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QUERIES ON HOMOGENEOUS POLYNOMIALS AND CLOSED ANALYTIC SUBSETS OF AN INFINITE-DIMENSIONAL COMPLEX PROJECTIVE SPACE

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Abstract: Let V be a complex Banach space and $X \subset \mathbf{P}(V)$ a closed analytic subset. For any sufficiently general n-dimensional subspace of $\mathbf{P}(V)$, let $d_{X,n}$ be the minimal degree of the polynomials needed to define $(X \cap A)_{red}$. For any integer $z \geq d_{X,n}$ let $m_{X,n,z}$ be the minimal number of homogeneous degree z polynomials needed to define $(X \cap A)_{red}$. Here we give some elementary properties and raise some questions on these sequences of integers and their relation to the continuous homogeneous polynomials on V vanishing on X.

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1. Queries

Let V be a complex Banach space space and $\mathbf{P}(V)$ the projective space parametrizing all one-dimensional linear subspaces of V. It is well-known that a closed analytic subset X of $\mathbf{P}(V)$ may have bad properties (e.g. any compact and metrizable topological space has a structure of analytic set for someseparable

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Banach space (see [4], Proposition II.1.3). However, if X is locally finitely determined (i.e. given locally by finitely many holomorphic equations in a Banach manifold), then it is nice: $\mathrm{Sing}(X)$ is a proper closed analytic subset of X (see [4], Theorem III.3.1.1), there is a decomposition of X into irreducible components and it is "algebraic", i.e. the zero-locus of continuous homogeneous polynomials on V. From now on, we will always assume that X is a finitely determined closed analytic subset of $\mathbf{P}(V)$ (see [4], Theorem III.2.3.1). We will often assume that X is irreducible, i.e. that the set X_{reg} of its smooth points is connected. For every integer $m \geq 0$ let $G(m+1, V)$ denote the Grassmannian of all $(m+1)$-linear subspaces of V. By Hahn-Banach every $A \in G(m+1, V)$ is closed and it has a closed supplement. We will always see each $A \in G(m+1, V)$ as a closed m-dimensional linear subspace of $\mathbf{P}(V)$.

Proposition 1. *Let V be a complex Banach space space and $X \subset \mathbf{P}(V)$ of finite definition such that each irreducible component of X has codimension m. Set $d := \deg(X)$. Then X is set-theoretically the zero-locus of a family of continuous degree d homogeneous polynomials on V.*

Proof. Taking if necessay a closed linear subspace of V we reduce to the case in which X is not contained in a closed hyperplane of $\mathbf{P}(V)$. If $m = 1$ we may take just one homogeneous degree d homogeneous polynomial with X as its zero-locus (even scheme-theoretically) by the definition of degree and [4], Theorem III.2.3.1: take a continuous homogeneous polynomial $f \neq 0$ vanishing on X and with minimal possible degree; the existence of at least one polynomial $\neq 0$ vanishing on X, being [4], Theorem III.2.3.1. We will adapt the proof of [3], Theorem 1. Fix any finite-dimensional linear subspace $W \subset V$ and set $e := \dim(W) - 1$. Assume $e \geq 0$. By Hanh-Banach W has a closed supplement in V and we fix one such supplent. Hence $M + W = V$ and $M \cap W = \{0\}$. Let g be any degree x homogeneous polynomial on V. We may expand g into monomials with variable in W; each such non-zero monomial, say of degree a, has as the coefficient a homogeneous polynomial of degree $x - a$ in M. Call g_W the sum of all such coefficients for various monomials in W with maximal degree $< x$, with the convention $g_W = 0$ if there is no such monomial. We will call g_W the cone of g with respect to W; g_W does not depend from the choice of M. For any zero-locus $Y = \cap_{j\in J}\{g_j = 0\} \subset \mathbf{P}(V)$ of continuous homogeneous polynomials, let $Y_W := \cap_{j\in J}\{(g_j)_W = 0\} \subset \mathbf{P}(V)$ be the cone of Y with vertex W or with vertex $\mathbf{P}(W)$. If Y is finitely determined and with finite codimension in $\mathbf{P}(V)$, then Y_W has the same properties and $Y \subseteq Y_W$, $\mathrm{codim}(Y_W) \geq \mathrm{codim}+e$ and $\deg(Y_W) \leq \deg(Y)$ (hint: use the finite-dimensional case). As in the finite-dimensional case we may find a family of finite-dimensional linear spaces W_j,

$j \in J$, such that $\dim(W_j) = m$, X_{W_j} is a hypersurface and $X = \cap_{j \in J} X_{W_j}$. If $\deg(X_{W_j}) = d$ for every j, we are done. If $\deg(W_j) = d - k$ for some $k > 0$, take all hypersurfaces $X_{W_j} \cup D_k$, where D_k is the k-power of a closed hyperplane of $\mathbf{P}(V)$. □

Proposition 2. *In the set-up of Proposition 1 there is an integer $s \le d^m - d + 1$ and s continuous homogeneous polynomials $f_1, \dots, f_s$ such that $\{f_1 = \dots = f_s = 0\}$ is the union of X and some lower dimensional analytic set.*

Proof. By Proposition 1 there is a family $\{g_i\}_{i \in I}$ of degree d continuos hoomogeneous polynomial such that $X = \cap_{i \in I} \{g_i = 0\}$ (set-theoretically). Since X has pure codimension m there are m element of $\{g_i\}_{i \in I}$, say $f_1, \dots, f_m$ such that $Z := \cap_{i=1}^m \{f_i = 0\}$ has pure codimension m. We have $\deg(Z) \le d^m$ and hence Z is the union of X and a finite set (perhaps empty) of m-codimensional irreducible subvarieties $Y_{m+1}, \dots, Y_s$. Fix any $f_j \in \{g_i\}_{i \in I}$, $m + 1 \le j \le s$, not vanishing at some point of Y_j. □

Remark 1. Let V be a complex locally convex and Hausdorff and $X \subset \mathbf{P}(V)$ a closed analytic subset which is locally of finite definition in the sense of [2], Chapter 5 of Part II. Hence there is a locally finite decomposition of X into irreducible components. For each of these components (say Y) it is possible to define its codimension in $\mathbf{P}(V)$ (perhaps $+\infty$) in the following way: choose any $P \in Y_{reg}$ and call codimension the codimension of the tangent space $T_P Y$ in $T_P \mathbf{P}(V)$. Assume that all these codimensions are finite and that they are equal. Call m these codimensions. The proofs of Proposition 1 and Proposition 2 works verbatim IF we assume that X is the zero-locus of a family of continuous homogeneous polynomials on V. Without this assumption we do not know (in general) to define the cones. It would be nice to have an extension of [4], Theorem III.2.3.1, to more general locally convex topological vector spaces.

Fix V, X, m and d as in the statement of Proposition 1 and Proposition 2 (or Remark 1). We will always assume that V is infinite-dimensional. Let d_X denote the minimal integer x such that X is set-theoretically the zero-locus of a family of continuous homogeneous polynomials of degree x on V. As in the proof of Proposition 1 it is easy to check that d_X is the minimal integer such that X is set-theoretically the zero-locus of a family of continuous homogeneous polynomials of degree $\le d_X$ on V. For any integer $n \ge m$ it easy to see the existence of a "large" subset (e.g. a dense subset) $U_n \subseteq G(n + 1, V)$ such that $A \cap X$ is nice for every of degree d (see [1] for much more). Let $d_{X,n}$ be the minimal integer y such that $(A \cap X)_{red}$ is the zero-locus of homogeneous polynomials of degree y for a general $A \in U_n$. For any integer $z \ge d_{X,n}$. It is eaay to check that $d_{X,n} \leq d_{X,n+1}$ for every $n \geq m$. Thus by Proposition 1 we have $d_{X,n} \leq d$ for every $n \geq m$. Hence $d_{X,n} = d_{X,n+1}$ for all $n \gg 0$. Set $d_{X,\infty} := d_{X,n}$ for any $n \gg 0$. Since for any $A \in U_n$ and any $P \in \mathbf{P}(V)\backslash X$ there is $B \in G(n+2, V)$ containing both A and P, it is easy to check that X is the zero-locus of homogeneous polynomials of degree $d_{X,\infty}$. Obviously, $d_{X,\infty}$ is the minimal integer with that property. Let $m_{X,n,z}$ denote the minimal number of degree z homogeneous polynomials defining $(A \cap X)_{red}$ for a general $A \in U_n$.

Question 1. Study the sequence $\{d_{X,n}\}_{n \geq m}$ and in particular find the minimal integer $n = n_X$ such that $d_{X,n} = d_{X,\infty}$. Study the sequence $\{m_{X,n,d_{X,\infty}}\}_{n \geq n_X}$.

Question 2. Under what assumptions on V and X, the analytic set X is the zero-locus of finitely many continuous homogeneous polynomials.

Remark 2. Take as $X \subset \mathbf{P}(V)$ the complete intersection of m continuous homogeneous polynomials of degree $a_1 \geq \cdots \geq a_m \geq 2$. X is the zero-locus of a family of continuous homogeneous polynomials of degree a_1, $d_{X,\infty} = a_1$ and $n_X = m$. If V is infinite-dimensional and $a_1 > a_m$, then X cannot be the zero-locus of finitely many polynomials of degree a_1 because the sequence $\{m_{X,n,a_1}\}_{n \geq m}$ diverges.

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