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ON THE DESIGN OF A HIGH PRECISION ROBUST CONTROL SYSTEM

Adam Lozowicki[1], Antonio Tiano[2 §]

[1]Technical University of Szczecin
Ul. Klemensiwicza 6/1, 70-028 Szczecin, POLAND
e-mail: lozowicki@ps.pl

[2]Department of Informatics and Systems
University of Pavia
Via Ferrata 1, 27100 Pavia, ITALY
e-mail: antonio@control1.unipv.it

Abstract: The paper concerns the design of a state feedback control system with sufficiently large gain (called Tytus feedback). Such system can perform with a high degree of accuracy. Conditions are given for a class of stable or unstable plants are given, according to which it is possible to robust control with ε-*accuracy* plants described by a high order differential equation. Methods for checking whether the given above conditions are satisfied, are discussed in details for causal stationary linear systems. These considerations are next extended to non-linear plants.

AMS Subject Classification: 93B52
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1. Introduction

In many practical applications, questions about the possibility to obtain a high precision performance for a feedback control system are frequently posed. These problems can be solved by using a high gain (gain tending to infinity) in the

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§Correspondence author

feedback loop (see Lozowicki [5], [6]). The control system presented in this paper enables to control with ε-*accuracy* both stable and unstable plants P described by high degree differential equations. This result is possible by using a state feedback (called Tytus feedback). Considerations presented in the paper apply to the robust control of stable as well as unstable plants with uncertainty.

Let us consider a set of signals X regarded as a Banach space. For example X can be the set of signals with bounded energy L^2, or the set of signals with bounded mean power M (Marcinkiewicz space). Let the plant given by operation $y(t) = P(u(t))$ transform any set of signals $W \subset X$ into X. We assume additionally that operation P is a composition

$$P = P_n(...(P_2(P_1))) \,, \tag{1}$$

of the operations $x_1(t) = P_1\,(u(t)) = P_1(x_o(t)),\ \ x_i(t) = P_i(x_{i-1}(t)), ...\ , i = 2,$ ... n. We assume that P_i transform a sets of signals $W_i \subset X$ into X. Let the operation P be given by the set of equations

$$\begin{cases} x_1\,(t) = P_1\,(x_o\,(t))\ , \\ x_2\,(t) = P_2\,(x_1\,(t))\ , \\ \quad . \quad . \quad . \\ x_n\,(t) = P_n\,(x_{n-1}\,(t))\ , \\ y\,(t) = x_n\,(t)\ . \end{cases} \tag{2}$$

As a particular case the variables $\chi(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ can be considered as components of the state space vector generated by operation P. We will denote them by $\chi(t) = P(u(t))$. The plant $P + \Delta P$ with uncertainty will be defined in the following way:

$$\begin{cases} x_1\,(t) = P_1\,(x_o\,(t)) + \Delta P_1\,, \\ x_2\,(t) = P_2\,(x_1\,(t)) + \Delta P_2\,, \\ \quad . \quad . \quad . \\ x_n\,(t) = P_n\,(x_{n-1}\,(t)) + \Delta P_n\,. \end{cases} \tag{3}$$

We take under consideration the state feedback control system described by equations

$$\begin{cases} u\,(t) = ke\,(t)\ , \\ \delta\,(t) = \chi_o\,(t) - P\,(u\,(t))\ , \\ \chi_o\,(t) = P\,(u_o\,(t))\ , \end{cases} \tag{4}$$

where: $y_o(t) = x_{on}(t)$ and $\chi_o(t) = [x_{o1}(t), x_{o2}(t), ... \quad x_{on}(t)]^T$ is a reference signal vector; $e_i(t) = x_{oi}(t) - x_i(t)$, $\delta(t) = [e_1(t), e_2(t), ..., e_n(t)]^T = \chi_o(t) - \chi(t)$,

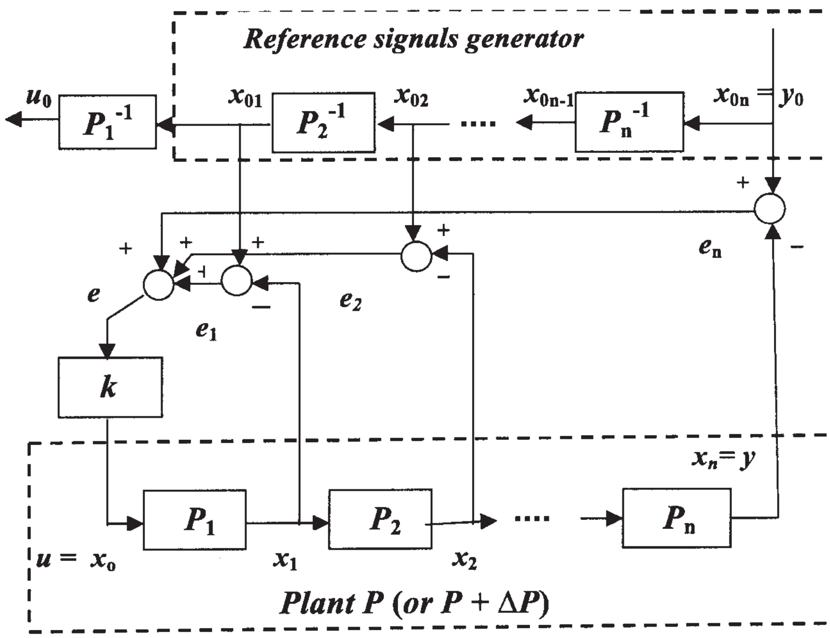

Figure 1: Feedback system controlling plant $P = P_n(...(P_2(P_1)))$ (or plant $P + \Delta P$) with ε -accuracy

$i = 1, 2, \ldots\ \ n,\ e(t) = \sum_{i=1}^{n} e_i(t)$, are error signals; $u(t) = x_o(t)$ is the control signal. The scheme of such system (Tytus system) is depicted in Figure 1.

We will assume that the reference signal $y_o(t) = x_{on}(t)$ is not a fixed signal but it can be modelled as belonging to the class

$$X\{P, m\} = \{y_o : \quad y_o = P(u), \text{ for same } u \in X, \ ||u||_X \leq \ m < \infty \ \} \tag{5}$$

In the particular case when the operation P is one-to-one, then the "ideal" control system should be a system generating (for any signal y_o) the control signal $u = P^{-1}(y_o)$. In this case we denote by $X\{P, m\}$ the set of all signals $y_o(t) \in X$ such that the operation P and constant $0 < m < \infty$ satisfy the relations

$$P^{-1}(y_o) \in X \quad \text{and} \quad ||P^{-1}(y_o)||_X \leq \ m < \infty \,, \tag{6}$$

i.e.

$$X\{P, m\} = \{y_o \in X : P^{-1}(y_o) \in X \text{ and } ||P^{-1}(y_o)||_X \leq \ m < \infty \ \} \,. \tag{7}$$

If the constant m is not determined exactly then the notation $X\{P\}$ will be used.

Definition 1. The plant P is controlled with *ε-accuracy* to signal $y_o(t) \in X\{P,m\}$ by the system (4) if there exist constants k_1, k_2 (k_1, k_2 depending on ε) such that for every $k \in [k_1, k_2)$ the inequality

$$||e_n||_X \leq \varepsilon \tag{8}$$

is satisfied.

2. The Case When $P_i = P$ $(i = 1)$

Now, we take under consideration the system (4) in the particular case, when $P_i = P$ (i.e. when $i = n = 1$). If the operation P_i is one-to-one, then the "ideal" control system is a system generating the control signal $x_{i-1} = P_i^{-1}(x_{oi})$, for any signal $x_{oi} \in X\{P_i\}$. We assume that the operation P_i transforms the set of signals $X\{P_i\}$ into Banach space X. Let k be a real number ($k \in \Re$). The first two equations of (4) can be rewritten in the form

$$x_{oi} - P_i(x_{i-1}) = \frac{1}{k} x_{i-1}\,, \tag{9}$$

for any given point $x_{oi} \in X\{P_i\}$.

Let X and Y be normed spaces. Let H map an open subset U of the Cartesian product $X \times Y$ into Y (i.e. $X \times Y \ni (x,y) \to H(x,y) \in Y$). We define a relation

$$\wp = \{(x,y) \in X \times Y : H(x,y) = 0\}\ .$$

The mapping H generates the implicit mapping X into Y in any neighbourhood of the point (x_o, y_o) if there exist two real numbers $r_1, r_2 > 0$, such that the relation G given by formula

$$G = \wp \cap (K(x_o, r_1) \times K(y_o, r_2))$$

maps X into Y. In this case we have $H(x, G(x)) = 0$.

We denote by $L(Y,Y)$ the space of all linear and continuous mapping which map Y into Y.

Theorem 1. (compare Graves [1], Hille and Phillips [2]) *Let X, Y be two Banach spaces and let H map the open subset V of the Cartesian product $X \times Y$ into Y (i.e. $X \times Y \supset V \ni (x,y) \to H(x,y) \in Y$). We assume additionally that* *there exists a point* (x_o, y_o) *such that* $H(x_o, y_o) = 0$ *and let* H *be continuously differentiable in any neighbourhood* $U \subset V$ *of the point* (x_o, y_o)*. If there exists* $\left(H'_y(x, y_o)\right)^{-1} \in L(Y.Y)$ *then the implicit mapping exists*

$$X \supset K(x_o, r_1) \ni x \to G(x) \in K(x_o, r_2) \subset Y\,,$$

generated by H*. The mapping* $G(x)$ *is continuous on the ball* $K(x_o, r_1)$*, differentiable at the point* x_o *and* $G'_y = -\left(H'_y(x_o, y_o)\right)^{-1} H'_x(x_o, y_o)$.

Theorem 1 is formulated for the derivative in the Frechet sense.

We apply the Graves Theorem to relation (9). We rewrite relation (9) in the form

$$H(x, y) = H(x_{i-1}, x_i) = x_{oi} - P_i(x_{i-1}) - \frac{1}{k}x_{i-1}\,. \tag{10}$$

Let relation (10) satisfy the assumptions of the Graves Theorem. Mapping $H(x_{i-1}, x_i)$ generate the implicit operation

$$X \ni G_k(x_{oi}) = x_{i-1} \in X, \qquad k \in [k_1, \infty)\,. \tag{11}$$

We assume additionally that the mapping P_i is continuous, one-to-one and that there exists a continuous inverse operation $x_{i-1} = P_i^{-1}(x_i)$. Under the above conditions, for any point $x_{oi} \in X\{Pi\}$ there exist two points $x_{oi-1}, x_{*i-1} \in X$ such that

$$x_{oi} - P_i(x_{*i-1}) - \frac{1}{k}x_{*i-1} = 0\,, \tag{12}$$

$$x_{oi} - P_i(x_{oi-1}) = 0\,, \tag{13}$$

or

$$G_k(x_{oi}) = x_{*i-1}\,, \tag{14}$$

$$P_i^{-1}(x_{oi}) = x_{oi-1}\,, \tag{15}$$

where $G_k(x_{oi})$ is the implicit mapping generated by (10). In this case Definition 1 is equivalent to Definition 1'.

Definition 1'. The plant P_i is controllable with ε-*accuracy* to signal $x_{oi}(t) \in X\{P_i, m\}$ by the system (4) if there exist two constants k_1, k_2 (k_1, k_2 depend on ε) such that for every $k \in [k_1, k_2)$ there exists an implicit mapping $G_k(x_{oi}) = x_{*i-1}$ generated by (10) such that the inequality

$$\left\|G_k(x_{oi}) - P_i^{-1}(x_{oi})\right\|_X \leq \varepsilon \tag{16}$$

is satisfied.

Theorem 2. *Let the operation* P_i *transform a set of signals* $X\{P_i\}$ *into* X *and let* P_i *fulfil the conditions:*

1^o for every $x_{oi}(t) \in X\{Pi\}$ *there exists an* $x_{*i-1}(t) \in X$ *such that* $x_{oi} - P_i\left(x_{*i-1}\right) - \frac{1}{k}x_{*i-1} = 0$.

2^o $P_i(x_{i-1})$ *is continuously differentiable.*

3^o $\left(P'_{ix}\left(x_{*i-1}\right) - \frac{1}{k}\right)^{-1} \in L\left(X, X\right)$, *for every* $k \in [k_1, \infty)$.

Then the system described by equations (4) controls with ε-accuracy the plant P_i *to a signal*

$$x_{oi}(t) \in X\{Pi\} \text{ for } k \in [k_1, k_2)\,.$$

Proof. From the equations (10) it results

$$\left\|G_k\left(x_{oi}\right) - P_i^{-1}\left(x_{oi}\right)\right\|_X = \left\|x_{*i-1} - x_{oi}\right\|_X \,. \tag{17}$$

Based on the equations (12) - (15) we can write

$$\left\|P_i\left(x_{i-1}\right) - P_i\left(x_{oi-1}\right)\right\|_X = \frac{1}{k}\left\|x_{oi}\right\|_X \,.$$

If for every $x_{oi}(t) \in X\{P_i\}$ there exists a solution $x_{*i-1}(t) \in X$ of the equations (4) (compare condition 1^o of Theorem 1) then a positive constant $m < \infty$ exists, such that the inequality

$$\left\|x_{*i-1}\right\|_X \leq m\left\|x_{oi}\right\|_X$$

is satisfied. The following estimation holds

$$\left\|P_i\left(x_{*i-1}\right) - P_i\left(x_{oi-1}\right)\right\|_X = \frac{m}{k}\left\|x_{oi}\right\|_X \,. \tag{18}$$

Comparing formulae (17), (18) and based on the continuity of the mapping P_i (continuity of P_i arise from continuously differentiability of P_i) we can write

$$\left\|G_k\left(x_{oi}\right) - P_i^{-1}\left(x_{oi}\right)\right\|_X \leq \varepsilon \text{ for } k > k_1. \qquad \square$$

Example 1. Let the plant P_i be described by the Urysohn operator in the form

$$P_i\left(x_{i-1}\right) = \int_\Omega K\left(t, \tau, x_{i-1}\left(\tau\right)\right) d\tau\,.$$

We assume that the kernel $K\left(t, \tau, x_{i-1}\left(\tau\right)\right)$ and its derivative $K'_x\left(t, \tau, x_{i-1}\left(\tau\right)\right)$ are continuous functions of the variables t, τ, x_{i-1}. Let for every $x_{oi}(t) \in X\{P_i\}$ and $k_o = k$ a solution exist $x_{*i-1}\left(t\right) \in X$ of the equation

$$\frac{1}{k}x_{*i-1}\left(t\right) = -\int_\Omega K\left(t, \tau, x_{i-1}\left(\tau\right)\right) d\tau + x_{o1}\left(t\right)\,. \tag{19}$$

The derivative of the mapping $H\left(x_{i-1},x_{i}\right)=-\int\limits_{\Omega}K\left(t,\tau,x_{i-1}\left(\tau\right)\right)d\tau+x_{o1}\left(t\right)-\frac{1}{k}x_{*i-1}\left(t\right)$ at the point $(x_{*i-1}(t),k_o)$ is equal to $H'_x\left(x_{i-1},k_o\right)h=-\frac{1}{k_o}h-\int\limits_{\Omega}K'_x\left(t,\tau,x_{*i-1}\left(\tau\right)\right)d\tau$. If the number $-\frac{1}{k_o}$ is not an eigenvalue of the kernel $\int\limits_{\Omega}K'_x\left(t,\tau,x_{i-1}\left(\tau\right)\right)$ then all assumptions of Theorem 1 are satisfied. For numbers k which are near to the number k_o there exists exactly one solution $x_{i-1}(t)$ of equation (19) and this solution is near to $x_{*i-1}(t)$.

In the particular case when the plant P_i is described by the linear equation

$$P_i\left(x_{i-1}\right)=\int\limits_0^t K\left(t-\tau\right)x_{i-1}\left(\tau\right)d\tau\,,$$

and the number $-\frac{1}{k_o}$ is not an eigenvalue of the kernel $K\left(t-\tau\right)$ then all assumptions of Theorem 2 are satisfied

It should be noticed that the above results are true for a continuously differentiable plant P in any open subset V (see assumptions of Theorem 1). From this we can conclude that a high precision robust control can be realized.

3. The Case When a Plant P_{i} is a Linear, Causal and Stationary

Let the linear, causal and stationary operation P_i map a set of signals from Banach space X into itself. Let now X be a Banach space L^2 (or Banach space M). We assume that the operation P_i is given by the formula

$$\left[P_i x_{i-1}\right](t)=\int\limits_0^t x_{i-1}\left(t-\tau\right)dh_i\left(\tau\right)\,, \tag{18 bis}$$

where $h_i(\tau)$ is a bounded variation function, or P_i is given by the transfer function $P_i(s)$. We rewrite the equations (4) in the form

$$\frac{1}{k}x_{i-1}\left(t\right)=x_i\left(t\right)-\int\limits_0^t x_{i-1}\left(t-\tau\right)dh_i\left(\tau\right)\,. \tag{19 bis}$$

Theorem 3. *If for a number $k\in[k_1,k_2)$, where k_1, k_2 are sufficiently large, and if for the operation P_i the inequality*

$$\inf_{res\geq 0}\left|\frac{1}{k}+P_i\left(s\right)\right|>0 \tag{20}$$

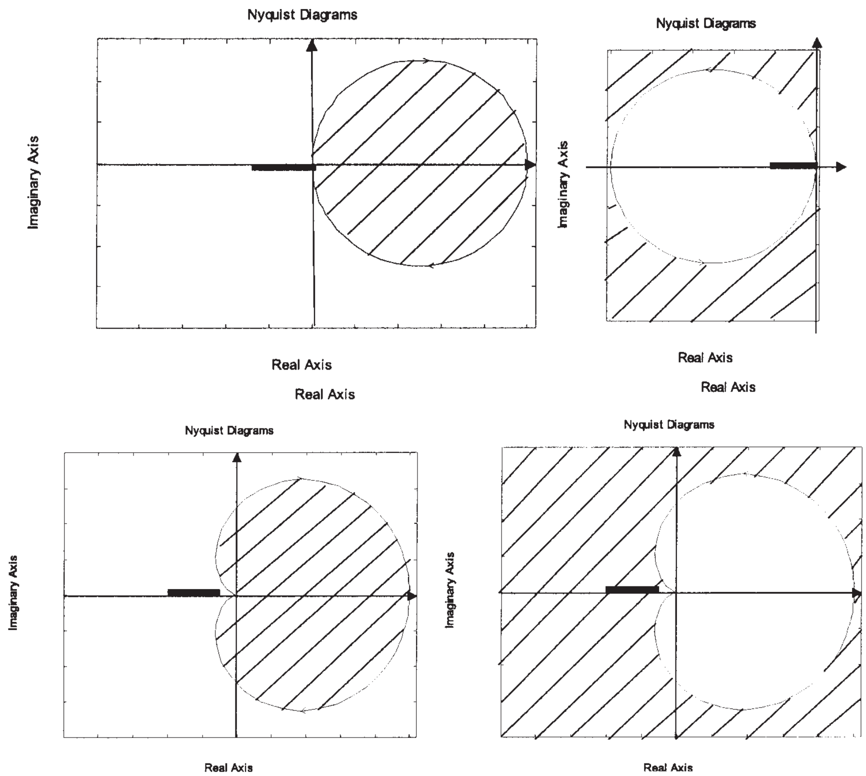

Figure 2: Spectrums of a first and second order element

is satisfied then the system given by equations (18 bis), (19 bis) controls with ε-accuracy the plant P_i to a signal $x_{oi}(t) \in L^2\{P_i\}$ (or $x_{oi}(t) \in M\{P_i\}$).

Proof of this theorem is a consequence of Theorem 2 and of the conditions for the existence of the solutions of the equation (19) (compare Lozowicki [6]). Theorem 3 is a particular case of Theorem 2.

Formula (20) in Theorem 2 has the following geometrical interpretation: The system described by the equations (18 bis) and (19 bis) controls with ε-*accuracy* plant P_i to a signal from class $L^2\{P_i\}$ (or $M\{P_i\}$) if the spectrum of operation $P_i(s)$ and interval $\left[-\frac{1}{k_1}, -\frac{1}{k_2}\right)$ are disjoint sets.

Example 2. Let the plant P_i be given by the transfer function

$$P_i\left(s\right) = \frac{c}{\left(as+b\right)^n}, \quad a,b,c \in \Re\,. \tag{21}$$

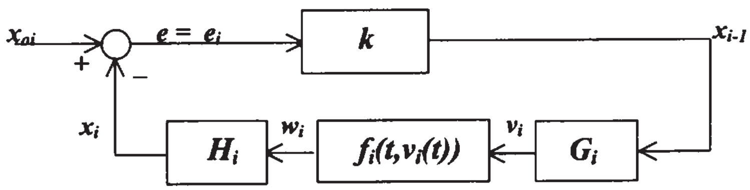

Figure 3: Scheme of the control system whith non-linear plant

The classes of reference signals $L^2\{P_i\}$ or $M\{P_i\}$ are given by the formulas:

$$L^2\{P_i\} = \left\{ x_{oi}\left(t\right) \in L^2 : \quad \sup_{res \geq 0} \left| \frac{(as+b)^n}{c} x_{oi}\left(s\right) \right| < \infty \right\} ,$$

or

$$M\{P_i\} = \left\{ x_{oi}\left(t\right) \in M : \quad \sup_{res \geq 0} \left| \frac{(as+b)^n}{c} x_{oi}\left(s\right) \right| < \infty \right\} .$$

For $n = 1$ and $n = 2$ $(a, b, c \in \Re)$ the spectrum of operation (21) and the interval $\left[-\frac{1}{k_1}, -\frac{1}{k_2}\right)$ are disjoint sets and the system (4) controls with ε-*accuracy* the plant given by formula (21). In the case $n \geq 3$ the assumptions of Theorem 3 are not satisfied and such systems are unstable.

The problem of a high precision robust control of the plant $P_i + \Delta P_i$ is very easy to solve. The range of changes of the parameters a, b, c of the considered plant $P_i + \Delta P_i$ (i.e. model error) can be very simply designed. For this purpose the geometrical interpretation of Theorem 3 can be useful.

4. Control Systems with a Nonlinear Plant P_i

Two methods for the analysis of non-linear systems are presented in the paper. The first one is based on an integral inequality. Such inequality has been used by Van der Schaft [8] for optimisation of non-linear feedback control systems and by Kudrewicz [4] to verify the stability of non-linear feedback systems. The second method applies the harmonically linearised form to the optimisation of non-linear plants (see Peyton Jones and Billings [3], Lozowicki [7]).

We assume that the operation describing the plant $x_i = P_i(x_{i-1})$ consists of the linear, causal and stationary part and of the non-linear part as in Figure 3. Let there exist real numbers λ and $\rho > 0$ such that

$$\int_0^T \left|\nu_i\left(t\right) - \lambda x_{i-1}\left(t\right)\right|^2 dt \leq \rho \int_0^T \left|x_{i-1}\left(t\right)\right|^2 dt \,, \tag{22}$$

for every $0 < T \leq \infty$. Let the linear part of the feedback system shown in Figure 3 be given by equations:

$$e_i(t) = x_i(t) - \int_o^t \nu_i(t-\tau)\, dh_i(\tau) \,, \tag{23}$$

$$x_{i-1}(t) = ke_i(t) \,, \quad k \in [k_1, k_2) \,. \tag{24}$$

Let the transfer function of the linear part of the nonlinear plant P_i be given by formula

$$A_i(s) = \int_0^\infty e^{-st} dh_i(\tau) \tag{25}$$

Theorem 4. *If for the system given by formulae (22)-(25) the inequality*

$$\inf_{res \geq 0} \left| \frac{1}{kA_i(s)} + \lambda \right| > \rho \tag{26}$$

is satisfied then there exists a positive constant $m < \infty$ such that

$$\int_0^T |x_{i-1}(t)|^2 \, dt \leq m \int_0^T |x_{oi}(t)|^2 \, dt \,, \quad \text{for every } \; T \leq \infty \,. \tag{27}$$

Proof. We rewrite equations (23), (24) in the form

$$\frac{1}{k} x_{i-1} = x_{oi} - A_i \nu_i \,. \tag{28}$$

Adding $\lambda A_i x_{i-1}$ to both sides of (28) we get

$$\frac{1}{k} x_{i-1} + \lambda A_i x_{i-1} = x_{oi} - A_i (\nu_i - \lambda x_{i-1}) \,.$$

From condition (26) we can conclude that

$$\inf_{res \geq 0} \left| \frac{1}{k} + \lambda A_i(s) \right| > 0 \,,$$

we rewrite the equations (23) and (24) in the equivalent form

$$x_{i-1} = \left(\frac{1}{k} I + \lambda A_i \right)^{-1} x_{oi} - \left(\frac{1}{k} I + \lambda A_i \right)^{-1} A_i (\nu_i - \lambda x_{i-1}) \,. \tag{29}$$

Computing the norms in spaces $X \quad (L^2(0,\infty)$ or $M)$ of both sides of (29) and using the inequality (22) we get

$$\begin{aligned}
&\|x_{i-1}\|_X \\
&\leq \left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} \right\|_X \|x_{oi}\|_X + \left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} A_i \right\|_X \|(\nu_i - \lambda x_{i-1})\|_X \\
&\qquad \leq \left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} \right\|_X \|x_{oi}\|_X + \left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} A_i \right\|_X \rho \|x_{i-1}\|_X \,,
\end{aligned}$$

i.e.

$$\|x_{i-1}\|_X \leq \frac{\left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} \right\|_X}{1 - \rho \left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} A_i \right\|_X} \|x_{oi}\|_X \,. \tag{30}$$

Based on the inequality (26) we can write

$$\begin{aligned}
\left\| \left(\frac{1}{k} I + \lambda A_i \right)^{-1} A_i \right\|_X &\leq \sup_{res \geq 0} \left| \frac{A_i(s)}{\frac{1}{k} + \lambda A_i(s)} \right| \\
&= \left(\inf_{res \geq 0} \left| \frac{1}{kA_i(s)} + \lambda \right| \right)^{-1} < \frac{1}{\rho} \,. \qquad (31)
\end{aligned}$$

From the inequalities (30) and (31) it results that there exists a positive constant $m < \infty$, such that $\|x_{i-1}\|_X \leq m \|x_{oi}\|_X$. □

Remark 1. The inequality (26) implies that: the spectrum of operation $\frac{1}{k} A_i^{-1}$ for $k \in [k_1, k_2]$ and the area enclosed by a circle with radius ρ and centre $(-\lambda, j0)$ are disjoint sets.

Remark 2. Condition (26) can be rewritten in the form

$$\inf_{res \geq 0} re \frac{1 + NkA_i(s)}{1 + nkA_i(s)} > 0 \,, \tag{32}$$

where

$$\lambda = \frac{N+n}{2} \,, \quad \rho = \frac{N-n}{2} \,.$$

Remark 3. The relation (22) is fulfilled if the static, non-linear part of the plant lies in the area between straight lines $v_i = Nx_{i-1}$ and $v_i = nx_{i-1}$ as in Figure 4.

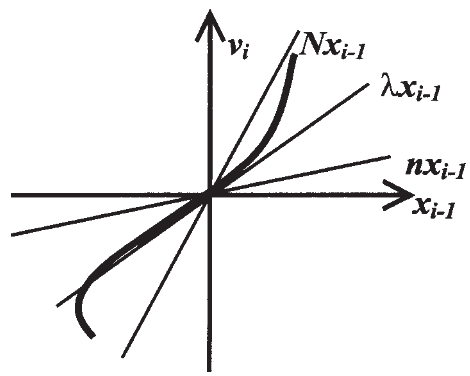

Figure 4: The geometrical interpretation of relation (22)

Remark 4. Theorem 4 does not guarantee that the solution of equations (23), (24) and (25) exist, but if such solution exists and if $x_{oi}(t) \in L^2$ then equally $x_{i-1}(t) \in L^2$.

We assume that the non-stationary, non-linear element be given by function $\nu_i(t) = f_i(t, x_{i-1}(t))$. Let the inequality

$$nx_{i-1}^2 \leq x_{i-1} f_i\left(t, x_{i-1}\right) \leq N x_{i-1}^2\,, \text{ for every } x_{i-1} \text{ and } t \geq 0, \tag{33}$$

be satisfied. The inequality (33) is a particular case of the inequality (22). The system controlling with *ε-accuracy* the plant P_i has the form

$$\frac{1}{k} x_{i-1}\left(t\right) = x_{o1}\left(t\right) - \int_0^t f_i\left(t-\tau, x_{i-1}\left(t-\tau\right)\right) dh_i\left(\tau\right)\,, \quad \text{for } k \in [k_1, k_2)\,. \tag{34}$$

The assumptions of Theorem 4 are fulfilled if the real numbers n and N satisfy relation (32).

Let plant $P_i(x_{i-1})$ consist of the two linear, stationary parts $G_i(s)$ and $H_i(s)$ and of the non-linear part $w_i(t) = f_i(t, v_i(t))$ (see Figure 5). If the inequality (33) is satisfied then relation (32) takes the form

$$\inf_{res \geq 0} re \frac{1 + NkG_i\left(s\right) H_i\left(s\right)}{1 + nkG_i\left(s\right) H_i\left(s\right)} > 0 \tag{35}$$

Let the system expressed by (23), (24) (25) have the form (34). Let the assumptions of Theorem 4 be satisfied. The following statement can be prooved (see Kudrewicza [4]):

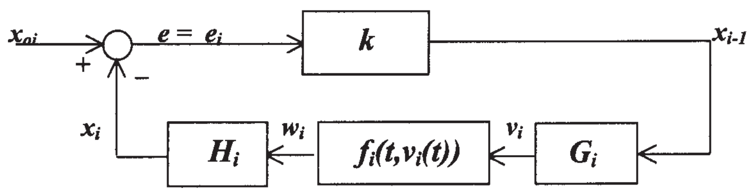

Figure 5: Scheme of the feedback system with two linear parts

If $h_i(t)$ in the equality (34) is a bounded variation function for $t \in [0,\infty)$ and if it is continuous at the point $t = 0$ then for every $x_{oi}(t) \in L^2(0,T)$, $T < \infty$ there exists exactly one solution $x_{i-1}(t) \in L^2(0,T)$ of equation (34).

For the analysis of a high precision robust control in the Banach space M the following theorem can be useful.

Theorem 5. *For numbers $\rho > 0$, $\lambda \in \Re$ and for the measurable functions $u_1(t)$, $u_2(t)$ and for every $T > 0$ let the inequality*

$$\int_o^T \left| f_i\left(t, u_1\left(t\right)\right) - f_i\left(t, u_2\left(t\right)\right) - \lambda\left(u_1\left(t\right) - u_2\left(t\right)\right)\right|^2 dt \leq \rho \int_o^T \left| u_1\left(t\right) - u_2\left(t\right)\right|^2 dt\,,$$

be satisfied. If

$$\inf_{res \geq 0} \left| \frac{1}{kA_i\left(s\right)} + \lambda \right| > \rho\,,$$

then equation (34) has exactly one solution $x_{i-1}(t) \in M$ for every $x_{oi}(t) \in M$.

Example 3. We take under consideration the two plants described by equations

$$(x_{i-1}(t))^3 = Tv_i'(t) + v_i(t),\ T > 0 \tag{36}$$

and

$$\arctan(x_{i-1}(t)) = av_i''(t) + bv_i'(t) + c, \quad a, b, c > 0\,. \tag{37}$$

The transfer functions of the linear parts have the forms, respectively

$$A_{i1}\left(s\right) = \frac{1}{Ts+1}, \qquad A_{i2}\left(s\right) = \frac{1}{as^2+bs+c}\quad .$$

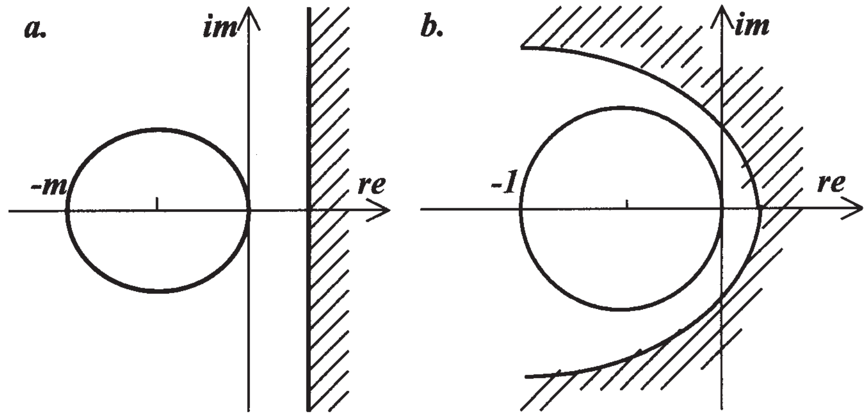

Figure 6: The geometrical interpretation of the considered conditions for controllability with $\varepsilon - accuracy$

It is easy to verify that conditions 2^o and 3^o of Theorem 2 are fulfilled. For the bounded signal $x_{oi}(t) \in L^2$ and for the solutions $x_{i-1}(t)$ of the equation (36) there exists a constant $m > 0$ such that the relation

$$0 \leq x_{i-1}(t)x_{i-1}^3(t) \leq \; mx_{i-1}^2(t)$$

is satisfied. For the equation (37) the inequality

$$0 \leq \arctan(x_{i-1}(t)) \leq x_{i-1}^2(t)$$

holds. In this case we do not assume that the signal $x_{oi}(t)$ is bounded. The plant (36) is controlled with ε-*accuracy*. Constant ε depend on the velocity feedback gain $k \in [k_1, k_2)$ (see Figure 6a). The controllability with ε-*accuracy* of the plant (37) depends on the coefficients a, b, c (see Figure 6b).

5. The Harmonically Linearization Method

We take under consideration the control system shown in Figure 1, i.e. the system described by equations (4) in the case when $i = 1$. Let the plant P be given by the non-linear operation $x_i(t) = P_i(x_{i-1}(t))$. Let P_i map a set of signals from Banach space L^2 (or M) into itself and let the condition: $x_i(t) \equiv 0$ for $x_{i-1}(t) \equiv 0$ be satisfied. We will assume that k is a sufficiently large positive number. The equations (4) can be transformed to the form

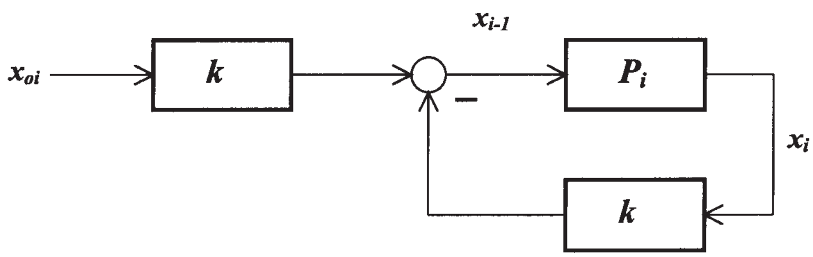

Figure 7: The equivalent system with the non-linear plant $P_i (i = 1)$

$$x_{oi} - x_i = \frac{1}{k}\left(\left(\frac{1}{k} + P_i\right)^{-1} (x_{oi})\right) . \tag{38}$$

Computing the norms of both sides of (38) we get

$$\|x_{oi} - x_i\|_2 = \left\|\frac{1}{k}\left(\frac{1}{k} + P_i\right)^{-1}\right\|_\infty \|(x_{oi})\|_2 . \tag{39}$$

In this case the system shown in Figure 1 is equivalent to the system shown in Figure 7. If for k sufficiently large the condition

$$\left\|\frac{1}{k}\left(\frac{1}{k} + P_i\right)^{-1}\right\|_\infty < \infty$$

is satisfied then $\|e_i\|_2 = \|x_{oi} - x_i\|_2 \leq \varepsilon$ and system (4) controls with ε-*accuracy* the plant P_i.

The describing function of the non-linear element P_i (compare Lozowicki [7], Peyton Jones and Billings [3]) will be denoted by $\underline{P}_i$. The approximate, linearized equations, which correspond to equations (4), have the form

$$\begin{cases} e = x_{oi} - x_i \, , \\ x_{i-1} = kx_{oi} - kx_i \, , \\ x_i = \underline{P}_i x_{i-1} \, , \end{cases} \tag{40}$$

or the form

$$x_{oi} - x_i = \frac{1}{k}\left(\left(\frac{1}{k} + \underline{P}_i\right)^{-1} x_{oi}\right) . \tag{41}$$

Computing the norms of both sides of (41) we get

$$\|x_{oi} - x_i\|_2 = \left\| \frac{1}{k} \left(\frac{1}{k} + \underline{P}_i \right)^{-1} \right\|_\infty \|x_{oi}\|_2 \ . \tag{42}$$

Based on the above considerations the following theorem can be proved.

Theorem 6. (compare Lozowicki [7]) *Let X be a Banach space $L^2(0,\infty)$ (or M). We assume that the non-linear operation $xi(t) = P_i(x_{i-1}(t))$, mapping the set of signals from Banach space X into X, has a uniformly, continuous, bounded derivative. If for $k \in [k_1, k_2)$ a solution of the system (40) exists and the inequality*

$$\inf_{res \geq 0} \left| \frac{1}{k} + \underline{P}_i\,(s) \right| > 0\,, \tag{43}$$

is satisfied, then the system shown in Figure 1 or Figure 7 controls with ε-accuracy the non-linear plant P_i to signal $x_{oi}(t) \in X\{P_i\}$.

6. System with State Feedback

Now we take under consideration the superposition P of the operations P_i given by the formulas (1) and (2) (or(3)) We put (1) and (2) into equations (4). In this case we get:

$$\begin{aligned} u(t) &= k\{(P_2^{-1}(x_{o2}(t)) - P_1(u(t))) + (P_3^{-1}(x_{o3}(t)) - P_2(P_1(u(t)))) + ... \\ &\quad + (x_{on}(t) - P_n(...(P_1(u(t)))))\} \end{aligned}$$

$$\begin{aligned} d(t) &= [P_2^{-1}(x_{o2}(t)) - P_1(u(t)), P_3^{-1}(x_{o3}(t)) - P_2(P_1(u(t))), ..., \\ &\quad x_{on}(t) - P_n(...(P_1(u(t))))]' \end{aligned}$$

$$\begin{aligned} \chi_o(t) &= [P_1(u_o(t)), P_2(P_1(u_o(t))), ..., P_n(...(P_1(u_o(t))))]' \\ &= [P_2^{-1}(x_{o2}(t)), P_3^{-1}(x_{o3}(t)), ..., x_{on}\,(t)]'\,. \end{aligned}$$

Equations (4) for linear, causal and stationary plants, take the form

$$\begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} u\,(s) = k \begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ ... & ... & ... & ... \\ 1 & 1 & ... & 1 \end{bmatrix} \begin{bmatrix} e_1\,(s) \\ e_2\,(s) \\ ... \\ e_n\,(s) \end{bmatrix} ,$$

$$\begin{bmatrix} e_1(s) \\ e_2(s) \\ ... \\ e_n(s) \end{bmatrix} = \begin{bmatrix} x_{o1}(s) \\ x_{o2}(s) \\ ... \\ x_{on}(s) \end{bmatrix} - \begin{bmatrix} P_1(s) & 0 & ... & 0 \\ 0 & P_1(s)P_2(s) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & P_1(s)P_2(s)...P_n(s) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} u(s) \; ,$$

$$\begin{bmatrix} x_{o1}(s) \\ x_{o2}(s) \\ ... \\ x_{on}(s) \end{bmatrix} = \begin{bmatrix} P_2^{-1}(s)...P_n^{-1}(s) & 0 & ... & 0 \\ 0 & P_3^{-1}(s)...P_n^{-1}(s) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} y_o \, . \tag{44}$$

Formally for $k \to \infty$, the system shown in Figure 1 controls with *ε-accuracy* the plant P. The following theorem gives the conditions for controllability with *ε-accuracy* of plant P.

Theorem 7. *If all of the operations: $P_1 : W \to X\{P_1\} \subset X$, $P_i : X\{P_{i-1}\} \to X\{P_i\} \subset X$, $i = 2, 3, ...n$ satisfy assumptions of Theorem 2 then the control system shown in Figure 1 controls with ε-accuracy the plant P given by formulae (1) or (2) to a reference signal $\chi_o(t)$ from class $X\{P\}$.*

Example 4. Let an unstable plant P (with zero initial conditions) be given by the transfer function

$$\frac{y(s)}{u(s)} = \frac{1}{s^3 - 0.5s^2 - 3s - 2} \, . \tag{45}$$

We define the state space vector as follows:

$$\begin{gathered} x_3(t) = y(t) \, , \\ x_2'(t) = x_3(t) \, , \\ x_1'(t) = x_2(t) \, , \\ y'''(t) = -0.5y''(t) - 3y'(t) - 2y(t) + u(t) \, , \end{gathered} \tag{46}$$

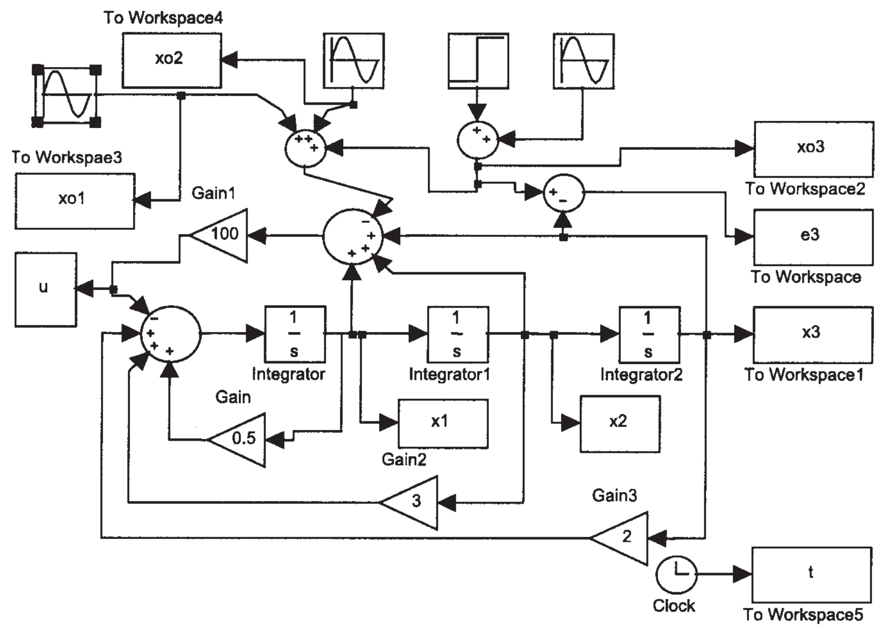

Figure 8: The *SIMULINK* scheme of the system controlling with *ε-accuracy* the unstable plant

i.e.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \,. \tag{47}$$

Based on formulas (46) and (47), *MATLAB-SIMULINK* simulation of the classical analog model of the plant P has been used. The full scheme of such system is shown in the Figure 8. The simulation examples (see Figure 9) have been made for the feedback gain $k = 100$ and reference signal

$$\chi_o(t) = \begin{bmatrix} x_{o1}(t) \\ x_{o2}(t) \\ x_{o3}(t) \end{bmatrix} = \begin{bmatrix} y_o''(t) \\ y_o'(t) \\ y_o(t) \end{bmatrix} = \begin{bmatrix} 9\cos(3t) \\ 3\sin(3t) \\ 1-\cos(3t) \end{bmatrix} \tag{48}$$

These results confirm a high quality of the above presented control system for a given reference signal χ_o . Signal y_o from L^2 space or from M space should have bounded first and second derivatives). Other simulations have been made for different denominators of the transfer function (45) and the same reference signal (48). The results of simulations confirm the robustness and high precision of the considered control system.

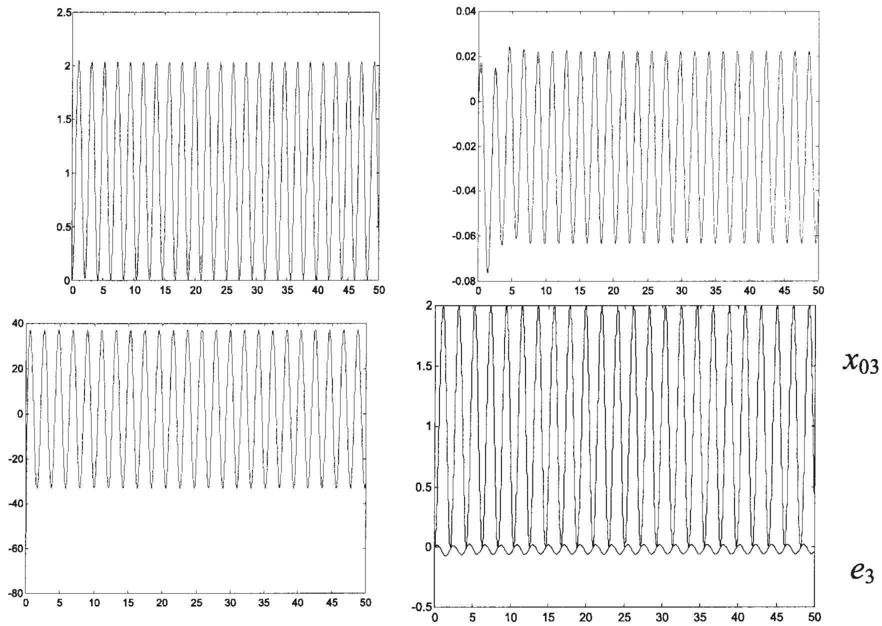

Figure 9: Results of the simulations for unstable plant and $k = 100$. In the first row, from left to right, there are plotted the reference signal $y_0(t) = x_{03}(t) = -\cos(3t) + 1$ and the error signal $e_3(t)$. In the second row from left to right there are plotted the control signal $u(t)$ and the comparison of reference signal $x_{03}(t)$ and error signal $e_3(t)$.

Example 5. Let the plant $P = P_4(P_3(P_2(P_1)))$ be described by the transfer function:

$$P(s) = \frac{1}{(s^2+s+1)} \frac{(s+2)}{(s^2+2s+1)} \frac{1}{(s^2+3s+1)} \frac{1}{(s+1)} = P_4(s) P_3(s) P_2(s) P_1(s) \,. \quad (49)$$

Notice that the functions $P_1(s)$, $P_2(s)$, $P_3(s)$ and $P_4(s)$ fulfil the assumptions of Theorem 3. Based on Theorem 7 we can conclude that plant given by the transfer function (49) is controlled with *ε-accuracy* by the system shown in Figure 1. The results of simulations in the case when $k = 100$ are shown in Figure 10.

Conclusion 1. *It is worth noting the important influence for the quality of the control of the considered systems of vector χ_o. In particular it is very*

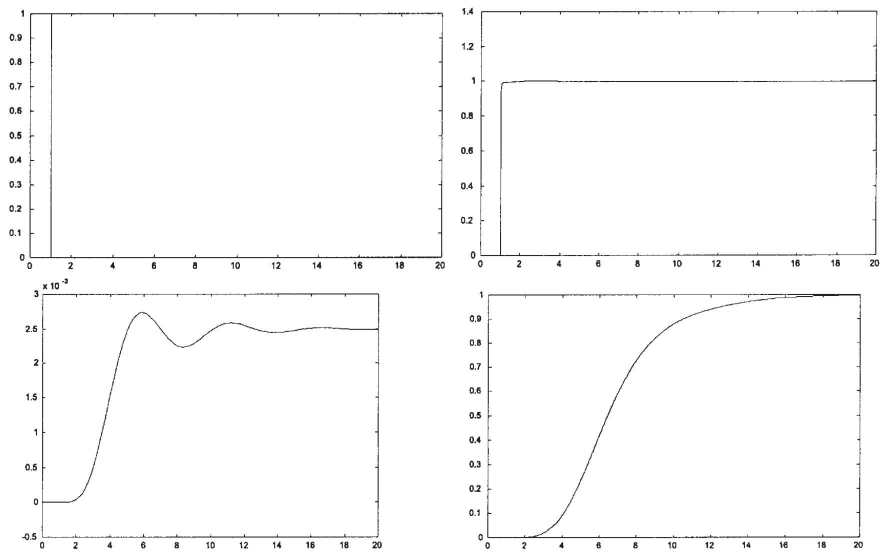

Figure 10: Results of simulations for $P(s) = \dfrac{(s+2)}{(s^2+s+1)(s^2+2s+1)(s^2+3s+1)(s+1)}$, and $k = 100$. In the first row, from left to right, there are plotted the reference signal $y_0(t) = x_0$ (ideal step) and the control input $u(t)$. In the second row, from left to right there are plotted the error signal $e_4(t) = x_{04}(t) - x_4(t)(\times 10^{-3})$ and the reference signal $x_{04}(t) = y_0(t)$.

important to note the accuracy of realization of the component $x_{on-1}(t)$ of vector χ_o (i.e. the $n-1$ derivative of the reference signal $y_o(t)$).

Conclusion 2. *The robustness property of the considered control system enables to control with ε-accuracy also plants with uncertainty* $P + \Delta P$.

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