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OPTIMAL FILTERING FOR NONLINEAR SYSTEMS OVER LINEAR OBSERVATIONS WITH TIME DELAY

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Abstract: In this paper, the optimal filtering problem for nonlinear systems over linear observations with time delay is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance. As a result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are first derived. The procedure for obtaining a closed system of the filtering equations for a polynomial state over linear observations with delay is then established, which yields the explicit closed form of the filtering equations in the particular case of a bilinear system state. In the example, performance of the designed optimal filter for bilinear states over linear observations with delay is verified against the best filter available for bilinear states over linear observations without delays and the conventional extended Kalman-Bucy filter.

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1. Introduction

Although the general optimal solution of the filtering problem for nonlinear state and observation equations confused with white Gaussian noises is given by the Kushner equation for the conditional density of an unobserved state with respect to observations [20], there are a very few known examples of nonlinear systems, where the Kushner equation can be reduced to a finite-dimensional closed system of filtering equations for a certain number of lower conditional moments. The most famous result, the Kalman-Bucy filter [16], is related to the case of linear state and observation equations, where only two moments, the estimate itself and its variance, form a closed system of filtering equations. However, the optimal nonlinear finite-dimensional filter can be obtained in some other cases, if, for example, the state vector can take only a finite number of admissible states [26] or if the observation equation is linear and the drift term in the state equation satisfies the Riccati equation $df/dx+f^2=x^2$ (see [7]). The complete classification of the "general situation" cases (this means that there are no special assumptions on the structure of state and observation equations), where the optimal nonlinear finite-dimensional filter exists, is given in [27].

The optimal filtering problem for linear system states and observations without delays was solved in 1960s, and this already mentioned closed form solution is known as the Kalman-Bucy filter [16]. However, the related optimal filtering problem for linear states with delay has not been solved in a closed form, regarding as a closed form solution a closed system of a finite number of ordinary differential equations for any finite filtering horizon. The optimal filtering problem for time delay systems itself did not receive so much attention as its control counterpart, and most of the research was concentrated on the filtering problems with observation delays (the papers [2, 17, 14, 21] could be mentioned to make a reference). On the other hand, the duality of the control and filtering problems for linear systems implies that the optimal state estimation for the system with delays is related to the optimal LQR problem for systems with delays, which was extensively studied using various approaches (see [12, 9, 3, 25, 1] and references therein). There also exists a considerable bibliography related to the robust control and filtering problems for time delay systems (such as [11, 22]). Comprehensive reviews of theory and algorithms for time delay systems are given in [18, 19, 23, 11, 10, 8].

In this paper, the optimal filtering problem for nonlinear systems over linear observations with time delay is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance [24]. As the first result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are derived. It is then proved that a closed finite-dimensional system of the optimal filtering equations with respect to a finite number of filtering variables can be obtained if the state equation is polynomial, the observations are linear, and the observation matrix is invertible. In this case, the corresponding procedure for designing the optimal filtering equations is established. Finally, the closed system of the optimal filtering equations with respect to two variables, the optimal estimate and the error variance, is derived in the explicit form for the particular case of a bilinear state equation.

In the illustrative example, performance of the designed optimal filter for bilinear states over linear observations with delay is verified against the best filter available for bilinear states over linear observations without delays [5] and the conventional extended Kalman-Bucy filter. The simulation results show a definite advantage of the designed optimal filter in regard to proximity of the estimate to the real state value.

The paper is organized as follows. Section 2 presents the filtering problem statement for a nonlinear system state over linear observations with delay. The Ito differentials for the optimal estimate and the error variance are derived in Section 3. Section 4 establishes the procedure for obtaining a closed system of the filtering equations for a polynomial state over linear observations with delay, which yields the explicit closed form of the filtering equations in the particular case of a bilinear system state. In Section 5, performance of the obtained optimal filter for bilinear states over linear observations with delay is verified in the illustrative example against the best filter available for bilinear states over linear observations without delays and the conventional extended Kalman-Bucy filter.

2. Filtering Problem for Nonlinear State over Delayed Observations

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes. The partially observed F_t-measurable random process $(x(t), y(t))$ is described by a nonlinear differential equation for the system state

$$dx(t) = f(x,t)dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{1}$$

and a linear delay differential equation for the observation process:

$$dy(t) = (A_0(t) + A(t)x(t-h) + B(t)dW_2(t), \tag{2}$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the observation process, the initial condition $x_0 \in R^n$ is a Gaussian vector such that x_0, $W_1(t)$, $W_2(t)$ are independent. The observation process $y(t)$ depends on delayed state $x(t-h)$, where h is a delay shift, which assumes that collection of information on the system state for the observation purposes is possible only at after a certain time lag h. The nonlinear function $f(x,t)$ is supposed to be measurable in t and Lipshitzian in x, which ensures [13] existence and uniqueness of the solution of the equation $\dot{x}(t) = f(x,t)$. It is assumed that $A(t)$ is a nonzero matrix and $B(t)B^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of time of appropriate dimensions.

The estimation problem is to find the estimate of the system state $x(t)$ based on the observation process $Y(t) = \{y(s), 0 \leq s \leq t\}$, which minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))],$$

at each time moment t. In other words, our objective is to find the conditional expectation

$$m(t) = \hat{x}(t) = E(x(t) \mid F_t^Y).$$

As usual, the matrix function

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y]$$

is the error variance.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differential of the conditional expectation $E(x(t) \mid F_t^Y)$ and its variance $P(t)$ (cited after [24]) and given in the following section.

3. Optimal Filter for Nonlinear State over Delayed Observations

The optimal filtering equations could be obtained using from the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ (see [24])

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + E(x[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times \left(B(t)B^T(t)\right)^{-1}(dy(t) - E(\varphi_1(x) \mid F_t^Y)dt),$$

where $f(x)$ is the nonlinear drift term in the state equation and $\varphi_1(x)$ is the drift term in the observation equation equal to $\varphi_1(x) = A_0(t) + A(t)x(t-h)$. Upon performing substitution and noticing that $E(x(t-h) \mid F_t^Y) = E(x(t-h) \mid F_{t-h}^Y) = m(t-h)$ for any $h > 0$, the estimate equation takes the form

$$\begin{aligned} dm(t) &= E(f(x,t) \mid F_t^Y)dt + E(x(t)[A(t)(x(t-h) - m(t-h))]^T \mid F_t^Y) \\ &\quad \times (B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-h)) \\ &= E(f(x,t) \mid F_t^Y)dt + E(x(t)(x(t-h) - m(t-h))^T \mid F_t^Y)A^T(t) \\ &\quad \times (B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-h))dt). \end{aligned} \tag{3}$$

The equation (3) should be complemented with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$.

Let us transform the term $E(x(t)(x(t-h) - m(t-h))^T \mid F_t^Y)$ in the last equation. Denote as $x_1(t)$ the solution of the equation $\dot{x}_1(t) = f(x_1, t)$ with the initial condition $x_1(t_0) = x_0$. Then, the solution $x(t)$ of the equation (1) can be represented in the form

$$x(t) = x_1(t) + \int_{t_0}^{t} b(s)dW_1(s). \tag{4}$$

Let us now introduce the matrix $\Phi(\tau, t)$, which would serve as a nonlinear analog of the state transition matrix in the inverse time. Indeed, define $\Phi(\tau, t)$ as a such matrix that the equality $\Phi(\tau, t)x_1(t) = x_1(\tau)$, $\tau \le t$, holds for any $t, \tau \ge t_0$ and $\tau \le t$. Naturally, $\Phi(\tau, t)$ can be defined as the diagonal matrix with elements equal to $x_{1_i}(\tau)/x_{1_i}(t)$, where $x_{1_i}(t)$ are components of the vector $x_1(t)$, if $x_{1_i}(t) \neq 0$ almost surely. The definition of $\Phi(\tau, t)$ for the case of $x_{1_i}(t) = 0$ will be separately considered below.

Hence, using the representation (4) and the notion of the matrix $\Phi(\tau, t)$, the term $E(x(t)(x(t-h) - m(t-h))^T \mid F_t^Y)$ can be transformed as follows

$$E(x(t)(x(t-h) - m(t-h))^T \mid F_t^Y) = E((x(t) - m(t))(x(t-h))^T \mid F_t^Y)$$

$$= E((x(t) - m(t))(x_1(t-h) + \int_{t_0}^{t-h} b(s)dW_1(s))^T \mid F_t^Y)$$

$$= E((x(t) - m(t))(x_1(t-h))^T \mid F_t^Y)$$

$$= E((x(t) - m(t))(\Phi(t-h,t)x_1(t))^T \mid F_t^Y)$$

$$= E((x(t) - m(t))(x_1(t))^T \mid F_t^Y)(\Phi^*(t-h,t))^T$$

$$= E((x(t) - m(t))(x_1(t) + \int_{t_0}^{t-h} b(s)dW_1(s))^T \mid F_t^Y)(\Phi^*(t-h,t))^T$$

$$= E((x(t) - m(t))(x(t))^T \mid F_t^Y)(\Phi^*(t-h,t))^T = P(t)(\Phi^*(t-h,t))^T, \quad (5)$$

where $P(t) = E((x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y)$ is the error variance and $\Phi^*(t-h,t)$ is the state transition matrix in the inverse time for the process $x_1^*(t)$, that is the solution of the equation $\dot{x}_1^*(t) = f(x_1^*, t)$ with the initial condition $x_1^*(t_0) = m_0 = E(x(t_0) \mid F_{t_0}^Y)$.

Thus, in view of the transformation (5), the equation (3) for the optimal estimate takes the form

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + P(t)(\Phi^*(t-h,t))^T A^T(t) \quad (6)$$

$$\times (B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-h))dt),$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$.

Let us now define the matrix $\Phi(t-h,t)$ in the case of $x_{1_i}(t) = 0$ almost surely for one of the components of $x_1(t)$. Then, the corresponding diagonal entry of $\Phi_{ii}(t-h,t)$ can be set to 0 for any $h > 0$, because, for the component $x_i(t)$,

$$E(x_i(t)(x_j(t-h) - m_j(t-h))^T \mid F_t^Y) = E((x_{1_i}(t) + (\int_{t_0}^{t} b(s)dW_1(s))_i)$$

$$\times (x_j(t-h) - m_j(t-h))^T \mid F_t^Y)$$

$$= E(x_{1_i}(t)(x_j(t-h) - m_j(t-h))^T \mid F_t^Y) = 0,$$

almost surely for any $j = 1, \ldots, m$. Hence, the definition $\Phi_{ii}(\tau, t) = 0$, $\tau \le t$, leads to the same result as in the equation (5) and can be employed. The diagonal element $\Phi_{ii}^*(\tau, t)$ of the matrix $\Phi^*(\tau, t)$ is defined accordingly and set to 0 for $\tau \le t$, if the corresponding component of the process $x_1^*(t)$ is equal to zero at the moment t, $x_{1_i}^*(t) = 0$. Thus, the equation (6) for the optimal estimate $m(t)$ also holds for the complete definition of the matrix $\Phi^*(\tau, t)$.

To compose a closed system of the filtering equations, the equation (6) should be complemented with the equation for the error variance $P(t)$. For this purpose, the formula for the Ito differential of the variance $P(t) = E((x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y)$ could be used (cited again after [24]):

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y) + E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y)$$
$$+b(t)b^T(t) - E(x(t)[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y)\big(B(t)B^T(t)\big)^{-1}$$
$$\times E([\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]x^T(t) \mid F_t^Y))dt$$
$$+E((x(t) - m(t))(x(t) - m(t))[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y)$$
$$\times\big(B(t)B^T(t)\big)^{-1}(dy(t) - E(\varphi_1(x) \mid F_t^Y)dt),$$

where the last term should be understood as a 3D tensor (under the expectation sign) convoluted with a vector, which yields a matrix. Upon substituting the expressions for φ_1, the last formula takes the form

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y) + E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y)$$
$$+b(t)b^T(t) - (E(x(t)(x(t-h) - m(t-h))^T \mid F_t^Y)A^T(t)(B(t)B^T(t))^{-1}\times$$
$$A(t)E((x(t-h) - m(t-h))x^T(t)) \mid F_t^Y))dt + E((x(t) - m(t))(x(t) - m(t))$$
$$\times(A(t)(x(t-h) - m(t-h)))^T \mid F_t^Y)(B(t)B^T(t))^{-1}(dy(t) - A(t)m(t-h))dt).$$

Using the formula (5) for the term $E((x(t-h) - m(t-h))x^T(t)) \mid F_t^Y)$, the last equation can be represented as

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y) \tag{7}$$
$$+E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$-P(t)(\Phi^*(t-h,t))^T A^T(t)(B(t)B^T(t))^{-1}A(t)(\Phi^*(t-h,t))P(t))dt$$
$$+E(((x(t) - m(t))(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y)$$
$$\times(\Phi^*(t-h,t))^T A^T(t)(B(t)B^T(t))^{-1}(dy(t) - A(t)m(t-h))dt).$$

The equation (3) should be complemented with the initial condition $P(t_0) = E[(x(t_0) - m(t_0)(x(t_0) - m(t_0)^T \mid F_{t_0}^Y]$.

Thus, the equations (6) and (7) for the optimal estimate $m(t)$ and the error variance $P(t)$ form a non-closed system of the filtering equations for the nonlinear state (1) over linear observations with delay (2). In the next section, it will be shown that this system becomes a closed system of the filtering equations for some nonlinear functions $f(x,t)$ in the equation (1), in particular, for polynomial functions, if the observation matrix $A(t)$ in (2) is invertible.

4. Optimal Filter for Polynomial State over Delayed Observations

Let us make for the system (1), (2) the same assumptions that were made in [5] to obtain a closed system of the filtering equations for a polynomial state over linear non-delayed observations. Namely, assume that 1) the nonlinear function $f(x,t)$ is a polynomial function of the state x with time-dependent coefficients (since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$), and 2) the matrix $A(t)$ in the observation equation (2) is invertible for any $t \geq t_0$. As was shown in [5], under these assumptions, the terms $E(f(x,t) \mid F_t^Y)$ in (6) and $E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$ in (7) can be expressed as functions of $m(t)$ and $P(t)$ and, thereby, a closed system of the filtering equations can be obtained proceeding from (5) and (6).

The basic property established in [5] claims that, if the matrix $A(t)$ in the linear observation equation (2) is invertible, the random variable $x(t)-m(t)$ is conditionally Gaussian for any $t \geq t_0$. Two following conclusions can be made at this point.

First, since the random variable $x(t)-m(t)$ is conditionally Gaussian, the conditional third moment $E((x(t)-m(t))(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y)$ of $x(t)-m(t)$ with respect to observations, which stands in the last term of the equation (7), is equal to zero, because the process $x(t)-m(t)$ is conditionally Gaussian. Thus, the entire last term in (7) is vanished and the following variance equation is obtained

$$dP(t) = (E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) \tag{8}$$
$$+E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$-P(t)(\Phi^*(t-h,t))^T A^T(t)(B(t)B^T(t))^{-1}A(t)(\Phi^*(t-h,t))P(t))dt,$$

with the initial condition $P(t_0) = E[(x(t_0)-m(t_0)(x(t_0)-m(t_0)^T \mid F_{t_0}^Y]$.

Second, if the nonlinear function $f(x,t)$ is a polynomial function of the state x with time-dependent coefficients, the expressions of the terms $E(f(x,t) \mid F_t^Y)$ in (6) and $E((x(t)-m(t))\ (f(x,t))^T \mid F_t^Y)$ in (8) would also include only polynomial terms of x. Then, those polynomial terms can be represented as functions of $m(t)$ and $P(t)$ using the following property of Gaussian random variable $x(t)-m(t)$: all its odd conditional moments, $m_1 = E[(x(t)-m(t)) \mid Y(t)], m_3 = E[(x(t)-m(t)^3 \mid Y(t)], m_5 = E[(x(t)-m(t))^5 \mid Y(t)]$, ... are equal to 0, and all its even conditional moments $m_2 = E[(x(t)-m(t))^2 \mid Y(t)], m_4 = E[(x(t)-m(t))^4 \mid Y(t)], ...$ can be represented as functions of the variance $P(t)$. For example, $m_2 = P, m_4 = 3P^2, m_6 = 15P^3$, etc. After representing all polynomial terms in (6) and (8), that are generated upon expressing $E(f(x,t) \mid$

F_t^Y) and $E((x(t) - m(t))(f(x,t))^T \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$, a closed form of the filtering equations would be obtained. The corresponding representations for $E(f(x,t) \mid F_t^Y)$ and $E((x(t) - m(t))(f(x,t))^T \mid F_t^Y)$ have been derived in [5, 6] for certain polynomial functions $f(x,t)$.

In the next subsection, a closed form of the filtering equations will be obtained from (6) and (8) under the made assumptions, if the nonlinear function $f(x,t)$ in (1) is a bilinear polynomial. It should be noted, however, that application of the same procedure would result in designing a closed system of the filtering equations for any polynomial function $f(x,t)$ in the state equation (1).

4.1. Optimal Filter for Bilinear State over Delayed Observations

In a particular case, if the function

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T \tag{9}$$

is a bilinear polynomial, where x is now an n-dimensional vector, a_1 is an $n \times n$ - matrix, and a_2 is a 3D tensor of dimension $n \times n \times n$, the representations for $E(f(x,t) \mid F_t^Y)$ and $E((x(t) - m(t))(f(x,t))^T \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ are derived as follows (see [5])

$$E(f(x,t) \mid F_t^Y) = a_0(t) + a_1(t)m(t) \tag{10}$$

$$+a_2(t)m(t)m^T(t) + a_2(t)P(t),$$

$$E((x(t) - m(t))(f(x,t))^T \mid F_t^Y) = a_1(t)P(t) + P(t)a_1^T(t) \tag{11}$$

$$+2a_2(t)m(t)P(t) + 2(a_2(t)m(t)P(t))^T.$$

Substituting the expression (10) in (6) and the expression (11) in (8), the filtering equations for the optimal estimate $m(t)$ of the bilinear state $x(t)$ and the error variance $P(t)$ are obtained

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t))dt \tag{13}$$

$$+P(t)(\Phi^*(t-h,t))^T A^T(t)(B(t)B^T(t))^{-1}[dy(t) - A(t)m(t)dt],$$

$$m(t_0) = E(x(t_0) \mid F_t^Y)),$$

$$dP(t)$$

$$= (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + 2(a_2(t)m(t)P(t))^T \tag{14}$$

$$+ b(t)b^T(t))dt$$
$$- P(t)(\Phi^*(t-h,t))^T A^T(t)(B(t)B^T(t))^{-1} A(t)(\Phi^*(t-h,t))P(t)dt.$$

$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid F_t^Y)),$$

where $\Phi^*(t-h,t)$ is such a matrix that $\Phi(\tau,t)x_1^*(t) = x_1^*(\tau)$, $\tau \le t$, where $x_1^*(t)$ is the solution of the equation $\dot{x}_1^*(t) = f(x_1^*, t)$ with the initial condition $x_1^*(t_0) = m_0 = E(x(t_0) \mid F_{t_0}^Y)$, if $x_1^*(t) \neq 0$, and $\Phi^*(t-h,t) = 0$, if $x_1^*(t) = 0$. One of possible ways to define such a matrix $\Phi^*(t-h,t)$ by components is described in Section 3.

Thus, based on the general non-closed system of the filtering equations (6), (7) for a nonlinear state (1) over linear delayed observations (2), it is proved that the closed system of the filtering equations (6), (8) can be obtained for any polynomial state over linear delayed observations with an invertible observation matrix. Furthermore, the specific form (13), (14) of the closed system of the filtering equations corresponding a bilinear state is obtained. In the next section, performance of the designed optimal filter for a bilinear state over delayed observations is verified against the optimal bilinear filter for linear observations without delays, obtained in [5], and the conventional extended Kalman-Bucy filter.

5. Example

This section presents an example of designing the optimal filter for a quadratic state over linear observations with delay and comparing it to the best filter available for a quadratic state without delay, that has recently been obtained in [5], and to the conventional extended Kalman-Bucy filter.

Let the unobserved state $x(t)$ satisfies the quadratic equation

$$\dot{x}(t) = x^2(t), \quad x(0) = 1, \tag{15}$$

and the observation process is given by the linear delay-differential equation

$$y(t) = x(t - 0.8) + \psi(t), \tag{16}$$

where $\psi(t)$ is a white Gaussian noise, which is the weak mean square derivative of a standard Wiener process (see [24]). The equations (15) and (16) present the conventional form for the equations (1) and (2), which is actually used in practice [4].

The filtering problem is to find the optimal estimate for the quadratic state (15), using delayed linear observations (16) confused with independent and identically distributed disturbances modelled as white Gaussian noises. Let us set the filtering horizon time to $T = 4$.

The filtering equations (6), (8) take the following particular form for the system (15), (16)

$$\dot{m}(t) = m^2(t) + P(t) + P(t)\Phi^*(t - 0.8, t)[y(t) - m(t - 0.8)], \tag{17}$$

with the initial condition $m(0) = E(x(0) \mid y(0)) = m_0$,

$$\dot{P}(t) = 4P(t)m(t) - P^2(t)(\Phi^*(t - 0.8, t))^2, \tag{18}$$

with the initial condition $P(0) = E((x(0) - m(0))^2 \mid y(0)) = P_0$. The auxiliary variable $\Phi^*(t - 0.8, t)$ is equal to $\Phi^*(t - 0.8, t) = x_1^*(t - 0.8)/x_1^*(t)$ for $t \geq 0.8$, and $\Phi^*(t - 0.8, t) = 0$ for $t < 0.8$, where $x_1^*(t)$ is the solution of the equation

$$\dot{x}_1^*(t) = (x_1^*(t))^2,$$

with the initial condition $x_1^*(0) = m_0$.

The estimates obtained upon solving the equations (17)–(18) are compared to the estimates satisfying the following filtering equations for the quadratic state (15) over the linear observations (16) (obtained in [5]), without taking into account the gain adjustment provided by $\Phi^*(t - 0.8, t)$:

$$\dot{m}_1(t) = m_1^2(t) + P_1(t) + P_1(t)[y(t) - m_1(t - 0.8)], \tag{19}$$

with the initial condition $m_1(0) = E(x(0) \mid y(0)) = m_0$,

$$\dot{P}_1(t) = 4P_1(t)m_1(t) - P_1^2(t), \tag{20}$$

with the initial condition $P_1(0) = E((x(0) - m(0))^2 \mid y(0)) = P_0$.

Moreover, the estimates obtained upon solving the equations (17)–(18) are also compared to the estimates satisfying the following extended Kalman-Bucy filtering equations for the quadratic state (15) over the linear observations (16), obtained using the direct copy of the state dynamics (15) in the estimate equation and assigning the filter gain as the solution of the Riccati equation:

$$\dot{m}_2(t) = m_2^2(t) + P_2(t)[y(t) - m_2(t - 0.8)], \tag{21}$$

with the initial condition $m_2(0) = E(x(0) \mid y(0)) = m_0$,

$$\dot{P}_2(t) = 2P_2(t) - P_2^2(t), \tag{22}$$

with the initial condition $P_2(0) = E((x(0) - m(0))^2 \mid y(0)) = P_0$.

Numerical simulation results are obtained solving the systems of filtering equations (17)–(18), (19)–(20), and (21)–(22). The obtained values of the estimates $m(t)$, $m_1(t)$, and $m_2(t)$ satisfying the equations (17), (19), and (21), respectively, are compared to the real values of the state variable $x(t)$ in (15).

For each of the three filters (17)–(18),(19)–(20), and (21)–(22) and the reference system (15) involved in simulation, the following initial values are assigned: $x_0 = 1.1$, $m_0 = 0.1$, $P_0 = 1$. Gaussian disturbances $\psi_1(t)$ and $\psi_2(t)$ in (9) are realized using the built-in *MatLab* white noise function.

The following graphs are obtained: graph of the reference state variable $x(t)$ for the system (15); graph of the optimal filter estimate $m(t)$ satisfying the equations (17)–(18); graph of the estimate $m_1(t)$ satisfying the equations (19)–(20); graph of the estimate $m_2(t)$ satisfying the equations (21)–(22). The graphs of all those variables are shown on the entire simulation interval from $T = 0$ to $T = 4$ (Figure 1). Note that the gain matrices $P(t)$, and $P_1(t)$, do not converge to zero as time tends to infinity, since the polynomial dynamics of third order is stronger than the quadratic Riccati terms in the right-hand sides of the equations (18) and (20).

The following values of the reference state variable $x(t)$ and the estimates $m(t)$, $m_1(t)$, $m_2(t)$ are obtained at the reference time points $T = 1, 2, 3, 4$: for $T = 1$, $x(1) = 1.221$, $m(1) = 0.377$, $m_1(1) = 0.272$, $m_2(1) = 0.205$; for $T = 2$, $x(2) = 1.372$, $m(2) = 1.1$, $m_1(2) = 0.744$, $m_2(2) = 0.611$; for $T = 3$, $x(3) = 1.566$, $m(3) = 1.533$, $m_1(3) = 1.136$, $m_2(3) = 0.901$; for $T = 4$, $x(4) = 1.821$, $m(4) = 1.820$, $m_1(4) = 1.466$, $m_2(4) = 1.119$.

Thus, it can be concluded that the obtained optimal filter (17)–(18) for a quadratic state over linear observations with delay yield definitely better estimates than the conventional filter for a quadratic state over linear observations without delay (19)–(20) or the conventional extended Kalman-Bucy filter (21)–(22). Subsequent discussion of the obtained simulation results can be found in the following section.

6. Conclusions

The simulation results show that the values of the estimate calculated by using the obtained optimal filter for a quadratic state over linear observations with delay are noticeably closer to the real values of the reference variable than the values of the estimate satisfying the best available filter for a quadratic state over linear observations without delay or the estimate given by the conventional

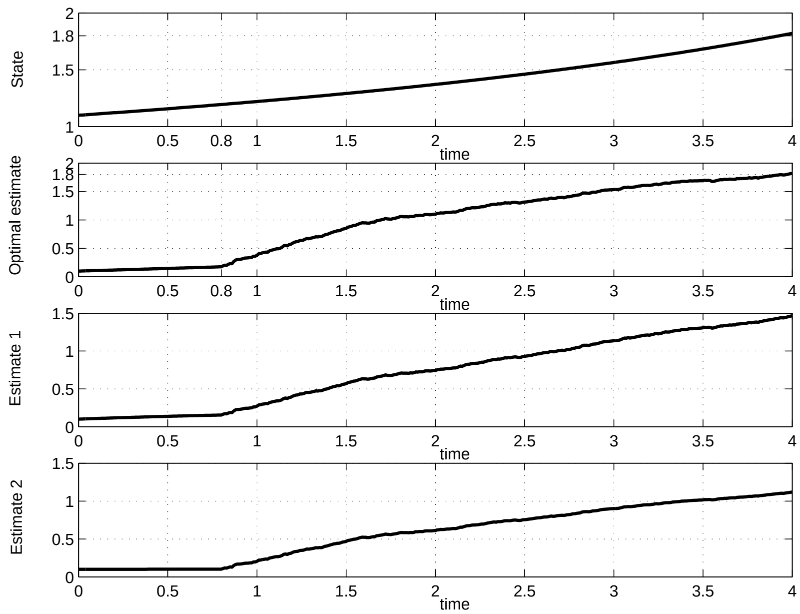

Figure 1: Graph of the reference state variable $x(t)$ (State); graph of the optimal filter estimate $m(t)$ satisfying the equations (17)–(18) (Optimal estimate); graph of the estimate $m_1(t)$ satisfying the equations (19)–(20) (Estimate 1); graph of the estimate $m_2(t)$ satisfying the equations (21)–(22) (Estimate 2) on the entire simulation interval $[0, 4]$.

extended Kalman-Bucy filter. This significant improvement in the estimate behavior is obtained due to the more careful selection of the filter gain matrix in the equations (19)–(20), as it should be in the optimal filter. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

It should however be noted that all filters treated in Example, including the obtained optimal filter (13)–(14), are not stable if the unobserved system state itself is unstable. The observability does not automatically yields stability here, because the innovations process, linear in the estimate m, is not able to compensate for polynomial terms of higher order in the drift part of the estimate equation, regardless of selection of the filter gain matrix. Thus, there exists the other problem of designing a stable polynomial filter, i.e., such that its error converges to zero as time tends to infinity, for a polynomial and observable but unstable state over linear observations.

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