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R-MAPS AND L-MAPS IN Q-ALGEBRAS

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Abstract: In this paper, we introduce the notion of positive implicativity in Q-algebras and study some relations between $R-(L-)$maps and positive implicativity.

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1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-

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algebras. Y.B. Jun, E.H. Roh and H.S. Kim (see [6]) introduced a new notion, called a BH-algebra, i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (V) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and showed that there is a maximal ideal in bounded BH-algebras. In [8], J. Neggers, S.S. Ahn and H.S. Kim introduced a new notion, called an Q-algebra, which is a generalization of the idea of $BCH/BCI/BCK$-algebras and they generalized some theorems discussed in BCI-algebras. Moreover, they introduced the notion of "quadratic" Q-algebra, and showed that every quadratic Q-algebra $(X; *, e), e \in X$, has a product of the form $x*y = x-y+e$, where $x, y \in X$ when X is a field with $|X| \geq 3$. In this paper, we introduce the notion of positive implicativity in Q-algebras and study some relations between $R-(L-)$ maps and positive implicativity.

2. Q-Algebras

A *Q-algebra* (see [8]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms :

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y \ \ \forall x, y, z \in X$.

For brevity we also call X a *Q-algebra*. In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Example 2.1. (see [8]) Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz | z \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are Q-algebras, where "$-$" is the usual substraction of integers.

Example 2.2. (see [8]) Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $X = (X; *, 0)$ is a Q-algebra.

Proposition 2.3. (see [8]) *If $(X;*,0)$ is a Q-algebra, then*
(IV) $(x*(x*y))*y=0$, *for any* $x,y\in X$.

Definition 2.4. Let X and Y be Q-algebras. A mapping $f:X\to Y$ is called a *homomorphism* if

$$f(x*y)=f(x)*f(y),\ \ \forall x,y\in X.$$

A homomorphism f is called a *monomorphism* (resp., *epimorphism*) if it is injective (resp., surjective). A bijective homomorphism is called an *isomorphism*. Two Q-algebras X and Y are said to be *isomorphic*, written by $X\cong Y$, if there exists an isomorphism $f:X\to Y$. For any homomorphism $f:X\to Y$, the set $\{x\in X|f(x)=0\}$ is called the *kernel* of f, denoted by $\operatorname{Ker}(f)$ and the set $\{f(x)|x\in X\}$ is called the *image* of f, denoted by $\operatorname{Im}(f)$. We denote by $\operatorname{Hom}(X,Y)$ the set of all homomorphisms of Q-algebras from X to Y.

Proposition 2.5. (see [8]) *Suppose $f:X\to X'$ is a homomorphism of Q-algebras. Then:*

(1) $f(0)=0'$,

(2) *f is isotone, i.e., if $x*y=0$, $x,y\in X$, then $f(x)*f(y)=0'$.*

Suppose that $f:X\to Y$ is a homomorphism of Q-algebras. For any $x,y\in X$, we define $x\sim y$ if and only if $f(x)=f(y)$. Now we prove that $\sim$ is an equivalence relation on X. Since $f(x)=f(x)$, $x\sim x$. This means that $\sim$ is reflexive Since $f(y)=f(x)$, $\sim$ is symmetric. If $x\sim y$ and $y\sim z$, then $f(x)=f(y)$ and $f(y)=f(z)$ and so $f(x)=f(z)$. Therefore $x\sim z$, i.e., $\sim$ is transitive. Thus $\sim$ is an equivalence relation on X. Furthermore we have the following lemma.

Lemma 2.6. *If $x\sim y$ and $u\sim v$, then $x*u\sim y*v$, hence $\sim$ is a congruence relation on a Q-algebra X.*

We denote $[x]_f:=\{y\in X|x\sim y\}=\{y\in X|f(x)=f(y)\}$ by the equivalence class of x determined by a homomorphism f. Then $[0]_f=\operatorname{Ker}f$. In fact, if $y\in[0]_f=\{y\in X|0\sim y\}$, then $f(0)=f(y)$. Since $f(0)=0'$, $f(y)=0'$ and so $y\in\operatorname{Ker}f$. Conversely, let $y\in\operatorname{Ker}f$. Then $f(y)=0'$. Since $f(0)=0'$, $f(0)=f(y)$ and so $0\sim y$. Hence $y\in[0]_f$.

Denote $X/f=\{[x]|x\in X\}$ and define that

$$[x]_f\circledast[y]_f=[x*y]_f.$$

Since $\sim$ is a congruence relation on X, the operation $\circledast$ is well-defined. In what follows, we will prove that $([x]_f;\circledast,[0]_f=\operatorname{Ker}f)$ is a Q-algebra. Let $[x]_f,[y]_f,[z]_f$ and $[0]_f\in X/f$. Then we have the following properties:

(1) $[x]_f \circledast [x]_f = [0]_f$,

(2) $[x]_f \circledast [0]_f = [x * 0]_f = [x]_f$,

(3) $([x]_f \circledast [y]_f) \circledast [z]_f = [x * y]_f * [z]_f = [(x * y) * z]_f = [(x * z) * y]_f = [x * z]_f * [y]_f = ([x]_f * [z]_f) * [y]_f$. Summarizing the above facts we have the following theorem.

Theorem 2.7. *Let $f : X \to Y$ be a homomorphism of Q-algebras. Then X/f is a Q-algebra with $[0]_f = \operatorname{Ker} f$. It is called the quotient Q-algebra determined by f.*

Corollary 2.8. *If $f : X \to Y$ is a homomorphism of Q-algebras and $x \sim y$ is the equivalence relation $f(x) = f(y)$, then the image of f is isomorphic to the quotient Q-algebra X/f, i.e., $X/f \cong \operatorname{Im} f$.*

3. *R*-maps and *L*-maps in *Q*-Algebras

In this section, we define R-maps and L-maps in Q-algebras and investigate their properties in Q-algebras.

Definition 3.1. Let X be a Q-algebra. For a fixed $a \in X$, we define a map $R_a : X \to X$ such that $R_a(x) = x * a$ for all $x \in X$, and we call R_a a *right map* on X. The set of all right maps on X is denoted by $\mathbb{R}(X)$. A *left map* is defined by a similar way, and denoted by $\mathbb{L}(X)$.

Definition 3.2. A right map R_a is said to be *idempotent* if $R_a \circ R_a = R_a$, i.e., $(x * a) * a = x * a$ for all $x \in X$.

Definition 3.3. A Q-algebra $(X; *, 0)$ is said to be *positive implicative* if it satisfies for all x, y and $z \in X$,

$$(x * z) * (y * z) = (x * y) * z.$$

Example 3.4. Let $X := \{0, a, b, 1\}$ be a set with the following table:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	1	1	0

Then $(X; *, 0)$ is a positive implicative Q-algebra.

Lemma 3.5. *Let $(X;*,0)$ be a Q-algebra. Then the following condition holds:*

$$(x * y) = (x * y) * y \quad \text{for any} \quad x, y \in X.$$

Proof. For any $x, y \in X$, $x * y = (x * y) * 0 = (x * y) * (y * y) = (x * y) * y$, since X is positive implicative. This completes the proof. □

Theorem 3.6. *If a Q-algebra X is positive implicative, then every right map on X is idempotent.*

Proof. Since X is positive implicative, $x * y = (x * y) * y$ for any $x, y \in X$, by Lemma 3.5. Hence $R_y(x) = (R_y \circ R_y)(x)$ and so $R_y = R_y \circ R_y$ for any $y \in X$. □

Theorem 3.7. *A Q-algebra X is positive implicative if and only if every right map on X is an endomorphism of X.*

Proof. If X is a positive implicative Q-algebra, then for each $a \in X$, $(x*y)*a = (x*a)*(y*a)$, i.e., $R_a(x*y) = R_a(x)*R_a(y)$. Thus R_a is an endomorphism.
The converse follows trivially. The proof is complete. □

Theorem 3.8. *If L_x is a homomorphism, then $x = 0$.*

Proof. Suppose that L_x is a homomorphism. Then $L_x(0) = L_x(0 * 0) = L_x(0) * L_x(0)$ and so $x = x * 0 = 0$. □

For a positive implicative Q-algebra X we define an operation $\circledast$ in $\mathbb{L}(X)$ as follows. For any $L_a, L_b \in \mathbb{L}(X)$ and any $x \in X$,

$$(L_a \circledast L_b)(x) := L_a(x) * L_b(x).$$

Using positive implicativity of X we know

$$(L_a \circledast L_b)(x) = (a * x) * (b * x) = (a * b) * x = L_{a*b}(x),$$

so $L_a \circledast L_b \in \mathbb{L}(X)$. □

The next theorem gives a characterization of a positive implicative Q-algebra by its left maps.

Theorem 3.9. *If X is a positive implicative Q-algebra, then $\mathbb{L}(X)$ is a positive implicative Q-algebra.*

Proof. For any $x \in X$, by positive implicativity of X we have

$$\begin{aligned} &((L_a \circledast L_b) \circledast L_c)(x) \\ &= ((a * x) * (b * x)) * (c * x)) \\ &= ((a * x) * (c * x)) * ((b * x) * (c * x)) \\ &= ((L_a \circledast L_c))(x) * ((L_b \circledast L_c))(x) \\ &= ((L_a \circledast L_c) \circledast (L_a \circledast L_c))(x), \end{aligned}$$

which implies

$$(L_a \circledast L_b) \circledast L_c = (L_a \circledast L_c) \circledast (L_b \circledast L_c).$$

It is easy to check that $((L)(X); \circledast, L_0)$ satisfies the axiom of Q-algebra. Therefore it is a positive implicative Q-algebra. □

Corollary 3.10. *If $f : X \to \mathbb{L}(X)$ is a homomorphism of Q-algebras and $\sim$ is the equivalence relation defined by $x \sim y$ if and only if $f(x) = f(y)$, i.e., $L_x = L_y$, then the image of f is isomorphic to to the quotient Q-algebra X/f, i.e., $X/f \cong \mathbb{L}(X)$, with $[0]_f = \mathrm{Ker}\, f$.*

Proof. We show that a map $f : X \to \mathbb{L}(X)$ defined by $f(x) = L_x$ is an epimorphism. Since for any $t \in X$, $f(x*y)(t) = L_{x*y}(t) = (x*y)*t = (x*t)*(y*t) = L_x(t) * L_y(t) = (L_x \circledast L_y)(t)$, it follows that f is a homomorphism. Clearly f is surjective. We denote by $[x]_f = \{y \in X | x \sim y\} = \{y \in X | L_x = L_y\}$, i.e., $[x]_f = \{y \in X | L_x(t) = L_y(t) \quad \text{for any} \quad t \in X\}$.

$$\begin{aligned} [0]_f &= \{y \in X | L_0(t) = L_y(t) \quad \text{for any} \quad t \in X\} \\ &= \{y \in X | 0 * t = y * t \quad \text{for any} \quad t \in X\} = \mathrm{Ker}\, f\,. \end{aligned}$$

By Corollary 2.8 we obtain, $X/f \cong \mathbb{L}(X)$. □

We define a binary operation $\circ$ on $\mathbb{R}(X)$ as follows: for any $R_a, R_b \in \mathbb{R}(X)$ and for any $x \in X$,

$$(R_a \circ R_b)(x) = R_a(R_b(x)).$$

Theorem 3.11. *Let X be a positive implicative Q-algebra. Then $(\mathbb{R}(X), \circ)$ is a commutative semigroup with zero element R_0.*

Proof. Let $R_a, R_b, R_c \in \mathbb{R}(x)$. Since $[(R_a \circ R_b) \circ R_c](x) = [R_a \circ (R_b \circ R_c)](x)$ for all $x \in X$, the associative law holds. Since $R_a \circ R_b(x) = R_b \circ R_a(x)$, the commutative law is true. Since $R_a \circ R_0 = R_0 \circ R_a = R_a$, R_0 is zero element. This completes the proof of the theorem. □

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