lambda} \int_E f d\mu + \mu(\{f > \lambda\}) \leq \frac{1}{\lambda} \int_{\{f > \lambda\}} f d\mu + \mu(E). \tag{5}$$

Lemma 3. *For any measure* ν *and sets* E, P_i *and* Q_j,

$$\sum_{i=1}^{S} \nu(P_i \cap E) = \nu(\cup_{i=1}^{S} P_i \cap E) + \sum_{j=2}^{S} \nu(Q_j \cap E), \tag{6}$$

where

$$Q_j = \cup_{\{m_1, \ldots, m_j\} \subset \{1, \ldots, S\}} (P_{m_1} \cap \cdots \cap P_{m_j}), \quad j = 2, \ldots, S.$$

Proof of Theorem. We prove the theorem by the same argument as in [1]. For $R > 0$, we define

$$\widetilde{M}_k^R f(x) \sup_{x \in B, \mathrm{rad}(B) < R} \frac{1}{\mu(kB)} \int_B f(y) d\mu,$$

where $\mathrm{rad}(B)$ is the radius of ball B.

For $\lambda > 0$, let $E_\lambda^R = \{x \in R^n; \widetilde{M}_k^R f(x) > \lambda\}$. By Lemma 1 we may assume $\mu(E_\lambda^R) < \infty$. For every $x \in E_\lambda^R$, there exists a ball $B(\widetilde{x}, r_x)$ such that

$$\frac{1}{\mu(B(\widetilde{x}, k r_x))} \int_{B(\widetilde{x}, r_x)} f d\mu > \lambda.$$

Since $\mathrm{rad}(B(\widetilde{x}, r_x)) < R$, we can apply Sawano Covering Lemma, that is, from $\{B(\widetilde{x}, r_x)\}_{x\in E_\lambda^R}$ we can choose disjoint subfamilies $\mathcal{B}_1, \ldots, \mathcal{B}_S$ such that

$$E_\lambda^R \subset \bigcup_{i=1}^{S} \bigcup_{B\in\mathcal{B}_i} kB.$$

Note that for each ball $B \in \mathcal{B}_i$,

$$\frac{1}{\mu(kB)} \int_B f d\mu > \lambda. \tag{7}$$

We have

$$\int_B f d\mu = \int_{B\setminus E_\lambda^R} f d\mu + \int_{B\cap E_\lambda^R} f d\mu \le \lambda\mu(B \setminus E_\lambda^R) + \int_{B\cap E_\lambda^R} f d\mu,$$

and by (7) we have

$$\lambda\mu(kB) \le \lambda\mu(B \setminus E_\lambda^R) + \int_{B\cap E_\lambda^R} f d\mu.$$

So we obtain

$$\lambda\mu(kB \cap E_\lambda^R) \le \int_{B\cap E_\lambda^R} f d\mu.$$

Because the balls in $\mathcal{B}_i$ are disjoint, we have

$$\lambda\mu(\cup_{B\in\mathcal{B}_i} kB \cap E_\lambda^R) \le \int_{\cup_{B\in\mathcal{B}_i} B\cap E_\lambda^R} f d\mu.$$

Let $F_i = \cup_{B\in\mathcal{B}_i} B$ and $\widetilde{F_i} = \cup_{B\in\mathcal{B}_i} kB$. Then we can write

$$\mu(\widetilde{F_i} \cap E_\lambda^R) \le \frac{1}{\lambda} \int_{F_i\cap E_\lambda^R} f d\mu. \tag{8}$$

Let

$$\begin{aligned} &G_j = \cup_{\{m_1,\ldots,m_j\}\subset\{1,\ldots,S\}}(F_{m_1} \cap \cdots \cap F_{m_j}),\\ \text{and}\quad &\widetilde{G}_j = \cup_{\{m_1,\ldots,m_j\}\subset\{1,\ldots,S\}}(\widetilde{F}_{m_1} \cap \cdots \cap \widetilde{F}_{m_j}), \end{aligned}$$

where $j = 2, \ldots, S$.

Applying Lemma 3 to $P_i = \widetilde{F}_i, Q_j = \widetilde{G}_j, E = E_\lambda^R$ and $\nu = \mu$, we have

$$\begin{aligned}\mu(\cup_{i=1}^S \widetilde{F}_i \cap E_\lambda^R) + \sum_{j=2}^S \mu(\widetilde{G}_j \cap E_\lambda^R) &= \sum_{i=1}^S \mu(\widetilde{F}_i \cap E_\lambda^R)\\ &\le \frac{1}{\lambda}\sum_{i=1}^S \int_{F_i \cap E_\lambda^R} f d\mu \quad \text{by} \quad (8).\end{aligned}$$

Applying Lemma 3 to $P_i = F_i, Q_j = G_j, E = E_\lambda^R$ and $\nu = f d\mu$, we have

$$\sum_{i=1}^S \int_{F_i \cap E_\lambda^R} f d\mu = \int_{\cup_{i=1}^S F_i \cap E_\lambda^R} f d\mu + \sum_{j=2}^S \int_{G_j \cap E_\lambda^R} f d\mu.$$

So we obtain

$$\begin{aligned}&\mu(\cup_{i=1}^S \widetilde{F}_i \cap E_\lambda^R) + \sum_{j=2}^S \mu(\widetilde{G}_j \cap E_\lambda^R)\\ &\qquad \le \frac{1}{\lambda}\Big(\int_{\cup_{i=1}^S F_i \cap E_\lambda^R} f d\mu + \sum_{j=2}^S \int_{G_j \cap E_\lambda^R} f d\mu\Big). \end{aligned} \tag{9}$$

By Lemma 2, we have

$$\begin{aligned}&\sum_{j=2}^S \frac{1}{\lambda}\int_{G_j \cap E_\lambda^R} f d\mu + (S-1)\mu(\{f > \lambda\})\\ &\qquad \le \frac{S-1}{\lambda}\int_{\{f>\lambda\}} f d\mu + \sum_{j=2}^S \mu(G_j \cap E_\lambda^R). \end{aligned} \tag{10}$$

Because $\mu(E_\lambda^R) = \mu(\cup_{i=1}^S \widetilde{F}_i \cap E_\lambda^R)$, by (9) and (10) we have

$$\begin{aligned}\mu(E_\lambda^R) + \sum_{j=2}^S \mu(\widetilde{G}_j \cap E_\lambda^R) \le &\frac{1}{\lambda}\int_{\cup_{i=1}^S F_i \cap E_\lambda^R} f d\mu + \frac{S-1}{\lambda}\int_{\{f>\lambda\}} f d\mu\\ &+ \sum_{j=2}^S \mu(G_j \cap E_\lambda^R) - (S-1)\mu(\{f > \lambda\}).\end{aligned}$$

Since $G_j \le \widetilde{G}_j$, we obtain

$$\mu(E_\lambda^R) + (S-1)\mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{E_\lambda^R} f d\mu + (S-1)\frac{1}{\lambda}\int_{\{f>\lambda\}} f d\mu,$$

and by letting $R \to \infty$, we prove the desired result. □

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