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ON SOME NEW MAJORIZED RESULTS ON HILBERT INTEGRAL INEQUALITY

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Abstract: The Hilbert-type inequalities play an important role in analysis and its applications. In this paper we get some new improvements and generalizations of the Hilbert-type inequalities.

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1. Introduction

Here, and in the rest of our paper we suppose that all integrals converge.

Xie Hongzheng, Lu Zhongxue and Xin Yumei have proved the following inequalities (see [1]):

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Theorem A. *Let $0 < a < b < \infty$. If f, g are real functions, then:*

$$(1)\quad \int_a^b\int_a^b \frac{f(x)g(y)}{x+y}\,dxdy < \left(\pi - 4\arctan\sqrt{\frac{a}{b}}\right)\left[\int_a^b f^2(t)\,dt\int_a^b g^2(t)\,dt\right]^{\frac{1}{2}}, \quad (1.1)$$

$$(2)\quad \int_0^b\int_0^b \frac{f(x)g(y)}{x+y}\,dxdy < \left[\int_0^b\left(\pi - 2\arctan\sqrt{\frac{t}{b}}\right)f^2(t)\,dt\int_0^b\left(\pi - 2\arctan\sqrt{\frac{t}{b}}\right)g^2(t)\,dt\right]^{\frac{1}{2}}, \quad (1.2)$$

$$(3)\quad \int_a^\infty\int_a^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \left[\int_a^\infty\left(\pi - 2\arctan\sqrt{\frac{a}{t}}\right)f^2(t)\,dt\int_a^\infty\left(\pi - 2\arctan\sqrt{\frac{a}{t}}\right)g^2(t)\,dt\right]^{\frac{1}{2}}. \quad (1.3)$$

Bicheng Yang and T. M. Rassias have proved the following inequality (see [2]):

Theorem B. *Let $0 < a < b < \infty$, $\lambda > 0$. If f is a real function, then:*

$$\int_a^b y^{\lambda-1}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^2 dy < \left\{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right]\right\}^2\int_a^b t^{1-\lambda}f^2(t)\,dt, \quad (1.4)$$

where B is beta-function.

In our paper we shall give some generalizations and improvements of these two theorems, as well as some related results.

2. Main Results

Theorem 1. *Let $0 \leq a < b \leq \infty$, $\lambda > 0$, $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If f, g are real functions, then the following inequality holds:*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < \left[\int_a^b f^p(x)x^{p-1-\frac{p\lambda}{2}} T_\lambda\left(\frac{a}{x},\frac{b}{x}\right)dx\right]^{\frac{1}{p}} \times \left[\int_a^b g^q(x)x^{q-1-\frac{q\lambda}{2}} T_\lambda\left(\frac{a}{x},\frac{b}{x}\right)dx\right]^{\frac{1}{q}}, \tag{2.1}$$

where

$$T_\lambda(x,y) = \int_x^y \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt. \tag{2.2}$$

Theorem 2. *Let $0 < a < b < \infty$, $\lambda > 0$, $p > 1$. If f is a real function then the following inequality holds:*

$$\int_a^b y^{\frac{\lambda p}{2}-1}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^p dy < \left\{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right]\right\}^p \int_a^b f^p(x)x^{p-1-\frac{p\lambda}{2}}\,dx, \tag{2.3}$$

where B is beta-function.

Proof of Theorem 1. We start with the equality:

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy = \int_a^b \int_a^b \frac{f(x)\frac{x^{\frac{2-\lambda}{2q}}}{y^{\frac{2-\lambda}{2p}}} \cdot g(y)\frac{y^{\frac{2-\lambda}{2p}}}{x^{\frac{2-\lambda}{2q}}}}{(x+y)^{\frac{\lambda}{p}}(x+y)^{\frac{\lambda}{q}}}\,dxdy. \tag{2.4}$$

Using (2.4) and Hölder inequality, we have:

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy \\
&\le \left[\int_a^b\int_a^b \frac{f^p(x)x^{\frac{p(2-\lambda)}{2q}}}{(x+y)^\lambda y^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{p}} \left[\int_a^b\int_a^b \frac{g^q(y)y^{\frac{q(2-\lambda)}{2p}}}{(x+y)^\lambda x^{\frac{2-\lambda}{2}}}\,dxdy\right]^{\frac{1}{q}} \\
&= \left[\int_a^b f^p(x)x^{\frac{p(2-\lambda)}{2q}}\left(\int_a^b \frac{y^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dy\right)dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_a^b g^q(y)y^{\frac{q(2-\lambda)}{2p}}\left(\int_a^b \frac{x^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dx\right)dy\right]^{\frac{1}{q}} \\
&= \left[\int_a^b f^p(x)x^{\frac{p(2-\lambda)}{2q}} I_x\,dx\right]^{\frac{1}{p}} \left[\int_a^b g^q(y)y^{\frac{q(2-\lambda)}{2p}} I_y\,dy\right]^{\frac{1}{q}},
\end{aligned} \tag{2.5}$$

where we denote:

$$I_x = \int_a^b \frac{y^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dy, \qquad I_y = \int_a^b \frac{x^{\frac{\lambda-2}{2}}}{(x+y)^\lambda}\,dx.$$

Using the change of variables $y = xt$, $dy = x\,dt$, we have for I_x:

$$I_x = \int_{\frac{a}{x}}^{\frac{b}{x}} \frac{x^{\frac{\lambda-2}{2}}t^{\frac{\lambda-2}{2}}}{(x+xt)^\lambda}x\,dt = x^{-\frac{\lambda}{2}}\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt = x^{-\frac{\lambda}{2}}T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big), \tag{2.6}$$

and similarly:

$$I_y = y^{-\frac{\lambda}{2}}T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big), \tag{2.7}$$

where we denote:

$$T_\lambda(x,y) = \int_x^y \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt, \qquad 0 \le x < y \le \infty. \tag{2.8}$$

Substitying (2.6) and (2.7) in (2.5) we finally obtain:

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy \le$$

$$\left[\int_a^b f^p(x) x^{\frac{p(2-\lambda)}{2q}-\frac{\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}} \left[\int_a^b g^q(y) y^{\frac{q(2-\lambda)}{2p}-\frac{\lambda}{2}} T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\,dy\right]^{\frac{1}{q}}$$

$$= \left[\int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}} \left[\int_a^b g^q(y) y^{q-1-\frac{q\lambda}{2}} T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\,dy\right]^{\frac{1}{q}}$$

and thus Theorem 1 is proved. □

Remark 1. For $p=2$, $a=0$, $b=\infty$, $T_\lambda(0,\infty)=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ (2.1) changes to well-known Hilbert inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\int_0^\infty t^{1-\lambda} f^2(t)\,dt \int_0^\infty t^{1-\lambda} g^2(t)\,dt\right]^{\frac{1}{2}}.$$

Let us examine some properties of the function $T_\lambda(x,y)$. Using the substitution $t=s^{-1}$ from (2.8) we obtain:

$$T_\lambda(x,y) = \int_x^y \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt = -\int_{\frac{1}{x}}^{\frac{1}{y}} \frac{s^{\frac{2-\lambda}{2}}}{\left(1+\frac{1}{s}\right)^\lambda}\cdot\frac{ds}{s^2} = \int_{\frac{1}{y}}^{\frac{1}{x}} \frac{s^{\frac{\lambda-2}{2}}}{(1+s)^\lambda}\,ds$$

$$= T_\lambda\Big(\frac{1}{y},\frac{1}{x}\Big). \qquad (2.9)$$

Thus $T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big) = T_\lambda\Big(\frac{x}{b},\frac{x}{a}\Big)$, and in particular $T_\lambda\Big(0,\frac{b}{x}\Big) = T_\lambda\Big(\frac{x}{b},\infty\Big)$.

In the special case of $\lambda=1$ we can express $T_1(x,y)$ as follows:

a) if $0=a<b<\infty$:

$$T_1\Big(0,\frac{b}{x}\Big) = T_1\Big(\frac{x}{b},\infty\Big) = \int_{\frac{x}{b}}^\infty \frac{dt}{\sqrt{t}(1+t)}$$

$$= \int_{\sqrt{\frac{x}{b}}}^\infty \frac{2\,du}{1+u^2} = \pi - 2\arctan\sqrt{\frac{x}{b}}, \qquad x<b; \qquad (2.10)$$

b) if $0 < a < b = \infty$:

$$T_1\left(\frac{a}{x}, \infty\right) = \int_{\frac{a}{x}}^{\infty} \frac{dt}{\sqrt{t}(1+t)} = \pi - 2\arctan\sqrt{\frac{a}{x}}, \qquad x > a; \tag{2.11}$$

c) if $0 < a < b < \infty$:

$$\begin{aligned} T_1\left(\frac{a}{x}, \frac{b}{x}\right) &= \int_{\frac{a}{x}}^{\frac{b}{x}} \frac{dt}{\sqrt{t}(1+t)} = 2\left(\arctan\sqrt{\frac{b}{x}} - \arctan\sqrt{\frac{a}{x}}\right) \\ &= 2\left(\operatorname{arccot}\sqrt{\frac{x}{b}} - \arctan\sqrt{\frac{a}{x}}\right) \\ &= \pi - 2\arctan\sqrt{\frac{a}{x}} - 2\arctan\sqrt{\frac{x}{b}}, \qquad a < x < b. \end{aligned} \tag{2.12}$$

In the last case, from $\frac{d}{dx}T_1\left(\frac{a}{x}, \frac{b}{x}\right) = \frac{1}{\sqrt{x}}\left[\frac{\sqrt{a}}{a+x} - \frac{\sqrt{b}}{b+x}\right]$ we conclude that $T_1\left(\frac{a}{x}, \frac{b}{x}\right)$ is strictly increasing on $(a, \sqrt{ab})$ and strictly decreasing on $(\sqrt{ab}, b)$. Hence:

$$\begin{aligned} T_1\left(\frac{a}{x}, \frac{b}{x}\right) \le T_1\left(\frac{a}{\sqrt{ab}}, \frac{b}{\sqrt{ab}}\right) = T_1\left(\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}}\right) \\ = \pi - 4\arctan\sqrt[4]{\frac{a}{b}}, \qquad a < x < b. \end{aligned} \tag{2.13}$$

In consequence of last four relations and Theorem 1 we have the following corollary.

Corollary 1. *Let* $0 < a < b < \infty$, $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. *If* f, g *are real* *functions, then:*

(1)
$$\int_a^b\int_a^b \frac{f(x)g(y)}{x+y}\,dxdy < \left[\int_a^b f^p(t)t^{\frac{p}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{a}{t}} - 2\arctan\sqrt{\frac{t}{b}}\right)dt\right]^{\frac{1}{p}} \times \left[\int_a^b g^q(t)t^{\frac{q}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{a}{t}} - 2\arctan\sqrt{\frac{t}{b}}\right)dt\right]^{\frac{1}{q}} < \left(\pi - 4\arctan\sqrt[4]{\frac{a}{b}}\right)\left(\int_a^b f^p(t)t^{\frac{p}{2}-1}\,dt\right)^{\frac{1}{p}}\left(\int_a^b g^q(t)t^{\frac{q}{2}-1}\,dt\right)^{\frac{1}{q}}; \quad (2.14)$$

(2)
$$\int_0^b\int_0^b \frac{f(x)g(y)}{x+y}\,dxdy < \left[\int_0^b f^p(t)t^{\frac{p}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{t}{b}}\right)dt\right]^{\frac{1}{p}} \times \left[\int_0^b g^q(t)t^{\frac{q}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{t}{b}}\right)dt\right]^{\frac{1}{q}}; \quad (2.15)$$

(3)
$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \left[\int_a^\infty f^p(t)t^{\frac{p}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{a}{t}}\right)dt\right]^{\frac{1}{p}} \times \left[\int_a^\infty g^q(t)t^{\frac{q}{2}-1}\left(\pi - 2\arctan\sqrt{\frac{a}{t}}\right)dt\right]^{\frac{1}{q}}. \quad (2.16)$$

Remark 2. For $p = q = 2$ inequalities (2.15) and (2.16) change to (1.2) and (1.3) respectively. Hence (2.15) and (2.16) are generalizations of (1.2) and (1.3). Inequality (2.14) is a generalization and improvement of (1.1), because the constant factor $K_1 = \pi - 4\arctan\sqrt[4]{\frac{a}{b}}$ in (2.14) is better than the constant factor $K_2 = \pi - 4\arctan\sqrt{\frac{a}{b}}$ in (1.1), $K_1 < K_2$.

In the special case of $\lambda = 2$ we have for $T_2(x, y)$:

$$T_2(x,y) = \int_x^y \frac{dt}{(1+t)^2} = \frac{1}{1+x} - \frac{1}{1+y}. \tag{2.17}$$

Using (2.1) we have the following corollary.

Corollary 2. *Let $0 < a < b < \infty$, $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If f, g are real functions, then:*

(1) $$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^2}\,dxdy < (b-a)\left[\int_a^b \frac{f^p(t)}{(t+a)(t+b)}\,dt\right]^{\frac{1}{p}} \left[\int_a^b \frac{g^q(t)}{(t+a)(t+b)}\,dt\right]^{\frac{1}{q}}; \tag{2.18}$$

(2) $$\int_0^b \int_0^b \frac{f(x)g(y)}{(x+y)^2}\,dxdy < b\left[\int_0^b \frac{f^p(t)}{t(t+b)}\,dt\right]^{\frac{1}{p}} \left[\int_0^b \frac{g^q(t)}{t(t+b)}\,dt\right]^{\frac{1}{q}}; \tag{2.19}$$

(3) $$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^2}\,dxdy < \left[\int_a^\infty \frac{f^p(t)}{t+a}\,dt\right]^{\frac{1}{p}} \left[\int_a^\infty \frac{g^q(t)}{t+a}\,dt\right]^{\frac{1}{q}}. \tag{2.20}$$

Proof of Theorem 2. Let $0 < a < b < \infty$. First we investigate the extremal value of $T_\lambda\left(\frac{a}{x}, \frac{b}{x}\right)$ when $x \in [a, b]$. Denote

$$T_\lambda\left(\frac{a}{x}, \frac{b}{x}\right) = \Phi(x) = \int_{\frac{a}{x}}^{\frac{b}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\,dt.$$

Then:

$$\Phi : [a, b] \to \mathbf{R}^+,$$

$$\Phi'(x) = x^{\frac{\lambda-2}{2}}\left[\frac{a^{\frac{\lambda}{2}}}{(x+a)^\lambda} - \frac{b^{\frac{\lambda}{2}}}{(x+b)^\lambda}\right].$$

We conclude: $\Phi'(\sqrt{ab}) = 0$, Φ is strictly increasing on $(a, \sqrt{ab})$ and is strictly decreasing on $(\sqrt{ab}, b)$. Hence:

$$\begin{aligned} T_\lambda\Big(\frac{a}{x}, \frac{b}{x}\Big) &\le T_\lambda\Big(\frac{a}{\sqrt{ab}}, \frac{b}{\sqrt{ab}}\Big) = T_\lambda\Big(\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}}\Big) \\ &= T_\lambda(0,\infty) - T_\lambda\Big(0, \sqrt{\frac{a}{b}}\Big) - T_\lambda\Big(\sqrt{\frac{b}{a}}, \infty\Big) \\ &= T_\lambda(0,\infty) - T_\lambda\Big(\sqrt{\frac{b}{a}}, \infty\Big) - T_\lambda\Big(\sqrt{\frac{b}{a}}, \infty\Big) \\ &= B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big) - 2T_\lambda\Big(\sqrt{\frac{b}{a}}, \infty\Big). \end{aligned} \tag{2.21}$$

In [3] Gavrea proved the inequality:

$$\int_\alpha^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda}\, dt > \frac{1}{2}\alpha^{-\frac{\lambda}{2}} B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big), \qquad \alpha > 1. \tag{2.22}$$

Using (2.22) we obtain from (2.21):

$$\begin{aligned} T_\lambda\Big(\frac{a}{x}, \frac{b}{x}\Big) &< B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big) - 2\cdot\frac{1}{2}\Big(\sqrt{\frac{b}{a}}\Big)^{-\frac{\lambda}{2}} B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big) \\ &= B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\left[1 - \Big(\frac{a}{b}\Big)^{\frac{\lambda}{4}}\right]. \end{aligned} \tag{2.23}$$

We continue our proof with the equality:

$$\begin{aligned} J &= \int_a^b y^{\frac{\lambda p}{2}-1}\left[T_\lambda\Big(\frac{a}{y}, \frac{b}{y}\Big)\right]^{1-p}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^p dy \\ &= \int_a^b y^{\frac{\lambda p}{2}-1}\left[T_\lambda\Big(\frac{a}{y}, \frac{b}{y}\Big)\right]^{1-p}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p-1}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right) dy \\ &= \int_a^b g(y)\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right) dy = \int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}\, dxdy, \end{aligned} \tag{2.24}$$

where we denote

$$g(y) = y^{\frac{\lambda p}{2}-1}\left[T_\lambda\Big(\frac{a}{y}, \frac{b}{y}\Big)\right]^{1-p}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p-1}, \; a \le y \le b. \tag{2.25}$$

By applying (2.1) and (2.25) we have:

$$J \le \left[\int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}} \left[\int_a^b g^q(y) y^{q-1-\frac{q\lambda}{2}} T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\,dy\right]^{\frac{1}{q}}$$

$$= \left[\int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}}$$

$$\times \left[\int_a^b y^{q(\frac{\lambda p}{2}-1)} \left[T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\right]^{q(1-p)} \left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{q(p-1)}\right.$$

$$\left. \times\, y^{q-1-\frac{q\lambda}{2}} T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\,dy\right]^{\frac{1}{q}}$$

$$= \left[\int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}}$$

$$\times \left[\int_a^b y^{\frac{\lambda p}{2}-1} \left[T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\right]^{1-p} \left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p} dy\right]^{\frac{1}{q}}.$$

Thus we obtain:

$$J \le \left[\int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx\right]^{\frac{1}{p}} \cdot J^{\frac{1}{q}},$$

wherefrom it follows:

$$J = \int_a^b y^{\frac{\lambda p}{2}-1} \left[T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\right]^{1-p} \left(\int_a^b \frac{f(x)\,dx}{(x+y)^\lambda}\right)^{p} dy$$

$$\le \int_a^b f^p(x) x^{p-1-\frac{p\lambda}{2}} T_\lambda\Big(\frac{a}{x},\frac{b}{x}\Big)\,dx. \quad (2.26)$$

As $1-p<0$ from (2.23) we have:

$$\left[T_\lambda\Big(\frac{a}{y},\frac{b}{y}\Big)\right]^{1-p} > \left\{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right]\right\}^{1-p}. \quad (2.27)$$

Using (2.27) and (2.23) in (2.26) we finally obtain:

$$\int_a^b y^{\frac{\lambda p}{2}-1}\left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{1-p}\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right]^{1-p}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^{\lambda}}\right)^{p} dy$$
$$< \int_a^b y^{\frac{\lambda p}{2}-1}\left[T_\lambda\left(\frac{a}{y},\frac{b}{y}\right)\right]^{1-p}\left(\int_a^b \frac{f(x)\,dx}{(x+y)^{\lambda}}\right)^{p} dy$$
$$< \int_a^b f^p(x)x^{p-1-\frac{p\lambda}{2}}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{\lambda}{4}}\right] dx$$

wherefrom it follows (2.3). □

Remark 3. For $p = 2$ inequality (2.3) changes to (1.4). Hence (2.3) is a generalization of (1.4).

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