International Journal of Pure and Applied Mathematics
Volume 12 No. 3 2004, 333-356

REGULAR METHODS OF SUMMABILITY AND ABSTRACT URYSOHN-TYPE OPERATORS

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Abstract: We study the summability properties for a class of nonlinear integral operators of the form $(T_w f)(s) = \int_{H_w} K_w(s, t, f(t)) d\mu_w(t)$ in some function spaces.

AMS Subject Classification: 40C10, 47G10, 46E30, 40J05
Key Words: locally compact spaces, Orlicz spaces, nonlinear integral operators, sampling series

1. Introduction

Summability represents an important tool in approximation theory. It originates from the classical Fourier analysis. We can quote, for example, the classical Cesaro-summability for Fourier series and the well-known Fejer Theorem. In general, given a family of operators $(T_w)_{w>0}$, one studies the assumptions under which a convergent family $(f_w)_{w>0}$ is transformed into a convergent family $(T_w f_w)_{w>0}$, in some sense. Let us note that this is not true in general as it is shown by examples in Section 6, by using the net of generalized sampling operators (see [24], [11], [9], [12]).

Received: February 28, 2004

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The study of summability methods by means of a net of nonlinear integral operators given by

$$(\widetilde{T}_w f)(s) = \int_G K_w(t-s, f(t))d\mu(t) = \int_G K_w(t, f(t+s))d\mu(t), \quad s \in G, \qquad (I)$$

where G is a locally compact Hausdorff topological group and μ is invariant, started in some general function spaces by the work of J. Musielak in [21] and later on by B. Tomasz in [25] and [26]. Other contributions to strong summability in function spaces are given in [19] and [22] (see also [5]). Further investigations are obtained by A. Waszak in [27] (see also [4]).

In this paper we will consider a general family of non convolution-type operators

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t))d\mu_{H_w}(t), \quad s \in G, \ w > 0, \qquad (II)$$

where G is a locally compact Hausdorff topological space provided with a regular measure μ_G, $(H_w)_{w>0}$ is a net of closed subspaces of G such that $\overline{\cup_{w>0} H_w} = G$, provided with a regular measure μ_{H_w} for every $w > 0$, and $f \in \operatorname{Dom} \mathbf{T} = \cap_{w>0} \operatorname{Dom} T_w$, where $\operatorname{Dom} T_w$ is the set on which $T_w f$ is well defined as a μ_G-measurable function of $s \in G$.

In [3] we studied approximation properties of (II) in Orlicz spaces. Here we apply such operators to summability problems of a family of functions $(f_w)_{w>0}$. The convergence is meant in pointwise, uniform and modular sense. In particular, in the last case, we investigate the regularity of T_w-method with respect to different modulars in Orlicz spaces. Theorem 2 below gives this regularity property in suitable subclasses of the Orlicz space and in Section 6, using again generalized sampling operators, we show that it is not true in general for the whole Orlicz space.

An important tool is an embedding theorem which shows that, under suitable assumptions, the Orlicz space is completely embedded in $\operatorname{Dom} \mathbf{T}$. Choosing G and H_w in a suitable way, we can obtain a very large number of examples, from convolution-type operators like (I) to discrete sampling-type operators (see [6], [7], [8], [5]). Then we obtain a general theory for the study of sampling operators which has interesting applications to signal analysis, see [9], [11], [12], [13], [15]. Finally we furnish some examples of kernels $(K_w)_{w>0}$ which satisfy the required assumptions.

2. Preliminaries

Let G be a locally compact Hausdorff topological space, provided with its family of Borel sets $\mathcal{B}$. Let μ_G be a regular measure defined on $\mathcal{B}$. Moreover let $H \subset G$ be a nonempty closed subset of G and let μ_H be another regular measure on the Borel σ-algebra generated by the set $\{A \cap H : A \text{ open set of } G\}$.

We will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset G \times G$ which generates the topology of G (see [28]). For every $s \in G$ and $U \in \mathcal{U}$ we put $U_s = \{z \in G : (s, z) \in U\}$ and, by local compactness, we assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets.

We will denote by $X(G)$ the space of all real-valued measurable functions on G with equality μ_G-almost everywhere. Let us denote by $C(G)$ the space of all bounded and continuous functions $f : G \to \mathbb{R}$ and by $C_c(G)$ the space of all continuous functions with compact support.

Let us recall that a function $f : G \to \mathbb{R}$ is uniformly continuous on G if for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|f(t) - f(s)| < \varepsilon$ for every $(s, t) \in U$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is continuous, nondecreasing function and $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$. We denote by $\widetilde{\Psi}$ the subset of Ψ consisting of functions ψ satisfying the following further assumption:

There is a constant $0 < M \leq 1$ such that $\psi(au) \geq Ma\psi(Mu)$, for every $u \geq 0$ and $0 \leq a \leq 1$.

Let us note that every concave function $\psi \in \Psi$ belongs to $\widetilde{\Psi}$, with $M = 1$.

For a given $\psi \in \Psi$, let $\mathcal{K}_\psi$ be the class of all functions $K : G \times H \times \mathbb{R} \to \mathbb{R}$ such that the following conditions hold:

K.1) $K(\cdot, \cdot, u)$ is measurable on $G \times H$ for every $u \in \mathbb{R}$ and $K(s, t, 0) = 0$, for every $(s, t) \in G \times H$.

K.2) K is $(L, \psi)-$Lipschitz, i.e. there are a function $L : G \times H \to \mathbb{R}_0^+$ and a constant $D > 0$ such that $L(s, t)$ is globally measurable on $G \times H$,

$$0 < \beta(s) := \int_H L(s, t) d\mu_H(t) \leq D,$$

for all $s \in G$ and

$$|K(s, t, u) - K(s, t, v)| \leq L(s, t)\psi(|u - v|),$$

for every $s \in G$, $t \in H$, $u, v \in \mathbb{R}$.

If $K \in \mathcal{K}_\psi$ we take into consideration the following nonlinear integral operator

$$(Tf)(s) = \int_H K(s,t,f(t))d\mu_H(t),$$

with $s \in G$, $f \in \mathrm{Dom}\, T$ where $\mathrm{Dom}\, T$ is the subset of $X(G)$ on which Tf is well defined as a μ_G-measurable function of $s \in G$.

In order to introduce the notion of Orlicz space, we give the following concepts.

Let Φ be the class of all functions $\varphi : I\!R_0^+ \to I\!R_0^+$ such that

i) φ is continuous, non decreasing,

ii) $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

Moreover we denote by $\widetilde{\Phi}$ the subspace of Φ whose elements are convex functions.

For $\varphi \in \Phi$, we define the functional

$$\varrho_G^\varphi(f) = \int_G \varphi(|f(s)|)d\mu_G(s),$$

for every $f \in X(G)$.

As it is well known, ϱ_G^φ is a modular on $X(G)$ and the subspace

$$L^\varphi(G) = \{f \in X(G) : \varrho_G^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ (see [20], [23]). If $\varphi \in \widetilde{\Phi}$, then ϱ_G^φ is a convex modular.

By $L^\varphi(H)$ we denote the space of all functions $f \in X(G)$ such that the restriction $f_{|H}$ belongs to the Orlicz space generated by $\varrho_H^\varphi(f) = \int_H \varphi(|f(t)|)d\mu_H(t)$.

In [3] we proved that $C(G) \subset \mathrm{Dom}\, T$ and in an analogous way we have that $L^\infty(G) \subset \mathrm{Dom}\, T$.

Now we can prove the following simple embedding result.

Proposition 1. *Let $\psi \in \widetilde{\Psi}$ and $K \in \mathcal{K}_\psi$. If for every $s \in G$ and $f \in L^\varphi(G)$ we have*

$$\int_H L(s,t)|f(t)|d\mu_H(t) < +\infty,$$

then

$$L^\varphi(G) \subset \mathrm{Dom}\, T.$$

Proof. By assumptions we have that $(Tf)(s)$ is a measurable function of the variable $s \in G$. Now let $s \in G$ be fixed. We have, for every $f \in L^{\varphi}(G)$,

$$|K(s,t,f(t))| \leq L(s,t)\psi(|f(t)|).$$

By the properties of $\psi \in \widetilde{\Psi}$, for every $u \geq 1$ there results

$$u\psi(1) \geq M\psi(Mu),$$

for $M \in]0,1]$. Let us put

$$A = \{t \in H : |f(t)| \geq M\},$$

and $B = H \setminus A$. We obtain

$$\begin{aligned}|K(s,t,f(t))| &\leq L(s,t)\psi(|f(t)|) = L(s,t)\psi(|f(t)|)\chi_A(t)\\ &+ L(s,t)\psi(|f(t)|)\chi_B(t) \leq L(s,t)|f(t)|M^{-2}\psi(1) + L(s,t)\psi(M),\end{aligned}$$

and then

$$\int_H |K(s,t,f(t))| d\mu_H(t) \leq \psi(1)M^{-2}\int_H L(s,t)|f(t)|d\mu_H(t) + \psi(M)D.$$

Thus we proved that

$$\int_H |K(s,t,f(t))| d\mu_H(t) < +\infty,$$

for $s \in G$.

Remark 1. Let us suppose that $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an N-function, i.e. it is convex, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and

$$\lim_{u \to 0^+} \frac{\varphi(u)}{u} = 0, \quad \lim_{u \to +\infty} \frac{\varphi(u)}{u} = +\infty.$$

The function $\varphi^* : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, defined by the formula

$$\varphi^*(u) = \sup_{v>0}(uv - \varphi(v)),$$

is called the conjugate to φ in the sense of Young and it is also a N-function (see [20], [23]).

We denote by $L^{\varphi}(G)$ and $L^{\varphi^*}(G)$ the Orlicz spaces generated by the following functionals

$$\varrho_G^{\varphi}(f) = \int_G \varphi(|f(s)|)d\mu_G(s), \quad \varrho_G^{\varphi^*}(f) = \int_G \varphi^*(|f(s)|)d\mu_G(s),$$

for $f \in X(G)$.

Using the Young inequality

$$uv \leq \varphi(u) + \varphi^*(v)$$

for $u, v \geq 0$, we have for every $\lambda > 0$

$$\lambda^2 L(s,t)|f(t)| \leq \varphi(\lambda|f(t)|) + \varphi^*(\lambda L(s,t)).$$

Then we obtain

$$\int_H L(s,t)|f(t)|d\mu_H(t) \leq \frac{1}{\lambda^2}\int_H \varphi(\lambda|f(t)|)d\mu_H(t) + \frac{1}{\lambda^2}\int_H \varphi^*(\lambda L(s,t))d\mu_H(t) = \frac{1}{\lambda^2}\varrho_H^{\varphi}(\lambda f) + \frac{1}{\lambda^2}\varrho_H^{\varphi^*}(\lambda L(s,\cdot)).$$

If $f \in L^{\varphi}(H)$, we have $\varrho_H^{\varphi}(\lambda f) < +\infty$ for sufficiently small $\lambda > 0$.

If we suppose that $L(s,\cdot) \in L^{\varphi^*}(H)$, then also $\varrho_H^{\varphi^*}(\lambda L(s,\cdot)) < +\infty$ for sufficiently small $\lambda > 0$. This means that the assumption of the previous proposition is satisfied if $f \in L^{\varphi}(G) \cap L^{\varphi}(H)$ and $L(s,\cdot) \in L^{\varphi^*}(H)$. So we have the following corollary.

Corollary 1. *Let φ and φ^* be a pair of N-functions conjugate in the sense of Young. Let $\psi \in \widetilde{\Psi}$ and $K \in \mathcal{K}_{\psi}$. Moreover let $L(s,\cdot) \in L^{\varphi^*}(H)$ for every $s \in G$ and let*

$$(Tf)(s) = \int_H K(s,t,f(t))d\mu_H(t).$$

Then

$$L^{\varphi}(G) \cap L^{\varphi}(H) \subset \operatorname{Dom} T.$$

3. Summability Methods

Let $\psi \in \Psi$ be fixed and let $I\!\!K = (K_w)_{w>0} \subset \mathcal{K}_\psi$ be a family of functions. We denote by $I\!\!L = (L_w)_{w>0}$ the corresponding class of functions for which the Lipschitz condition holds for any $w > 0$.

For a given $I\!\!K = (K_w)_{w>0} \subset \mathcal{K}_\psi$ we will consider the family $\mathbf{T} = (T_w)_{w>0}$ of operators defined by

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t)) d\mu_{H_w}(t),$$

with $s \in G$, $f \in \operatorname{Dom} \mathbf{T} = \cap_{w>0} \operatorname{Dom} T_w$.

For every $w > 0$, H_w is a nonempty closed set with $H_w \subset G$ and $\overline{\cup_{w>0} H_w} = G$. Every H_w is provided with the regular measure $\mu_{H_w} = \mu_w$.

We will say that $I\!\!K$ is *singular* if the following assumptions hold (see [3]):

1) There is $D > 0$ such that for every $s \in G$ and $w > 0$ we have

$$0 < \beta_w(s) = \int_{H_w} L_w(s, t) d\mu_w(t) \leq D.$$

2) For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w \to +\infty} \int_{H_w \setminus U_s} L_w(s, t) d\mu_w(t) = 0.$$

3) For every $s \in G$ and for every $u \in I\!\!R$ we have

$$\lim_{w \to +\infty} \int_{H_w} K_w(s, t, u) d\mu_w(t) = u.$$

We will say that $I\!\!K$ is *uniformly singular* if the conditions 2) and 3) are replaced by the following ones

2') For every $U \in \mathcal{U}$ we have

$$\lim_{w \to +\infty} \int_{H_w \setminus U_s} L_w(s, t) d\mu_w(t) = 0$$

uniformly with respect to $s \in G$.

3') We have

$$\lim_{w\to+\infty}\int_{H_w} K_w(s,t,u)d\mu_w(t) = u$$

uniformly with respect to $s \in G$ and $u \in C$, where C is any compact subset of $\mathbb{R}\setminus\{0\}$.

Remark 2. Let us note that the condition $\overline{\cup_{w>0}H_w} = G$ is necessary. Indeed this condition comes from the assumptions of singularity, which involve the subsets H_w throughout the pointwise convergence Theorem 5 in [3].

In this section we investigate the convergence problem of the transformed net $(T_w f_w)_{w>0}$.

First we obtain the following summability theorem

Theorem 1. *Let $\psi \in \Psi$, $\mathbb{K} = (K_w)_{w>0} \subset \mathcal{K}_\psi$ be singular and $(f_w)_{w>0} \subset L^\infty(G)$. If $f_w \to f$ uniformly in G, then*

$$\lim_{w\to+\infty}(T_w f_w)(s) = f(s)$$

at every $s \in G$ in which f is continuous.

Moreover let $\mathbb{K} = (K_w)_{w>0} \subset \mathcal{K}_\psi$ be uniformly singular and $(f_w)_{w>0} \subset C(G)$ be a net of uniformly continuous functions. If $f_w \to f$ uniformly in G then

$$\lim_{w\to+\infty}||T_w f_w - f||_\infty = 0.$$

Proof. We prove only the second part of the theorem. In an analogous way one can obtain the first one. By Lemma 1 of [3] we have that $C(G) \subset \mathrm{Dom}\,\mathbf{T}$. By the fact that $f_w \in C(G)$ for every $w > 0$ and by uniform convergence, we have $f \in C(G)$ and so $f \in \mathrm{Dom}\,\mathbf{T}$. This means that we can evaluate $T_w f$ for every $w > 0$. We obtain

$$\begin{aligned}
|(T_w f_w)(s) - f(s)| &\le |(T_w f_w)(s) - (T_w f)(s)| + |(T_w f)(s) - f(s)| \\
&\le \int_{H_w} |K_w(s,t,f_w(t)) - K_w(s,t,f(t))| d\mu_w(t) + |(T_w f)(s) - f(s)| \\
&\le \int_{H_w} L_w(s,t)\psi(|f_w(t) - f(t)|)d\mu_w(t) + |(T_w f)(s) - f(s)| = I_1 + I_2.
\end{aligned}$$

Now we estimate I_1.

By uniform convergence for every $\varepsilon > 0$ there is $\overline{w}$ such that, for every $w \ge \overline{w}$ and for every $t \in H_w$,

$$|f_w(t) - f(t)| < \varepsilon.$$

So we have

$$I_1 \leq \psi(\varepsilon) \int_{H_w} L_w(s,t) d\mu_w(t) \leq \psi(\varepsilon) D,$$

and then $I_1 \to 0$ uniformly if $w \to +\infty$. Now we consider I_2. We have

$$I_2 = |(T_w f)(s) - f(s)| \leq ||T_w f - f||_\infty.$$

By uniform convergence we have that f is also uniformly continuous and so by Theorem 5 of [3] we have finally that $I_2 \to 0$ uniformly as $w \to +\infty$. The theorem is now proved. □

In order to obtain a convergence result in Orlicz spaces, we introduce the following notion. We say that a sequence $(f_w)_{w>0} \subset L^\varphi(G)$ is modularly convergent to $f \in L^\varphi(G)$ if there is $\lambda > 0$ such that

$$\lim_{w \to +\infty} \varrho^\varphi_G[\lambda(f_w - f)] = 0.$$

We also can say that $(f_w)_{w>0}$ is ϱ^φ_G-convergent. This notion extends the norm-convergence in L^p spaces and it is weaker than the convergence induced by Luxenburg norm generated by ϱ^φ_G, see [20] and [23]. In the following, if $\eta \in \Phi$, we define the modular functionals

$$\varrho^\eta_G(f) = \int_G \eta(|f(s)|) d\mu_G(s), \quad \varrho^\eta_{H_w}(f) = \int_{H_w} \eta(|f(t)|) d\mu_w(t),$$

and we denote the corresponding Orlicz spaces by $L^\eta(G)$ and $L^\eta(H_w)$.

From now on we will denote by $\mathcal{X}_\psi$ the subclass of all kernels $\mathbb{K} = (K_w)_{w>0} \subset \mathcal{K}_\psi$ satisfying the further conditions:

a) for every $w > 0$ there exists $d_w > 0$ such that

$$d_w \leq \beta_w(s) = \int_{H_w} L_w(s,t) d\mu_w(t),$$

for every $s \in G$,

b) for every $w > 0,\ t \in H_w$

$$\int_G L_w(s,t) d\mu_G(s) \leq \gamma_w \leq D,$$

where $(\gamma_w)_{w>0}$ is a bounded net of positive constants.

In particular we have

$$\int_G \frac{L_w(s,t)}{\beta_w(s)} d\mu_G(s) \leq \frac{\gamma_w}{d_w} := \xi_w,$$

for every $t \in H_w$ and $w > 0$.

We remark that if we consider operators of convolution type of the form

$$(T_w f)(s) = \int_G K_w(t-s, f(t)) d\mu_G(t),$$

where $(G,+)$ is a topological group and $G = H_w$ for every $w > 0$, we have that $\mathcal{K}_\psi = \mathcal{X}_\psi$.

In the following we will assume the further condition: let $\varphi \in \Phi$ be such that there is a function $\eta \in \Phi$ with

$$(\varphi \circ \psi) \leq \eta, \tag{1}$$

where $\psi \in \Psi$ is the function in the definition of Lipschitz condition for the kernel K.

Let $(f_w)_{w>0} \subset L^\eta(G) \cap \operatorname{Dom} \mathbf{T}$ be a family of functions. We say that $(f_w)_{w>0}$ is $\varrho^\eta_{H_w}$−convergent to a function $f \in L^\eta(G)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho^\eta_{H_w}[\lambda(f_w - f)] = 0.$$

In the following we will assume $\lambda = 1$. If $\lambda < 1$ we only slightly modify inequality (1) (see [5]).

Given $\{H_w\}_{w>0} \subset G$ and a constant $P > 0$, we denote by $\mathcal{L}_{H_w}(P)$ the subset of $L^\eta(G)$ whose elements f satisfy the following condition

$$\limsup_{w\to+\infty} \xi_w \varrho^\eta_{H_w}(f) \leq P \varrho^\eta_G(f).$$

Let $\mathcal{F} \subset L^\varphi(G)$ be a non empty subset. A family $(f_w)_{w>0}$ of functions with $f_w \in \operatorname{Dom} \mathbf{T}$ is $(\mathbf{T}, \varrho^\varphi_G)$-summable to a function $f \in \mathcal{F}$ if $T_w f_w \to f$ modularly with respect to ϱ^φ_G.

We will say that this method of summability is $(\varrho^\eta_{H_w}, \varrho^\varphi_G)$-*conservative* in $\mathcal{F}$, if for every family $(f_w)_{w>0} \subset \operatorname{Dom} \mathbf{T} \cap L^\eta(G)$, $\varrho^\eta_{H_w}$-convergent to a function $f \in \mathcal{F}$, there exists a function $g \in L^\varphi(G)$ such that the net $(T_w f_w)_{w>0}$ is ϱ^φ_G-convergent to g. If we have $g = f$ we will say that the method is *regular* from $L^\eta(G)$ to $\mathcal{F}$. Finally we will say that the method is *locally regular* from $L^\eta(G)$ to $\mathcal{F}$ if the ϱ^φ_G-convergence to f is local in the sense that there exists a constant $\lambda > 0$ such that

$$\varrho^\varphi_G[\lambda(T_w f_w - f)\chi_S] \to 0, \quad w \to +\infty,$$

being S any compact subset of G.

We are now ready to prove a regularity property of the $\mathbf{T}$-method, given by the family $(T_w)_{w>0}$. At first we introduce the following subspace of $L^{\varphi}(G)$

$$\mathcal{F} = \{f \in L^{\varphi+\eta}(G) : f - C_c(G) \subset \mathcal{L}_{H_w}(P), \text{ for some } P > 0\}.$$

We have the following theorem.

Theorem 2. *Let $\psi \in \widetilde{\Psi}$, $\varphi \in \widetilde{\Phi}$ and $\eta \in \Phi$ be functions satisfying property (1) and $\mathbb{K} \in \mathcal{X}_{\psi}$ be singular.*

We assume that the family of functions $\mathbb{L}$ satisfies the following assumption: for every $f \in \mathcal{F}$, $w > 0$ and $s \in G$ there results

$$\int_{H_w} L_w(s,t)|f(t)|d\mu_w(t) < +\infty.$$

Then the $\mathbf{T}$-method given by the family of operators $(T_w)_{w>0}$ is locally regular from $L^{\eta}(G)$ to $\mathcal{F}$ in the sense of the previous definitions.

Proof. Let $(f_w)_{w>0}$ be a family of functions in $\operatorname{Dom}\mathbf{T} \cap L^{\eta}(G)$ which is $\varrho^{\eta}_{H_w}$-convergent to a function $f \in \mathcal{F}$. By Proposition 1 we have that $\mathcal{F} \subset \operatorname{Dom}\mathbf{T} = \bigcap_{w>0} \operatorname{Dom} T_w$. Then we have

$$\varphi(|(T_w f_w)(s) - f(s)|) \leq \frac{1}{2}\varphi(2|(T_w f_w)(s) - (T_w f)(s)|) + \frac{1}{2}\varphi(2|(T_w f)(s) - f(s)|).$$

For $2\lambda D \leq 1$, we obtain

$$\int_G \varphi(\lambda|(T_w f_w)(s) - f(s)|\chi_S) d\mu_G(s) \leq \frac{1}{2}\int_S \varphi(2\lambda|(T_w f_w)(s) - (T_w f)(s)|) d\mu_G(s) + \frac{1}{2}\int_S \varphi(2\lambda|(T_w f)(s) - f(s)|) d\mu_G(s) = I_1 + I_2.$$

By Theorem 9 of [3] we have that $I_2 \to 0$. Now we consider I_1. We have, by

Jensen inequality and property (1)

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_S \varphi(2\lambda|(T_wf_w)(s)-(T_wf)(s)|)d\mu_G(s)\\
&\leq \frac{1}{2}\int_S \varphi[2\lambda\int_{H_w} L_w(s,t)\psi(|f_w(t)-f(t)|)d\mu_w(t)]d\mu_G(s)\\
&\leq \frac{1}{2}\int_S\{\int_{H_w}\varphi[2\lambda D\psi(|f_w(t)-f(t)|)]\frac{L_w(s,t)}{D}d\mu_w(t)\}d\mu_G(s)\\
&\leq \frac{1}{2}\int_S\{\int_{H_w}\eta(|f_w(t)-f(t)|)\frac{L_w(s,t)}{D}d\mu_w(t)\}d\mu_G(s)\\
&= \frac{1}{2}\int_{H_w}\{\int_S\eta(|f_w(t)-f(t)|)\frac{L_w(s,t)}{D}d\mu_G(s)\}d\mu_w(t)\\
&\leq \frac{1}{2}\int_{H_w}\eta(|f_w(t)-f(t)|)d\mu_w(t)=\frac{1}{2}\varrho^\eta_{H_w}(f_w-f).
\end{aligned}$$

By the fact that the net $(f_w)_{w>0}$ is modularly convergent with respect to $\varrho^\eta_{H_w}$, we obtain that $I_1 \to 0$ and so the theorem is completely proved. □

4. Applications to Urysohn Operators

In this section, we apply the general theory developed in the previous sections to families of operators of the form

$$(\widetilde{T}_wf)(s)=\int_G K_w(s,t,f(t))d\mu_G(t), \quad s\in G, \tag{2}$$

i.e. we consider the particular case of $H_w = G$ and $\mu_w = \mu_G$ for every $w > 0$. We limit ourselves to consider the case of regularity with respect to the Orlicz modulars ϱ^φ_G and ϱ^η_G.

In [1] we studied approximation properties for families of integral operators of the previous type in abstract modular spaces.

In this present setting we can reformulate Proposition 1 for an operator $\widetilde{T}$ of the form $(\widetilde{T}f)(s)=\int_G K(s,t,f(t))d\mu_G(t)$, with $s\in G$ and G instead of H.

Corollary 1 assumes, in this instance, a simpler form. In fact we have the following corollary.

Corollary 2. *Let φ and φ^* be a pair of N-functions conjugate in the sense of Young. Let $\psi\in\widetilde{\Psi}$ and $K\in\mathcal{K}_\psi$. Moreover let $L(s,\cdot)\in L^{\varphi^*}(G)$ for* *every* $s \in G$ *and let*

$$(\widetilde{T}f)(s) = \int_G K(s,t,f(t))d\mu_G(t).$$

Then

$$L^{\varphi}(G) \subset \mathrm{Dom}\, T.$$

Concerning singularity assumptions for $I\!\!K = (K_w)_{w>0}$ we have the same conditions 1), 2), 3) and 2'), 3'), with G instead of H_w and $\mu_w = \mu_G$ for every $w > 0$, (see also [1]).

Summability method for $(T_w f_w)_{w>0}$ with respect to pointwise and uniform convergence is analogous to Theorem 1.

The situation is different, if we consider modular convergence. Indeed, in this case, we have

$$\mathcal{L}_{H_w}(P) = L^{\eta}(G),$$

for P sufficiently large, if we assume that the net $(\xi_w)_{w>0}$ is bounded.

We remark that the net $(\xi_w)_{w>0}$ is bounded if the net $(d_w)_{w>0}$ is bounded from below by a positive constant $d > 0$.

Now we reformulate the notions about summability given in the previous section. Let $\mathcal{F} \subset L^{\varphi}(G)$ be a non empty subset. A family $(f_w)_{w>0}$ of functions with $f_w \in \mathrm{Dom}\,\mathbf{T}$ is $(\mathbf{T}, \varrho^{\varphi}_G)$-summable to a function $f \in \mathcal{F}$ if $T_w f_w \to f$ modularly with respect to ϱ^{φ}_G.

This method of summability is $(\varrho^{\eta}_G, \varrho^{\varphi}_G)$-*conservative* in $\mathcal{F}$, if for every net $(f_w)_{w>0} \subset L^{\eta}(G) \cap \mathrm{Dom}\,\mathbf{T}$ ϱ^{η}_G- convergent to a function $f \in \mathcal{F}$, there exists a function $g \in L^{\varphi}(G)$ such that the net $(T_w f_w)_{w>0}$ is ϱ^{φ}_G- convergent to g. If $g = f$ the method is *regular* from $L^{\eta}(G)$ to $\mathcal{F}$. The method is *locally regular* from $L^{\eta}(G)$ to $\mathcal{F}$ if the ϱ^{φ}_G-convergence to f is local in the sense that there exists a constant $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho^{\varphi}_G[\lambda(T_w f_w - f)\chi_S] = 0,$$

with S any compact subset of G.

So we can formulate the following summability result, which shows that the method given by $(T_w)_{w>0}$ is locally regular from $L^{\eta}(G)$ to $\mathcal{F} = L^{\varphi+\eta}(G)$.

Corollary 3. *Let* φ *and* φ^* *be a pair of N-functions conjugate in the sense of Young. Let* $\psi \in \widetilde{\Psi}$, $\eta \in \Phi$ *such that property (1) is satisfied and* $I\!\!K \in \mathcal{X}_{\psi}$ *be singular with* $d_w \geq d > 0$ *for every* $w > 0$. *We assume that for every* $s \in G$ *and* $w > 0$, $L_w(s,\cdot) \in L^{\varphi^*}(G)$. *Then the* $\mathbf{T}$*-method given by the family of operators* $(T_w)_{w>0}$ *is locally regular from* $L^{\eta}(G)$ *to* $L^{\varphi+\eta}(G)$.

Now we shortly discuss the linear case, i.e. operators of the form

$$(\widetilde{T}_w f)(s) = \int_G \widetilde{K}_w(s,t) f(t) d\mu_G(t),$$

where

$$K_w(s,t,u) = \widetilde{K}_w(s,t)u, \quad u \in I\!R,$$

with $\widetilde{K}_w : G \times G \to I\!R$ measurable functions on $G \times G$.

Then $I\!K = (\widetilde{K}_w)_{w>0}$ is (L,ψ)-Lipschitz with $\psi(u) = u$, $u \geq 0$ and $L_w(s,t) = |\widetilde{K}_w(s,t)|$. In this instance the notions of singularity and uniform singularity can be reformulated assuming the following conditions:

$\widetilde{\mathbf{K}}.\mathbf{1}$) There is $D > 0$ such that for every $s \in G$ and $w > 0$

$$\int_G |\widetilde{K}_w(s,t)| d\mu_G(t) \leq D.$$

$\widetilde{\mathbf{K}}.\mathbf{2}$) For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{G\setminus U_s} |\widetilde{K}_w(s,t)| d\mu_G(t) = 0.$$

$\widetilde{\mathbf{K}}.\mathbf{3}$) For every $s \in G$

$$\lim_{w\to+\infty} \int_G \widetilde{K}_w(s,t) d\mu_G(t) = 1.$$

In an analogous way we obtain conditions 2') and 3') for uniform singularity. Considering linear operator $\widetilde{T}$ of the form

$$(\widetilde{T} f)(s) = \int_G \widetilde{K}(s,t) f(t) d\mu_G(t),$$

with $s \in G$ we have that Proposition 1 is obviously satisfy and Corollary 2 holds with $L(s,\cdot) = |\widetilde{K}(s,\cdot)|$.

About summability with respect to modular convergence, here we have $\varphi = \eta$. For $I\!K = (\widetilde{K}_w)_{w>0} \subset \mathcal{X}_\psi$, we have $\mathcal{L}_{H_w}(P) = L^\varphi(G)$, if $\widetilde{\beta_w}(s) = \|\widetilde{K}_w(s,\cdot)\|_1 \geq d$, for some positive constant d.

The definitions of regularity take now the following form. A family $(f_w)_{w>0} \subset L^\varphi(G) \cap \mathrm{Dom}\,\mathbf{T}$ is $(\mathbf{T}, \varrho^\varphi_G)$- summable to a function $f \in L^\varphi(G)$ if $(T_w f_w)_{w>0}$ is ϱ^φ_G-convergent to f.

We say that this method of summability is ϱ^φ_G-conservative if for every family $(f_w)_{w>0} \subset L^\varphi(G) \cap \mathrm{Dom}\,\mathbf{T}$ ϱ^φ_G-convergent to a function $f \in L^\varphi(G)$, there exists a function $g \in L^{\varphi}(G)$ such that the net $(T_w f_w))_{w>0}$ is ϱ^{φ}_G-convergent to g. If we have $g = f$ we say that the method is regular in $L^{\varphi}(G)$. Moreover we say that the method is locally regular in $L^{\varphi}(G)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho^{\varphi}_G[\lambda(T_w f_w - f)\chi_S] = 0,$$

with S compact subset of G.

We obtain, finally, the following result.

Corollary 4. *Let φ and φ^* be a pair of N-functions conjugate in the sense of Young and $K \in \mathcal{X}_\psi$ be singular with $\widetilde{\beta_w}(s) \geq d > 0$, for every $w > 0$, $s \in G$. We assume that for every $s \in G$ and $w > 0$, $|\widetilde{K}_w(s,\cdot)| \in L^{\varphi^*}(G)$. Then the* **T***-method given by the family of operators $(\widetilde{T}_w)_{w>0}$ is locally regular in $L^{\varphi}(G)$.*

5. Applications to Discrete Operators

Let $G = (\mathbb{R}, +)$ and $H = (\boldsymbol{Z}, +)$ with μ_G the Lebesgue measure on $\mathbb{R}$ and μ_H the counting measure on $\boldsymbol{Z}$.

We obtain the following nonlinear discrete operator

$$(Tf)(s) = \sum_{k=-\infty}^{+\infty} K(s, k, f(k)),$$

with $s \in \mathbb{R}$ (see e.g. [2]).

Let us apply the embedding result in Orlicz spaces generating by functions $\varphi \in \Phi$. In this case the modular functionals are of the form

$$\varrho^{\varphi}_{\mathbb{R}}(f) = \int_{-\infty}^{+\infty} \varphi(|f(s)|)ds, \quad \varrho^{\varphi}_{\boldsymbol{Z}}(f) = \sum_{k=-\infty}^{+\infty} \varphi(|f(k)|),$$

and the corresponding Orlicz spaces are denoted as before $L^{\varphi}(\mathbb{R})$ and $L^{\varphi}(\boldsymbol{Z})$ respectively.

About embedding theorem we have the following corollaries.

Corollary 5. *Let $\psi \in \widetilde{\Psi}$ and $K \in \mathcal{K}_\psi$. If for every $s \in \mathbb{R}$ and $f \in L^{\varphi}(\mathbb{R})$ we have*

$$\sum_{k=-\infty}^{+\infty} L(s,k)|f(k)| < +\infty,$$

then

$$L^{\varphi}(\mathbb{R}) \subset \operatorname{Dom} T.$$

Corollary 6. *Let φ and φ^* be a pair of N-functions conjugate in the sense of Young. Let $\psi \in \widetilde{\Psi}$ and $K \in \mathcal{K}_\psi$. Moreover let $L(s,\cdot) \in L^{\varphi^*}(\boldsymbol{Z})$ for every $s \in I\!\!R$ and let*

$$(Tf)(s) = \sum_{k=-\infty}^{+\infty} K(s,k,f(k)).$$

Then

$$L^{\varphi}(I\!\!R) \cap L^{\varphi}(\boldsymbol{Z}) \subset \mathrm{Dom}\, T.$$

If K is linear, that is $K(s,k,u) = \widetilde{K}(s,k)u$ the operator takes the form

$$(\widetilde{T}f)(s) = \sum_{k=-\infty}^{+\infty} \widetilde{K}(s,k)f(k), \quad s \in I\!\!R,$$

and for the particular choice $\widetilde{K}(s,k) = F(s-k)$ the operator is a generalized sampling series (see [13]).

Now for every $w > 0$ we put $H_w = \frac{1}{w}\boldsymbol{Z}$ and we consider a family of operators

$$(T_w f)(s) = \sum_{k=-\infty}^{+\infty} K_w(s, \frac{k}{w}, f(\frac{k}{w})),$$

with $s \in I\!\!R$.

In this setting, for $\psi \in \Psi$, we say that the kernel $I\!\!K = (K_w)_{w>0} \subset \mathcal{K}_\psi$ is singular if:

1) there exists $D > 0$ such that

$$0 < \beta_w(s) = \sum_{k=-\infty}^{+\infty} L_w(s, \frac{k}{w}) \le D,$$

for every $s \in I\!\!R$ and $w > 0$.

2) For every $s \in I\!\!R$ and for every $\delta > 0$ we have

$$\lim_{w\to+\infty} \sum_{|sw-k|>\delta w} L_w(s, \frac{k}{w}) = 0.$$

3) For every $s \in I\!\!R$ and $u \in I\!\!R$

$$\lim_{w\to+\infty} \sum_{k=-\infty}^{+\infty} K_w(s, \frac{k}{w}, u) = u.$$

In an analogous way we define the notion of uniform singularity for the family $I\!K$.

Concerning summability in $L^\infty(I\!R)$ and $C(I\!R)$ we have the following corollary.

Corollary 7. *Let $\psi \in \Psi$, $I\!K = (K_w)_{w>0} \subset \mathcal{K}_\psi$ be singular and $(f_w)_{w>0} \subset L^\infty(I\!R)$. If $f_w \to f$ uniformly in $I\!R$ then*

$$\lim_{w\to+\infty} \sum_{k=-\infty}^{+\infty} K_w(s, \frac{k}{w}, f_w(\frac{k}{w})) = f(s)\,,$$

at every $s \in I\!R$, where f is continuous. Moreover let $I\!K = (K_w)_{w>0} \subset \mathcal{K}_\psi$ be uniformly singular and $(f_w)_{w>0} \subset C(I\!R)$ be a net of uniformly continuous functions. If $f_w \to f$ uniformly in $I\!R$ then

$$\lim_{w\to+\infty} ||T_w f_w - f||_\infty = 0.$$

In order to prove a regularity theorem in Orlicz spaces, we need the following modulars. If $\eta \in \Phi$ we put

$$\varrho^\eta_{I\!R}(f) = \int_{-\infty}^{+\infty} \eta(|f(s)|)ds, \quad \varrho^\eta_{\frac{1}{w}\boldsymbol{Z}}(f) = \sum_{k=-\infty}^{+\infty} \eta(|f(\frac{k}{w})|)\,,$$

and we denote the corresponding Orlicz spaces by $L^\eta(I\!R)$ and $L^\eta(\frac{1}{w}\boldsymbol{Z})$.

In this case the family of kernels $I\!K = (K_w)_{w>0} \subset \mathcal{X}_\psi$ satisfies the following conditions:

a) for every $w > 0$ there is $d_w > 0$ such that

$$d_w \le \beta_w(s) = \sum_{k=-\infty}^{+\infty} L_w(s, \frac{k}{w})\,,$$

for every $s \in I\!R$,

b) for every $w > 0$, $k \in \boldsymbol{Z}$

$$\int_{-\infty}^{+\infty} L_w(s, \frac{k}{w})ds \le \gamma_w \le D\,,$$

with $(\gamma_w)_{w>0}$ a bounded net.

In particular we have

$$\int_{-\infty}^{+\infty} \frac{L_w(s, \frac{k}{w})}{\beta_w(s)} ds \leq \frac{\gamma_w}{d_w} = \xi_w,$$

for every $w > 0$ and $k \in \boldsymbol{Z}$.

For a given constant $P > 0$, the class $\mathcal{L}_{\frac{1}{w}\boldsymbol{Z}}(P)$ consists of the functions $f \in L^\eta(I\!R)$ such that

$$\limsup_{w \to +\infty} \xi_w \sum_{k=-\infty}^{+\infty} \eta(|f(\frac{k}{w})|) \leq P \int_{-\infty}^{+\infty} \eta(|f(s)|) ds.$$

The notions of summability and regularity take now the following form. Let $\mathcal{F} \subset L^\varphi(I\!R)$ be a non empty subset. A family $(f_w)_{w>0}$ of functions with $f_w \in \operatorname{Dom} \mathbf{T}$ is $(\mathbf{T}, \varrho^\varphi_{I\!R})$-summable to a function $f \in \mathcal{F}$ if $T_w f_w \to f$ modularly with respect to $\varrho^\varphi_{I\!R}$. We say that this method of summability is $(\varrho^\eta_{\frac{1}{w}\boldsymbol{Z}}, \varrho^\varphi_{I\!R})$- conservative in $\mathcal{F}$, if for every family $(f_w)_{w>0} \subset \operatorname{Dom} \mathbf{T} \cap L^\eta(I\!R)$, $\varrho^\eta_{\frac{1}{w}\boldsymbol{Z}}$-convergent to a function $f \in \mathcal{F}$, there exists a function $g \in L^\varphi(I\!R)$ such that the net $(T_w f_w)_{w>0}$ is $\varrho^\varphi_{I\!R}$-convergent to g. If $g = f$ we say that the method is regular from $L^\eta(I\!R)$ to $\mathcal{F}$. We say that the method is locally regular from $L^\eta(I\!R)$ to $\mathcal{F}$ if the $\varrho^\varphi_{I\!R}$- convergence to f is local, that is there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} \varrho^\varphi_{I\!R}[\lambda(T_w f_w - f)\chi_S] = 0\,,$$

with S any compact subset of G.

We are ready to obtain the following regularity property of the $\mathbf{T}$-method, considering the subspace of $L^\varphi(I\!R)$ given by

$$\mathcal{F} = \{f \in L^{\varphi+\eta}(I\!R) : f - C^\infty_c(I\!R) \subset \mathcal{L}_{\frac{1}{w}\boldsymbol{Z}}(P), \text{ for some } P > 0\ \}.$$

Corollary 8. *Let φ and φ^* be a pair of N-functions conjugate in the sense of Young. Let $\psi \in \widetilde{\Psi}$, $\eta \in \Phi$ such that property (1) is satisfied and $I\!K \in \mathcal{X}_\psi$ be singular. We assume that for every $s \in I\!R$ and $w > 0$ $L_w(s, \cdot) \in L^{\varphi^*}(\frac{1}{w}\boldsymbol{Z})$. Then the $\mathbf{T}$-method of summability is locally regular from $L^\eta(I\!R)$ to $\mathcal{F}$.*

Remark 3. In this setting we use $C^\infty_c(I\!R)$ instead of $C_c(I\!R)$. Indeed the density Theorem 8 in [3] holds if we take into consideration $C^\infty_c(I\!R)$, see [14]. Moreover let us note that the class $\mathcal{F}$ is intimately linked to the space of the functions of bounded variation over $I\!R$, in some specific situation, see [3].

A particular case is given by the (linear) generalized sampling series (see [11], [13]). In this case we have

$$(\widetilde{T}_w f)(s) = \sum_{k=-\infty}^{+\infty} K_w(s - \frac{k}{w}) f(\frac{k}{w}),$$

and, as in the classic version, if $K_w(z) = F(wz)$

$$(S_w f)(s) = \sum_{k=-\infty}^{+\infty} F(ws - k) f(\frac{k}{w}), \quad s \in I\!R,\ w > 0,$$

being $F \in C_c(I\!R)$ and $f \in C(I\!R)$. For a theory of these operators we refer to [11] and [13] for uniform convergence and [7], [17], [18] in Orlicz and modular spaces.

6. Examples

1. Let us consider the function space $L^p(I\!R)$, $p \geq 1$ and the net in $L^p(I\!R)$

$$f_w(t) = \chi_{[w,w+1]}(t), \quad w > 0,$$

being χ_A the characteristic function of the set A. It is easy to show that $(f_w)_{w>0}$ does not converge in $L^p(I\!R)$. However, taking the generalized sampling series $(S_w f_w)_{w>0}$, we have

$$(S_w f_w)(s) = \sum_{k \in I_w} F(ws - k), \quad s \in I\!R,$$

being $I_w = [[w^2] + 1, [w^2 + w]]$. As in the classical theory of the generalized sampling operators, the series $\sum_{k=-\infty}^{+\infty} |F(u - k)|$ is convergent uniformly with respect to $u \in I\!R$ and so we easily obtain that $|(S_w f_w)(s)| < \varepsilon$ for sufficiently large w. This means that $S_w f_w \to 0$ pointwise in $I\!R$. Taking now any compact subset $S \subset I\!R$, since $S_w f_w$ is uniformly bounded by $m_0(F) = \sup_{u \in I\!R} \sum_{k=-\infty}^{+\infty} |F(u - k)|$, it follows that $S_w f_w \to 0$ in $L^p_{loc}(I\!R)$. If the function F is such that there exists an integer $\overline{N} > 0$ such that

$$G(u) := \sum_{|k| > \overline{N}} |F(u - k)| = \mathcal{O}(|u|^{-\alpha}), \quad u \to \infty,$$

for $\alpha > 0$, we have

$$|(S_w f_w)(s)| \leq \sum_{k \in I_w} |F(ws-k)|,$$

and, for w sufficiently large and s such that $|s| \geq L$,

$$|(S_w f_w)(s)| \leq \frac{M}{|ws|^\alpha},$$

for some constants $L, M > 0$. Now, for $\alpha p > 1$, there results

$$\int_{I\!R} |(S_w f_w)(s)|^p ds \leq \int_{-L}^{L} |(S_w f_w)(s)|^p ds + \int_{|s|\geq L} \frac{M^p}{|ws|^{\alpha p}} ds$$
$$= \int_{-L}^{L} |(S_w f_w)(s)|^p ds + \frac{M^p}{w^{\alpha p}} \int_{|s|\geq L} \frac{1}{|s|^{\alpha p}} ds.$$

So we obtain $S_w f_w \to 0$ in $L^p(I\!R)$.

2. This example shows that we cannot take net of functions from the whole $L^p(I\!R)$ space. Indeed, let us consider a net of functions $(f_w)_{w>0}$ defined in such a way that:

$f_w \in C(I\!R)$ with $0 \leq f_w \leq 1$, $f_w(\frac{k}{w}) = 1$ for every $k \in \boldsymbol{Z}$. For every fixed $w > 0$ and $k \in \boldsymbol{Z}$ let us take the interval around $\frac{k}{w}$ of type $I_{k,w} = (\frac{k}{w} - \frac{1}{2^{|k|+2}w}, \frac{k}{w} + \frac{1}{2^{|k|+2}w})$ and we define $f_w(t) = 0$ for every $t \in (\cup_{k=-\infty}^{+\infty}[\frac{k}{w} - \frac{1}{2^{|k|+2}w}, \frac{k}{w} + \frac{1}{2^{|k|+2}w}])^c$ and linear with continuity on every $I_{k,w}$. Now, defining $g_w = (f_w)^{\frac{1}{p}}$, we have again $g_w(\frac{k}{w}) = 1$ and

$$||g_w||_{L^p} = \frac{1}{w} \sum_{k=0}^{+\infty} \frac{1}{2^{k+2}} < +\infty.$$

So $g_w \in L^p(I\!R)$ and $g_w \to 0$ in $L^p(I\!R)$. Applying now the operators S_w to g_w, there results

$$(S_w g_w)(s) = \sum_{k=-\infty}^{+\infty} F(ws-k) = 1, \quad s \in I\!R,$$

and so $S_w g_w$ does not converge in $L^p_{loc}(I\!R)$ to zero. Let us note that the functions g_w are not of bounded variation on $I\!R$.

Now we will consider some examples of kernels (see [3]) concerning not only singularity assumptions but also the condition which gives the embedding of the Orlicz space into $\operatorname{Dom} T$.

1. Let us assume that for every $w>0$ $G=H_w=\mathbb{R}$ and let $\mu_G=\mu_w$ be the Lebesgue measure.

In this case $\mathcal{L}_{H_w}(P)=\mathcal{L}_G(P)=\mathcal{L}_{\mathbb{R}}(P)$ is the Orlicz space $L^\eta(\mathbb{R})$ for P sufficiently large, if the net $(\xi_w)_{w>0}$ is bounded.

We consider the class of functions $L_w:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_0^+$ given by

$$L_w(s,t)=\frac{g_w(s)}{(g_w(s))^2+w^2t^2}\,,$$

where for every $w\geq 1$ we put $g_w(s)=g(ws)$, being g a positive measurable function such that $1/g\in L^1(\mathbb{R})$.

Moreover we take into consideration the family of operators

$$(T_wf)(s)=\int_{-\infty}^{+\infty}K_w(s,t,f(t))dt\,,$$

with

$$K_(s,t,u)=L_w(s,t)Q_w(u)\,,$$

where $Q_w:\mathbb{R}\to\mathbb{R}$ is a net of measurable functions such that Q_w is a (L,ψ)-Lipschitz function with respect to a fixed function $\psi\in\Psi$ and

$$\lim_{w\to+\infty}\frac{Q_w(u)}{w}=\frac{u}{\pi}.$$

For examples of Q_w one can see [5]. In [3] we obtained that $(K_w)_{w>0}\subset\mathcal{X}_\psi$. Now we prove that $L_w(s,\cdot)\in L^{\varphi^*}(\mathbb{R})$. We have

$$\begin{aligned}\int_{-\infty}^{+\infty}\varphi^*\left(\lambda\frac{g_w(s)}{(g_w(s))^2+w^2t^2}\right)dt&=\int_{-1}^{1}\varphi^*\left(\lambda\frac{g_w(s)}{(g_w(s))^2+w^2t^2}\right)dt\\&+\int_{|t|>1}\varphi^*\left(\lambda\frac{g_w(s)}{(g_w(s))^2+w^2t^2}\right)dt\leq 2\varphi^*\left(\lambda\frac{1}{g_w(s)}\right)\\&+\int_{|t|>1}\frac{g_w(s)}{(g_w(s))^2+w^2t^2}\varphi^*(\lambda)dt\\&=2\varphi^*\left(\lambda\frac{1}{g_w(s)}\right)+\varphi^*(\lambda)g_w(s)\int_{|t|>1}\frac{1}{w^2t^2}dt<+\infty\,,\end{aligned}$$

by using convexity of φ^*.

We can consider also, as different example, the family of functions

$$L_w(s,t)=\frac{2w}{\pi}\frac{e^{w(s+t)}}{e^{2ws}+e^{2wt}},$$

for $s, t \in I\!R$.

Here we have that easily $\gamma_w = \delta_w = 1$ and $I\!K \subset \mathcal{X}_\psi$ (see [3]). As before it is easy to show that $L_w(s, \cdot) \in L^{\varphi^*}(I\!R)$.

2. Now we take into consideration the sequence of Bernstein polynomials (see [16]). Here we have $G = [0,1]$ provided with the Lebesgue measure and for every $n \in I\!N$, $H_n = (j/n)_{j=0,\cdots,n}$ with the counting measure μ_n.

Let $\mathcal{R}[0,1]$ be the class of all Riemann integrable functions on $[0,1]$. Then the family of operators are of the form

$$(B_n f)(s) = \sum_{j=0}^{n} \binom{n}{j} f(\frac{j}{n}) s^j (1-s)^{n-j},$$

with $s \in [0,1]$ and $f \in \mathcal{R}[0,1]$.

The kernel of this operator is given by $K_n(s, j/n, u) = L_n(s, j/n)u$, with

$$L_n(s, \frac{j}{n}) = \binom{n}{j} s^j (1-s)^{n-j},$$

with $s \in [0,1]$ and $j = 0, \cdots, n$.

In [3] we obtained that $I\!K \subset \mathcal{X}_\psi$ (see also [16]). Moreover we have that for every $\lambda > 0$

$$\begin{aligned} \int_{H_n} \varphi^*(\lambda L_n(s, j/n)) d\mu_n &= \sum_{j=0}^{n} \varphi^* \left(\lambda \binom{n}{j} s^j (1-s)^{n-j} \right) \\ &\leq \sum_{j=0}^{n} \varphi^* \left(\lambda \binom{n}{j} \right) < +\infty. \end{aligned}$$

So for every $n \in I\!N$ and $s \in [0,1]$ we have $L_n(s, \cdot) \in L^{\varphi^*}(H_n)$.

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