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SEMIGROUPS OF HOLOMORPHIC MAPPINGS ON THE UNIT DISK WITH A BOUNDARY FIXED POINT

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Abstract: In this paper we describe the class of holomorphic generators of semigroups on the unit disk having boundary fixed point. In particular, we study the asymptotic behavior of semigroups of hyperbolic type. A classical problem of branching processes is to find conditions providing the convergence of exponential type. We give a geometrical solution of this problem in terms of the Robertson class of starlike functions with respect to a boundary point.

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1. Preliminaries

Let $\mathbb{C}$ be a complex plane, and let Δ be the open unit disk in $\mathbb{C}$. By $\mathrm{Hol}(\Delta, \mathbb{C})$

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we denote the class of all holomorphic functions on Δ. By $\mathrm{Hol}(\Delta)$ we denote the semigroup (with respect to the composition operation) of all holomorphic self-mappings of Δ.

Let A be either $\mathbb{N}^+ := \{0, 1, 2, \dots\}$ or $\mathbb{R}^+ := [0, \infty)$.

A family $\mathcal{S} = \{F_t\}_{t \in A} \subset \mathrm{Hol}(\Delta)$ *of holomorphic self-mappings of* Δ *is called a one-parameter semigroup if*

(i) $F_{t+s} = F_t \circ F_s$ *whenever* s, t *and* $s + t$ *belong to* A*;*

(ii) $F_0(z) = z$ *for all* $z \in \Delta$*, that is,* F_0 *is the identity operator on* Δ*.*

If $A = \mathbb{N}^+$ then $S = \{F_0, F_1, F_2, \dots, F_n, \dots\}$, $F_n \in \mathrm{Hol}(\Delta)$, is called a *one-parameter discrete semigroup*. Actually such a semigroup consists of iterates of a holomorphic self-mapping $F = F_1$, because of conditions (i) and (ii), i.e., $F_0 = I$, $F_n = F_1^{(n)}, \quad n = 1, 2, \dots$.

If $A = \mathbb{R}^+$ and the mapping $F_t : A \mapsto \mathrm{Hol}(\Delta)$ is continuous, we say that $\mathcal{S}$ is a *one-parameter continuous semigroup* of holomorphic self-mappings.

In other words, a one-parameter continuous semigroup of holomorphic self-mappings of Δ is a family $\mathcal{S} = \{F_t\}_{t \in A} \subset \mathrm{Hol}(\Delta)$ such that conditions (i), (ii) and

$$\lim_{t \to s} F_t(z) = F_s(z), \quad t > 0,\ s \geq 0,$$

hold.

It turns out that only the right continuity at zero of a semigroup implies continuity (right and left) on all of $\mathbb{R}^+ = [0, \infty)$. Moreover, in this case the semigroup is differentiable on $\mathbb{R}^+$ with respect to the parameter. For the one-dimensional case this nice result is due to E. Berkson and H. Porta [4] (see, also [14, 15, 1].

Proposition 1. (see Proposition 3.2.1 and Proposition 3.2.2 [19]) *Let* $\mathcal{S} = \{F_t\}_{t \geq 0}$ *be a one-parameter semigroup of holomorphic self-mappings of* Δ*, such that for each* $z \in \Delta$*:*

$$\lim_{t \to 0^+} F_t(z) = z. \tag{1.1}$$

Then for each $z \in \Delta$ *there exists the limit:*

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \tag{1.2}$$

which is a holomorphic function on Δ*. The convergence in (1.2) is uniform on each subset strictly inside* Δ*. Moreover, the semigroup* $\mathcal{S}$ *can be defined as the solution of the Cauchy problem:*

$$\begin{cases} \dfrac{\partial F_t}{\partial t} + f(F_t) = 0, \quad t \geq 0, \\ F_0 = z, \quad z \in \Delta. \end{cases} \tag{1.3}$$

Definition 1. The function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is said to be a generator if for any point $z \in \Delta$, the Cauchy problem (1.3) has a unique solution $\{u(t, z)\} \subset \Delta$ defined on the set $\mathbb{R}^+ \times \Delta$.

The family of all holomorphic generators on Δ will be denoted by $\mathcal{G}$. This set is a real cone in $\mathrm{Hol}(\Delta, \mathbb{C})$ [14, 19]. Different descriptions of $\mathcal{G}$ can be found in [2, 4, 8]. We formulate some of them here.

Proposition 2. *Let* $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. *Then:*
(i) $f \in \mathcal{G}$ *if and only if*

$$\operatorname{Re} f(z)\bar{z} \geq \operatorname{Re} f(0)\bar{z}(1 - |z|^2), \quad z \in \Delta;$$

(ii) $f \in \mathcal{G}$ *if and only if there exists a point* $\tau \in \overline{\Delta}$ *and a holomorphic function* $p \in \mathrm{Hol}(\Delta, \mathbb{C})$ *with* $\operatorname{Re} p(z \geq 0)$ *such that*

$$f(z) = (z - \tau)(1 - z\bar{\tau})p(z). \tag{1.4}$$

Formula (1.4) is usually called the Berkson–Porta representation of $f \in \mathcal{G}$.

Further, it is known (see, for example, [19]) that if for some $t_0 > 0$ a semigroup element F_{t_0} is not an elliptic automorphism then there exists a unique point $\tau \in \overline{\Delta}$ such that

$$\lim_{t \to \infty} F_t(z) = \tau, \quad z \in \Delta.$$

The point τ is called the *Denjoy–Wolff point* of the semigroup. Moreover, this point is a fixed point for all elements F_t, $t \geq 0$, of the semigroup, i.e.,

$$\lim_{r \to 1^-} F_t(r\tau) = \tau, \quad \tau \in \overline{\Delta}, \quad t \geq 0.$$

Proposition 3. (see [4, 19]) *Let* $f \in \mathcal{G}$ *have the form (1.4). If:*
(i) $\tau \in \Delta$ *and* $p(z) \neq ib$, $b \in \mathbb{R}$,
or
(ii) $\tau \in \partial\Delta$,
then τ *is the Denjoy–Wolff point of the semigroup* $\mathcal{S}$ *generated by* f.

Thus the Denjoy–Wolff point of a one-parameter continuous semigroup is an interior or boundary null point of its generator. As a matter of fact, a generator $f \in \mathcal{G}$ may have other null points on the boundary. To distinguish the Denjoy–Wolff point of a semigroup one can use the following assertion which is a consequence of Theorem 1 in [7].

Proposition 4. *Let* $\mathcal{S} = \{F_t\}_{t \in \mathbb{R}^+}$ *be a one-parameter continuous semigroup generated by* $f \in \mathcal{G}$.

(i) A point $\tau \in \Delta$ *is the Denjoy–Wolff point of* $\mathcal{S}$ *if and only if* $f(\tau) = 0$ *and* $\operatorname{Re} f'(\tau) > 0$.

(ii) A point $\tau \in \partial\Delta$ *is the Denjoy–Wolff point of* $\mathcal{S}$ *if and only if* $f(\tau) := \lim_{r\to 1^-} f(r\tau) = 0$ *and* $\operatorname{Re} f'(\tau) := \operatorname{Re}\lim_{r\to 1^-} \frac{f(r\tau)}{(r-1)\tau} \geq 0$. *Moreover, in this case* $f'(\tau)$ *is a real number.*

We are interested in the study of a class generators having a boundary null point which is not necessarily the Denjoy–Wolff point of generated semigroups. To do this we define the following subclass of $\mathcal{G}$ as follows:

$$\mathcal{G}[1] := \Big\{ f \in \mathcal{G} : \lim_{r\to 1^-} f(r) = 0 \text{ and } f'(1) := \lim_{r\to 1^-} \frac{f(r)}{r-1} \text{ exists} \Big\}. \tag{1.5}$$

A classical model in which these classes arise naturally is the model considered in the theory of Markov branching processes. More precisely.

Consider generating functions of homogeneous branching processes (see, for example, [3]).

Once again we denote by A either $\mathbb{N}^+$ or $\mathbb{R}^+ := [0, \infty)$. Let $\{Z(t),\ t \in A\}$ be a Markov branching process defined on A. If $A = \mathbb{N}^+$, this process is the known Galton–Watson process which can be describe as follows. It starts at time $t = 0$ with a single particle ($Z(0) = 1$). The first generation $Z(1)$ is a random variable with distribution of probabilities

$$P_n(1) = P\{Z(1) = n \,|\, Z(0) = 1\}, \quad n = 0, 1, \ldots.$$

Its generating function

$$F_1(z) = \sum_{n=0}^{\infty} P_n(1) z^n \in \operatorname{Hol}(\Delta)$$

is evidently a self-mapping of the open unit disk Δ. The number of "offspring" produced by a single "parent" particle at any given time is independent of the previous steps of the process, and of other particles existing at the present. The number of particles in the t-th generation is a random variable $Z(t)$. One can show (see [3]) that its generating function

$$F_t(z) = \sum_{n=0}^{\infty} P_n(t) z^n \in \operatorname{Hol}(\Delta), \quad t \in \mathbb{N}^+, \tag{1.6}$$

with

$$P_n(t) = P\{Z(t) = n \,|\, Z(0) = 1\}, \tag{1.7}$$

is t-fold iterate of $F_1(z)$. So, $\{F_t(z)\}_{t\in\mathbb{N}^+}$ is a discrete-time semigroup of holomorphic self-mappings of Δ.

If $A=\mathbb{R}^+$, a Markov branching process $\{Z(t),\ t\in\mathbb{R}^+\}$ can be considered as a continuous time analog of a Galton–Watson processes and can be interpreted as the number of "offspring" produced at any (continuous) time $t\in\mathbb{R}^+$. We concentrate now on generating functions $F_t,\ t\in\mathbb{R}^+$, of continuous branching processes defined by the same formulas (1.6)–(1.7) for all positive t, i.e.,

$$F_t(z)=\sum_{n=0}^{\infty}P_n(t)z^n\in\mathrm{Hol}(\Delta),\qquad t\in\mathbb{R}^+, \tag{1.6$'$}$$

where $P_n(t)$ are defined by (1.7). By definition, $P_n(t)\geq 0$, $\sum\limits_{n=0}^{\infty}P_n(t)\leq 1$. A branching process $Z(t)$ is called regular if the latter sum is equal to 1, and then the point $z=1$ is a fixed point for all of functions $F_t,\ t\geq 0$.

In this case $\mathcal{S}:=\{F_t(z)\}_{t\in\mathbb{R}^+}$ is a continuous-time semigroup of holomorphic self-mappings of Δ. The differential equation (1.3) in this situation is called the Kolmogorov backward equation (see [3]).

It is known that generators of such semigroups have the following form:

$$f(z)=a\left(z-\sum_{n=0}^{\infty}\tilde{p}_nz^n\right),$$

where the number $a>0$ and the so-called infinitesimal probabilities $\tilde{p}_n$ satisfying $0\leq\tilde{p}_n$ and $\sum\limits_{n=0}^{\infty}\tilde{p}_n=1$ are the total data of the process. Thus,

$$f(z)=\sum_{n=0}^{\infty}p_nz^n,$$

$$\text{where }\ p_1\geq 0,\ p_n\leq 0\ \text{for } n\neq 1 \text{ and } \sum_{n=0}^{\infty}p_n=0. \tag{1.8}$$

Representation (1.8) implies immediately that $f(1)=0$. The smallest root $q\in[0,1]$ of the equation

$$f(x)=0\,,$$

has a clear probability meaning:

$$q=\lim_{t\to\infty}P\left\{Z(t)=0\right\}\,,$$

and is the probability that the given process becomes extinct. The number q is called the *extinction probability* of the branching process.

Since $F_t(0) = P\{Z(t) = 0\}$ we have that q $(= \lim_{t\to\infty} F_t(0))$ is the Denjoy–Wolff point of the semigroup, so each generator of type (1.8) can be also represented by the Berkson–Porta formula (1.4) with $\tau = q$.

From the point of view of the theory of branching processes the following restriction is natural. We assume that for a generator $f \in \mathcal{G}$ of the form (1.8) the derivatives $f'(1)$ exists. This means that the first moment (the expectation) for a suitable Markov process exists. By $\mathcal{G}_p$ we denote the following class of generators related to branching processes:

$$\mathcal{G}_p := \Big\{ f(z) = \sum_{n=0}^{\infty} p_n z^n \in \mathcal{G} : p_1 \geq 0,\ p_n \leq 0 \text{ for } n \neq 1, \\ \sum_{n=0}^{\infty} p_n = 0 \text{ and the series } \sum_{n=1}^{\infty} n p_n \text{ converges} \Big\}.$$

Thus $\mathcal{G}_p$ is a subcone of the cone $\mathcal{G}[1]$ defined by (1.5) with $\tau = 1$. In the next section we study this situation in general.

2. Classifications of the Cone $\mathcal{G}[1]$

First we show that if $f \in \mathcal{G}[1]$ then $f'(1)$ is a real number. To do this we need the following representation theorem.

Theorem 1. *Let $f \in \mathcal{G}$. Then $f \in \mathcal{G}[1]$ if and only if it admits the representation*

$$f(z) = -(1-z)^2 \mathfrak{p}(z) + \frac{\beta}{2}(z^2 - 1), \tag{2.1}$$

where $\beta \in \mathbb{R}$ and $\mathfrak{p} \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\,\mathfrak{p} > 0$ and $\lim_{r\to 1^-} (1-r)\mathfrak{p}(r) = 0$.

Proof. Let f admit representation (2.1). Then

$$f(1) = \lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} \left[-(1-r)^2 \mathfrak{p}(r) + \frac{\beta}{2}(r^2 - 1) \right] = 0\,,$$

and

$$f'(1) := \lim_{r\to 1^-} \frac{f(r)}{r-1} = \lim_{r\to 1^-} \left[(1-r)\mathfrak{p}(r) + \frac{\beta}{2}(r+1) \right] = \beta$$

exists, i.e., $f \in \mathcal{G}[1]$.

Conversely. Suppose now that $f \in \mathcal{G}[1]$. Then the radial derivative

$$f'(1) := \lim_{r\to 1^-} \frac{f(r)}{r-1}$$

exists finitely. Denote $\beta = \operatorname{Re} f'(1)$ and define

$$h(z) := \frac{\beta}{2}(1-z^2) \quad \text{and} \quad g(z) := f(z) + h(z), \quad z \in \Delta. \tag{2.2}$$

Since $h \in \mathcal{G}[1]$ (actually, it is a generator of a group of hyperbolic automorphisms) and $\mathcal{G}[1]$ is a real cone, we have $g \in \mathcal{G}[1]$. Moreover,

$$\operatorname{Re} g'(1) = \operatorname{Re} f'(1) - \beta = 0.$$

By Proposition 4 the point $\tau = 1$ is the Denjoy–Wolff point for the semigroup generated by g and $g'(1)$ is a real number, that is, $g'(1) = 0$.

Further, writing g by the Berkson–Porta representation (1.4):

$$g(z) = -(1-z)^2 \mathfrak{p}(z),$$

we see that $\lim\limits_{r\to 1^-}(1-r)\mathfrak{p}(r) = 0$. Hence, (2.2) coincides with (2.1). The proof is complete. □

Remark. It can be shown by using the Riesz–Herglotz representation of the function $\mathfrak{p}$ in (2.1) that, in fact, the angular limit $\angle \lim\limits_{z\to 1}(1-z)\mathfrak{p}(z)$ exists and equals zero. So, if $f \in \mathcal{G}[1]$ then the angular limit

$$f'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} = \angle \lim_{z\to 1} f'(z)$$

also exists and is a real number. We denote $\beta = \beta_f = f'(1)$ for $f \in \mathcal{G}[1]$.

It is natural to differ now the following three cases: $\beta > 0$, $\beta = 0$ and $\beta < 0$.

If $\beta = f'(1) \geq 0$ then once again by Proposition 4 the point $\tau = 1$ is a Denjoy–Wolff point for the semigroup generated by f, and f can be represented by the Berkson–Porta formula:

$$f(z) = -(1-z)^2 p(z),$$

where $\operatorname{Re} p(z) \geq 0$. If, in addition, $f \in \mathcal{G}_p$, then the continuous-time branching process (see Section 1) related to the semigroup $\mathcal{S} := \{F_t(z)\}_{t\in\mathbb{R}^+}$ generated by f has the extinction probability

$$q = \lim_{t\to\infty} P\{Z(t) = 0\} = 1.$$

In the case $\beta > 0$ the semigroup generated by f converges to $\tau = 1$ with exponential rate of convergence, namely

$$|F_t(z) - 1| \leq C(z)\exp(-t\beta)$$

(see Section 4). In this situation we say that the semigroup is of hyperbolic type; if f, in particular, belongs to $\mathcal{G}_p$ the corresponding branching process is called subcritical.

If $\beta = 0$ the rate of convergence is more slow then exponential. In this case the generated semigroup is called of parabolic type; in the theory of branching processes this is a critical case.

If $\beta = f'(1) < 0$ then the point $z = 1$ is not an attractive point of the semigroup generated by f. Hence, it follows by the Denjoy–Wolff Theorem and Proposition 5 that f must have a different null point $\tau \neq 1$, which is the Denjoy–Wolff point for the semigroup $\mathcal{S}$ generated by f. If, in addition, $f \in \mathcal{G}_p$ this point (the extinction probability q) must lie on the interval $[0, 1)$; this case is called supercritical. Note also that in the last situation the Berkson–Porta representation does not related with the boundary null point $z = 1$ of f. However formula (2.1) proved above can be used for all values of β. In particular, we have the following consequence of Theorem 1.

Corollary 1. *Let $f \in \mathcal{G}_p$ be an infinitesimal generating function of a supercritical branching process with the first moment $\beta < 0$. Represent f by (2.1). Then the extinction probability q is a unique on the interval $[0, 1)$ root of the equation*

$$\mathfrak{p}(x) = \frac{\beta}{2}\frac{x+1}{x-1}.$$

For convenience we collect the above facts in the following table.

	$\beta = f'(1)$	General case	Probabilistic case
1.	$\beta > 0$	hyperbolic, $z = 1$ is a Denjoy–Wolff point	subcritical case
2.	$\beta = 0$	parabolic, $z = 1$ is a Denjoy–Wolff point	critical case
3.	$\beta < 0$	there is an additional root of $f(z) = 0$ in $\overline{\Delta}$, $z = 1$ is not a Denjoy–Wolff point	supercritical case, there is $0 \leq q < 1$ such that $f(q) = 0$

3. The Kœnigs Embedding Problem

A classical problem of analysis is, given a function F, to find a continuous-time semigroup $\{F_t(z)\}_{t\geq 0}$ such that $F_n(z) = F^n(z)$ $n \geq 1$. If F is a probability generating function, one can ask also whether F_t is a probability generating function for each $t \in \mathbb{R}^+$? In this case one says that the Galton–Watson process corresponding to F can be embedded in a Markov branching process.

If F has an interior fixed point, Kœnigs (1884) showed how the problem may be solved. Further, this problem has been considered by many mathematicians (see references in [5, 18, 19]).

In this section we discuss a solution of the problem in terms of the Robertson class of univalent functions (see Definition 4 below) for the case where F is a holomorphic self-mapping of the hyperbolic type.

Definition 2. We say that a function $F \in \mathrm{Hol}(\Delta)$ satisfies the Kœnigs embedding property (K-E-P), or just is embedable, if there is a one-parameter semigroup $\{F_t\}_{t\geq 0}$ of holomorphic self-mappings of Δ such that $F_1 = F$ (consequently, $F_n = F^n$).

Let $F \in \mathrm{Hol}(\Delta)$ be a holomorphic self-mapping of Δ such that F has no fixed point in Δ and $\xi = 1$ is a Denjoy–Wolff point for F. The set of such function we denote $\mathcal{B}[1]$. Assume that

$$F'(1) = \angle \lim_{z\to 1} F(z) = \alpha < 1,$$

i.e., F is a holomorphic self-mapping of hyperbolic type. Following the terminology of branching processes theory this case is subcritical.

Definition 3. If for some $\xi \in \Delta$ the limit

$$Q_\xi(z) = \lim_{n\to\infty} \frac{F^n(z) - 1}{F^n(\xi) - 1} \tag{3.1}$$

exists and is not constant we will call it the Kœnigs–Valiron function for F (or simply (K-V)–function).

Definition 4. (see [16]) A class of functions $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ normalized by conditions $h(0) = 1$ and $h(1) := \angle \lim_{z\to 1} h(z) = 0$ is said to be the class of Roberston if

$$\mathrm{Re}\left[2\frac{zh'(z)}{h(z)} + \frac{1+z}{1-z}\right] > 0. \tag{3.2}$$

We will denote this class by $\mathcal{R}(\Delta)$.

In 1981 M.S. Robertson conjectured that this class consists precisely of the univalent functions on Δ whose images are starlike with respect to the boundary point $0 = f(1)$ and lie in a half-plane.

This conjecture was proved by Lyzzaik [11] in 1984 and extended to the general class of starlike functions with respect to a boundary point by Silverman and Silvia [21] in 1990 (see also, [6], [19] and [10]).

Theorem 2. *Let $F \in \mathcal{B}[1]$ with $\alpha = F'(1) < 1$ be embedable. Then for each $\xi \in \Delta$ the Kœnigs–Valiron function Q_ξ exists and $\frac{Q_\xi(z)}{Q_\xi(0)} = Q_0(z)$ is a function of the class $\mathcal{R}(\Delta)$.*

Proof. First we note that for a given point $\xi \in \Delta$ the family of functions

$$Q_n(z) = \frac{F^n(z) - 1}{F^n(\xi) - 1}\,, \tag{3.3}$$

is a normal family.

Indeed, since

$$-\pi < \arg(F^n(z) - 1) - \arg(F^n(\xi) - 1) < \pi,$$

we have that each Q_n cannot obtain values in the left-hand real axis.

Therefore, by Montel Theorem $\{Q_n\}_{n=0}^{\infty}$ is normal, i.e., there is a subsequence $\{Q_{n_k}\}_{k=1}^{\infty}$ which converges locally uniformly on Δ either to a holomorphic function Q or to infinity.

But the last case is impossible because of the obvious equality

$$Q_n(\xi) = 1. \tag{3.4}$$

Setting, in particular, $\xi = 0$ we have

$$Q(0) = 1. \tag{3.5}$$

Now it follows by the Hurwitz theorem that $Q(z)$ is either a univalent function on Δ or a constant. To see that the last case is impossible we differentiating $Q_n(z)$ obtain

$$Q'(z) = \lim_{k\to\infty} \frac{(F^{n_k})'(z)}{F^{n_k}(\xi) - 1}\,, \tag{3.6}$$

and

$$\frac{Q'(z)}{Q(z)} = \lim_{k\to\infty} \frac{(F^{n_k})'(z)}{F^{n_k}(z) - 1}. \tag{3.7}$$

Further, if F is embedable we consider the net $\{Q_t\}_{t\geq 0}$ defined by

$$Q_t = \frac{F_t(z) - 1}{F_t(\xi) - 1}. \tag{3.8}$$

Since $F_n = F^n$ we have by the previous considerations that there is a subnet $\{Q_{t_k}\}_{k=1}^{\infty}$ $(t = n_k)$ which converge to Q. We will show that, in fact, the net $\{Q_t\}$ itself converges to Q, and Q is of class $\mathcal{R}(\Delta)$. To do this consider now the following expression

$$q_t(z) = \frac{(F_t)'(z)}{F_t(z) - 1}. \tag{3.9}$$

If we denote $f(z) = \lim_{t\to 0^+} \frac{z - F_t(z)}{t}$, the infinitesimal generator of the semigroup $\{F_t\}_{t\geq 0}$, then we have

$$\frac{\partial F_t(z)}{\partial t} = -f(F_t(z)). \tag{3.10}$$

On the other hand, it is known (see for example [14]) that $\{F_t\}_{t\geq 0}$ also satisfies the following differential equation

$$\frac{\partial F_t(z)}{\partial t} = -\frac{\partial F_t(z)}{\partial z} f(z). \tag{3.11}$$

Comparing (3.10) and (3.11) we obtain the following functional equation:

$$f(F_t(z)) = \frac{\partial F_t(z)}{\partial z} \cdot f(z). \tag{3.12}$$

By using (3.9) and (3.12) we get now

$$q_t(z) = \frac{f(F_t(z))}{f(z)} \cdot \frac{1}{F_t(z) - 1}. \tag{3.13}$$

Since $\xi = 1$ is the Denjoy–Wolff point for the semigroup $\{F_t\}_{t\geq 0}$ it follows by a continuous version of the Julia–Wolff–Carathéodory Theorem (see [7]) that the angular derivative

$$f'(1) = \angle \lim_{z\to 1} f'(z) = \angle \lim_{z\to 1} \frac{f(z)}{z - 1} \tag{3.14}$$

exists and is a negative real number $\beta \geq 0$. Moreover,

$$\frac{\partial F_t(z)}{\partial z}\Big|_{z=1} = \angle \lim_{z\to 1} \frac{\partial F_t(z)}{\partial z} = e^{-t\beta}. \tag{3.15}$$

Since $F_1 = F$ and $\alpha = F'(1) < 1$ we obtain from (3.15)

$$\beta = f'(1) = -\ln \alpha > 0. \tag{3.16}$$

In addition, it follows by (3.7), (3.9) and (3.13) that

$$\frac{Q'(z)}{Q(z)} = \lim_{k\to\infty} q_{t_k}(z) = \frac{1}{f(z)} \lim_{k\to\infty} \frac{f(F_{t_k}(z))}{F_{t_k}(z) - 1}. \tag{3.17}$$

Thus the limit

$$\lim_{k\to\infty} \frac{f(F_{t_k}(z))}{F_{t_k}(z) - 1} = f(z)\frac{Q'(z)}{Q(z)}$$

exists finitely. Since for each $z \in \Delta$ the sequence $\{F_{t_k}(z)\}_{k=1}^{\infty}$ converges to $\xi = 1$ as k goes to infinity, it follows by the Lindelöf Theorem (see, for example, [18]) and (3.14) that this limit does not depend on $z \in \Delta$ and equals to $f'(1) = \beta > 0$ (see (3.16)). Thus we have by (3.17) that the function $Q(z)$ defined by (3.6) satisfies the differential equation

$$\beta Q(z) = Q'(z) \cdot f(z). \tag{3.18}$$

Finally, it was shown in [19] (see also, [5]) that for each $\mu \in (0, 2\beta]$ equation (3.18) has a unique solution, normalized by the conditions $Q(0) = 1$ and $Q(1) = 0$, the image of which is a starlike domain with respect to the boundary point zero and lies in the minimal angle equals to $\frac{\mu}{\beta}\pi$.

Setting in our case $\xi = 0$ and $\mu = \beta$ we obtain that Q defined by (3.6) is of class $\mathcal{R}(\Delta)$. Moreover, it follows by the uniqueness of the solution of (3.18) that the limit in (3.6) does not depend on the choice of a convergent sequence $\{Q_{n_k}\}_{k=1}^{\infty}$.

Thus we have that, in fact,

$$Q_0(z) = Q(z) = \lim_{n\to\infty} \frac{F^n(z) - 1}{F^n(0) - 1},$$

and we are done.

The following is a converse assertion.

Theorem 3. *Let $F \in \mathcal{B}[1]$ and let its (K-V)–function $Q(z) = Q_0(z) \neq$ const be a well defined univalent function on Δ. Then:*

(i) $\alpha = F'(1) = \angle \lim_{z\to 1} F'(z) < 1$;

(ii) if $Q \in \mathcal{R}(\Delta)$, then F is embedable.

Proof. Since Q is not constant it follows that Q is univalent on Δ. Now we have by (3.1)

$$Q(F(z)) = \lim_{n\to\infty} \frac{F^{n+1}(z)-1}{F^n(0)-1} = \lim_{n\to\infty} \frac{F^{n+1}(z)-1}{F^{n+1}(0)-1}$$
$$= \lim_{n\to\infty} \frac{F^n(F(0))-1}{F^n(0)-1} \cdot Q(z) = Q(F(0)) \cdot Q(z).$$

On the other hand, setting $F'(0) = z_n$ we have by the Julia–Carathéodory Theorem that

$$\lim_{n\to\infty} \frac{F^n(F(0))-1}{F^n(0)-1} = \lim_{n\to\infty} \frac{F(z_n)-1}{z_n-1} = F'(1) = 0.$$

Since $Q(0) = 1$ and Q is univalent, $\alpha = Q(F(0))$ cannot be 1. Thus Q satisfies the following functional (Scröder) equation

$$Q(F(z)) = \lambda Q(z), \ \lambda = F'(1) < 1. \tag{3.19}$$

Now assume that $Q \in \mathcal{R}(\Delta)$.

Since $Q(\Delta)$ is a starlike domain one can define the semigroup

$$F_t(z) = Q^{-1}(e^{-\beta t} Q(z)), \tag{3.20}$$

where $\beta = -\ln\alpha > 0$. Then it follows by (3.19) and (3.20) that

$$F_1(z) = Q^{-1}(e^{-\beta} Q(z)) = Q^{-1}(\lambda Q(z)) = F(z).$$

The theorem is proved. □

Remark. Actually, in the proof of Theorem 2 (ii) we have used only the fact that (K-V)–function Q of F is a starlike function with respect to a boundary point. Different subclasses of this class (including the class of Robertson) were discussed in [21], [6] and [19]). Using, in particular, Proposition 5.6.1 in [19] one can formulate the following characterization of embedable self-mappings of the unit disk in the subcritical case.

Corollary 2. *Let $F \in \mathcal{B}[1]$. The following are equivalent:*
(i) $F'(1) = \angle\lim_{z\to 1} F(z) < 1$ and F is embedable;
(ii) there is $\lambda \in \left[0, \frac{1}{2}\right]$ such that the function $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by

$$h(z) = \frac{z}{(1-z)^{2-2\lambda}} \cdot Q(z),$$

where Q is the (K-V)–function for F, is a normalized starlike function of order λ with respect to the origin, i.e.,

$$h(0) = 0, \quad h'(0) = 1 \quad \textit{and} \quad \operatorname{Re} \frac{zh'(z)}{h(z)} > \lambda.$$

Remark. Two extremal cases $\lambda = 0$ and $\lambda = \frac{1}{2}$ described in (ii) are of special interest because they are connected to classical extremal functions of the classes starlike and convex functions, respectively.

Indeed, for $\lambda = 0$ we have that assertion (ii) is equivalent to the following one

(ii') the function h defined by

$$h(z) = K(z) \cdot Q(z),$$

where $K(z) = \frac{z}{(1-z)^2}$ is the Köebe function, is starlike with respect to the origin.

Setting $\lambda = \frac{1}{2}$ we obtain that (ii) as well as (ii') are equivalent to

(ii") the function h defined by

$$h(z) = C(z) \cdot Q(z),$$

where $C(z) = \frac{z}{1-z}$ is a convex function on Δ, is a starlike function of order $\frac{1}{2}$ with respect to the origin.

Note that (ii") preciesely means that Q belongs to the class $\mathcal{R}(\Delta)$ of Roberston.

Remark. We have seen already that for $F \in \mathcal{B}[1]$, with $\alpha = F'(1) < 1$ the (K-V)–function $Q(z)$ is a solution of the functional (Shröder) equation

$$Q(F(z)) = \alpha Q(z).$$

If F is embedable, then Q is also a unique solution of the equation

$$Q(F_t(z)) = e^{-t\beta} Q(z),$$

where $\beta = f'(1)$, f is the generator of the semigroup $\{F_t\}_{t \geq 0}$, $F_1 = F$, normalized by the condition: $Q(0) = 1$. Also $Q(1) := \angle \lim_{z \to 1} Q(z) = 0$.

The uniqueness can be seen by the uniqueness of the solution of (3.18) represented by

$$Q(z) = \exp\left(- \int_0^z \frac{\beta dz}{f(z)} \right). \tag{3.21}$$

Since (3.21) is equivalent (3.18), this representation of the (K-V)–function with some $f \in \mathcal{G}[1]$ with $\beta = f'(1)\,(= -\ln\alpha) > 0$ is also sufficient for F to be embedable.

By using some representations of functions of class $\mathcal{G}[1]$ one can formulate the following criteria for $F \in \mathcal{B}[1]$ to be embedable.

Corollary 3. *Let $F \in \mathcal{B}[1]$ with $\alpha = F'(1) < 1$, and let $Q \in \mathrm{Hol}(\Delta, \mathbb{C})$ be its (K-V)–function defined by*

$$Q(z) = \lim_{n\to\infty} \frac{F^n(z) - 1}{F^n(0) - 1}.$$

The following assertions are equivalent:

(i) F is embedable;

(ii) there exists $p \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\, p \geq 0$ and $\angle\lim\limits_{z\to 1} \frac{p(z)}{z-1} = p'(1) = (\ln\alpha)^{-1}$ such that Q admits the representation

$$Q(z) = \exp\left(-\int_0^1 \frac{\beta p(z)dz}{(1-z)^2}\right);$$

(iii) for any $c \in (0, -\frac{1}{2}\ln\alpha)$ there is $G \in \mathcal{B}[1]$ with $\gamma = G'(1) = \angle\lim\limits_{z\to 1} G(z) = \frac{2C}{\beta}$, $\beta = \ln\alpha$, such that (K-V)–function Q admits the representation

$$Q(z) = \exp\left(-\frac{2}{\gamma}\int_0^z \frac{1 - G(z)}{(1-z)^2(1+G(z))}dz\right).$$

Proof. The equivalence of assertion (i) and (ii) is just an interpretation of formula (3.21) (see Remark 3) in the spirit of the Berkson–Porta representation of the class $\mathcal{G}[1]$:

$$f(z) = -(1-z)^2 q(z)\,, \tag{3.22}$$

with $\mathrm{Re}\, q \geq 0$ everywhere.

Setting $p(z) = \frac{1}{q(z)}$ we get by (3.22) that

$$0 < \beta := f'(1) = \lim_{z\to 1} \frac{f(z)}{z-1} = -\lim_{z\to 1} \frac{(z-1)}{p(z)} = -\frac{1}{p'(1)} = \ln\alpha.$$

In turn (3.22) implies that (ii) is equivalent to (iii) if we present $q(z)$ in the form

$$q(z) = c\,\frac{1+G(z)}{1-G(z)}\,,$$

where $G \in \mathrm{Hol}(\Delta)$ is a self-mapping of Δ.

Since $c \in (0, \frac{\beta}{2})$ we have

$$0 < \beta = \angle \lim_{z\to 1} \frac{f(z)}{z-1} = c \cdot \lim_{z\to 1} \frac{(1-z)}{1-G(z)} \cdot (1+G(z)).$$

Since $|G(z)| < 1,\ z \in \Delta$, we have that the limit

$$\gamma = \angle \lim_{z\to 1} \frac{1-G(z)}{1-z} \tag{3.23}$$

exists and $\gamma = \frac{2c}{\beta} < 1$.

The reverse considerations complete our proof. □

4. Asymptotic Behavior of Semigroups

Let $f \in \mathcal{G}[1]$, i.e., $\lim\limits_{r\to 1^-} f(r) = 0$ and $\beta = f'(1) := \lim\limits_{r\to 1^-} \dfrac{f(r)}{r-1}$ exists (see (1.5)), and let $\{F_t\}_{t\geq 0}$ be a semigroup generated by f. Then $\lim\limits_{t\to\infty} F_t(z) = 1$ if and only if $\beta \geq 0$. Moreover, in this case it is known (see [7]) that

$$\frac{|1-F_t(z)|^2}{1-|F_t(z)|^2} \leq e^{-t\beta} \, \frac{|1-z|^2}{1-|z|^2}\,. \tag{4.1}$$

If $\beta > 0$ this estimate establishes the exponential rate of convergence F_t to the point $\tau = 1$ in terms of a non-Euclidean distance defined by

$$d(z,\tau) = \frac{|\tau - z|^2}{1-|z|^2}\,,$$

where $\tau \in \partial\Delta,\ z \in \Delta$.

The natural problem arises (see, for example, [3, 9]) is whether the norm convergence is exactly of exponential type, i.e.,

$$|F_t(z) - 1| \sim e^{-\beta t} \ ?$$

In other words, the question is whether the following limit

$$K(z) := \lim_{t\to\infty} \exp\left(t\beta\right)\, (1 - F_t(z))$$

exists and is not zero?

The following well-known fact is a very important result in the study of the probability asymptotic of branching processes (see, for example, [17]): *Let F_t, $t \geq 0$, be the generating function of a subcritical continuous branching process. Then*

$$1 - F_t(0) = Ke^{-\beta t}(1 + o(1)),$$

if, and only if, the (real) integral

$$\int_0^1 \frac{\beta x + f(1-x)}{xf(1-x)}\, dx$$

converges, where f is the infinitesimal generator of the semigroup $\{F_t\}_{t\geq 0}$ of the form (1.8). In this case the integral is equal to $-\log K$.

As a matter of fact, we will see below that this result does not depend on the probability nature of of the semigroup rather the boundary behavior of the Robertson class function Q defined by (4.3) which in the probability sense is closely related to the solution of the so-called Yaglom equation:

$$g(F_t(z)) = m_t g(z) + 1 - m. \tag{4.2}$$

Also note that for the discrete-time semigroups this problem has been studied in [22] and [13]. It turns out that for continuous semigroups the answer can be given in terms of the (K-V)–function

$$Q(z) = Q_0(z) = \lim_{n\to\infty} \frac{F^n(z) - 1}{F^n(0) - 1}$$

defined in the previous section.

First, we observe that actually we have proved the following fact:

Let $f \in \mathcal{G}[1]$ have a positive angular derivative $\beta := f'(1) > 0$ and generate a semigroup of holomorphic self-mappings $\{F_t\}_{t\geq 0}$. Then the limit

$$Q(z) := \lim_{t\to\infty} \frac{1 - F_t(z)}{1 - F_t(0)} \tag{4.3}$$

exists, and is a function of the Robertson class $\mathcal{R}(\Delta)$.

Moreover, this function satisfies the differential equation

$$\beta Q(z) = Q'(z) f(z), \tag{4.4}$$

which is equivalent to the (Schröder) equation:

$$Q(F_t(z)) = \lambda_t Q(z) \quad \text{for all } t \geq 0, \tag{4.5}$$

with $\lambda_t = e^{-t\beta}$.

Since we will employ the latter fact, we prove it separately.

Lemma 1. *Let $\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ be a semigroup of holomorphic self-mappings generated by $f \in \mathcal{G}$, and let μ be a complex number with $\mathrm{Re}\,\mu > 0$. A univalent function $\phi \in \mathrm{Hol}(\Delta, \mathbb{C})$ is a solution of the differential equation:*

$$\mu\phi(z) = \phi'(z)f(z) \tag{4.6}$$

if and only if ϕ satisfies the equation:

$$\phi(F_t(z)) = e^{-t\mu}\phi(z). \tag{4.7}$$

Thus, in this case ϕ is a spirallike function.

Proof. Suppose that (4.7) holds. Differentiating this equality by t at $t = 0^+$ we obtain that ϕ satisfies the differential equation (4.6) with $f = -\left.\frac{\partial F_t}{\partial t}\right|_{t=0^+}$.

Conversely. Suppose now that ϕ is a solution of (4.6). Denote by $\{G_t\}_{t\geq 0} \subset \mathrm{Hol}(\phi(\Delta))$ a family of holomorphic functions defined by

$$G_t(w) := \phi\left(F_t(\phi^{-1}(w))\right). \tag{4.8}$$

This family is a semigroup of holomorphic self-mappings of the domain $\phi(\Delta)$. Indeed,

$$G_0(w) = \phi\left(F_0(\phi^{-1}(w))\right) = \phi\left(\phi^{-1}(w)\right) = w$$

and

$$\begin{aligned} G_t(G_s(w)) &= \phi\left(F_t(\phi^{-1}(\phi(F_s(\phi^{-1}(w))))))\right) \\ &= \phi\left(F_t(F_s(\phi^{-1}(w)))\right) = \phi\left(F_{t+s}(\phi^{-1}(w))\right) = G_{t+s}(w). \end{aligned}$$

To find the generator of this semigroup, we just differentiate $G_t(w)$ at $t = 0^+$:

$$\begin{aligned} -\left.\frac{\partial G_t(w)}{\partial t}\right|_{t=0} &= -\phi'(\phi^{-1}(w))\left.\frac{\partial F_t(\phi^{-1}(w))}{\partial t}\right|_{t=0} \\ &= \phi'(\phi^{-1}(w))\, f(\phi^{-1}(w)) = \mu\phi(\phi^{-1}(w)) = \mu w. \end{aligned}$$

On the other hand, it follows by the uniqueness of the solution of the Cauchy problem:

$$\begin{cases} \dfrac{du(t)}{dt} + \mu u(t) = 0 \\ u(0) = w, \quad w \in \phi(\Delta), \end{cases}$$

that $G_t(w) = e^{-\mu t} w$. Setting $w = \phi(z)$ we get (4.7) by (4.8). The lemma is proved. □

We need the following assertion which is a consequence of Proposition 4.13 [12].

Lemma 2. *Let ϕ be a univalent function on the unit disk Δ, and let ϕ be conformal at the boundary point $\zeta \in \partial\Delta$ (i.e., $\exists \angle \phi'(\zeta) = s \neq 0, \infty$). Let $g \in \mathrm{Hol}(\Delta, \mathbb{C})$ satisfy $g(\Delta) \subset \phi(\Delta)$ and $g(\zeta) = \phi(\zeta)$. If there exists a sequence $\{z_n\} \subset \Delta$ nontangentially converges to ζ such that*

$$g'(z_n) \to \alpha \in \mathbb{C},$$

then

$$\angle \lim_{z \to \zeta} g'(z) = \alpha.$$

Proof. First we define

$$g_1(z) := \phi^{-1} \circ g(z).$$

This function is a well-defined self-mapping of Δ because of $g(\Delta) \subset \phi(\Delta)$. Moreover, since $g(\zeta) = \phi(\zeta)$ the point ζ is a boundary fixed point of g_1. Then by Proposition 4.13 [12] the limit

$$\angle \lim_{z \to \zeta} g_1'(z) = A \leq \infty$$

exists. Let now $\gamma : [0, \infty) \mapsto \Delta$ be a continuous curve in Δ such that $\gamma(t) \to \zeta$ nontangentially as $t \to \infty$ and $\gamma(n) = z_n$. We know already that

$$\lim_{t \to \infty} g_1'(\gamma(t)) = A.$$

Hence

$$\lim_{n \to \infty} g_1'(z_n) = \lim_{n \to \infty} g_1'(\gamma(n)) = A.$$

By definition $g = \phi \circ g_1$, and so $g'(z) = \phi' \circ g_1(z) \cdot g_1'(z)$. Therefore

$$\alpha = \lim_{n \to \infty} g'(z_n) = \lim_{n \to \infty} \Big(\phi' \circ g_1(z_n) \cdot g_1'(z_n) \Big) = s \cdot A.$$

Then $A \neq \infty$, and

$$\angle \lim_{z \to \zeta} g'(z) = \angle \lim_{z \to \zeta} \Big(\phi' \circ g_1(z) \cdot g_1'(z) \Big) = cA = \alpha.$$

The lemma is proved. □

Definition 5. Let f be a continuous on the open unit disk Δ function. We say that the integral $\int\limits_0^1 f(z)dz$ converges non-tangentially if the function

$$\varphi(z) = \int\limits_0^z f(z)dz$$

has a non-tangential limit at the boundary point $z = 1$.

Theorem 4. *Let $f \in \mathcal{G}[1]$, i.e.,*

$$f(z) = -(1-z)^2 \mathfrak{p}(z) + \beta/2(z^2 - 1), \tag{4.9}$$

where $\beta \in \mathbb{R}$ and $\mathfrak{p} \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\,\mathfrak{p} > 0$ and $\angle \lim\limits_{z \to 1} (1-z)\mathfrak{p}(z) = 0$. Suppose that $\beta > 0$. Let $\{F_t\}_{t \geq 0}$ be a semigroup of holomorphic self-mappings generated by f. Then:

(I) the limit

$$K(z) := \lim_{t \to \infty} \exp(t\beta)\ (1 - F_t(z))$$

exists.

(II) the function K is either identically zero or a univalent function with $\mathrm{Re}\, K > 0$.

(III) $K(z) \not\equiv 0$ if and only if one of the following assertions holds:

(a) the integral

$$\int\limits_0^1 \frac{\beta(1-z) + f(z)}{(1-z)f(z)}\, dz$$

converges non-tangentially;

(b) the integral

$$\int\limits_0^1 \mathfrak{p}(z)\, dz$$

converges non-tangentially;

(c) the function Q defined by (4.3) is conformal at the point $z = 1$.

In this case $K(z) = K(0)Q(z)$.

Proof. Any element $F_t,\ t \geq 0$, of the semigroup is a univalent self-mapping of the unit disk Δ. Therefore any function

$$K_t(z) := \exp(t\beta)\ (1 - F_t(z))$$

is univalent and satisfies $\operatorname{Re} K_t > 0$. So, the family $\{K_t\}_{t\geq 0}$ is normal. In addition, it follows by (4.1) that $\{K_t\}_{t\geq 0}$ is compact:

$$|K_t(z)| \leq \frac{|1-z|^2}{1-|z|^2}\,\frac{1-|F_t(z)|^2}{|1-F_t(z)|} \leq \frac{|1-z|^2}{1-|z|^2}\,\frac{1-|F_t(z)|}{|1-F_t(z)|}\,(1+|F_t(z)|) \leq 2\frac{|1-z|^2}{1-|z|^2}\,.$$

So, there exists a convergent sequence $\{K_{t_n}\}$. Moreover, the limit function $K := \lim\limits_{n\to\infty} K_{t_n}$ is either $K \equiv 0$ or $K(z) \neq 0,\ z \in \Delta$.

Also we see that

$$\frac{K_t'(z)}{K_t(z)} = \frac{-F_t'(z)}{1-F_t(z)} = \frac{1}{f(z)}\,\frac{f(F_t(z))}{F_t(z)-1} \to \frac{\beta}{f(z)} \quad \text{as } t\to\infty.$$

Thus, a limit function K satisfies the following equation:

$$\beta K(z) = K'(z) f(z). \tag{4.10}$$

Since (4.10) is a homogeneous linear differential equation it follows by (3.18) that K is equal to zero identically, or $K(z) = \alpha Q(z)$ for some $\alpha \neq 0$.

Suppose that there exists a sequence $\{K_{t_n}\}$ converging to a non-zero function K:

$$K(z) = \lim_{n\to\infty} K_{t_n} = \lim_{n\to\infty} \exp(t_n\beta)\,(1-F_{t_n}(z)) = \alpha Q(z).$$

In this case K is univalent function on the unit disk Δ. We show now that the angular derivative $K'(1)$ exists and is equal to -1. We already know that any limit function K satisfies the differential equation (4.10). Then by Lemma 1 K is a solution of Schröder equations (4.7). Therefore denoting $w_n = F_{t_n}(z)$ for fixed $z \in \Delta$ and using Schröder equation (4.7), differential equation (4.10) and representation (4.9) one can calculate

$$\begin{aligned}
0 &= \lim_{n\to\infty}\Big(e^{t_n\beta}(1-w_n) - e^{t_n\beta}K(w_n)\Big) \\
&= \lim_{n\to\infty} e^{t_n\beta}\left[(1-w_n) - \frac{K'(w_n)f(w_n)}{\beta}\right] \\
&= \lim_{n\to\infty} e^{t_n\beta}\left[(1-w_n) + \frac{K'(w_n)\left[(1-w_n)^2\mathfrak{p}(w_n) - \beta/2(w_n^2-1)\right]}{\beta}\right] \\
&= \lim_{n\to\infty} e^{t_n\beta}(1-w_n)\left[1 + \frac{K'(w_n)(1-w_n)\mathfrak{p}(w_n)}{\beta} + \frac{K'(w_n)(w_n+1)}{2}\right] \\
&= K(z)\lim_{n\to\infty}\left[1 + K'(w_n) + K'(w_n)\,\frac{(1-w_n)\mathfrak{p}(w_n)}{\beta}\right].
\end{aligned}$$

So,

$$\lim_{n\to\infty} K'(F_{t_n}(z)) = -1.$$

By Lemma 2 we conclude that the angular derivative $K'(1)$ exists and is equal to -1. This implies that if there exists a sequence $\{K_{t_n}\}$ converges to a non-zero function K, then the function $Q(z) = K(z)/\alpha$ has the angular derivative $Q'(1) = -1/\alpha \neq 0$, i.e., Q is conformal at the point $z = 1$. Moreover, such non-zero limit function is unique: $K(z) = -Q(z)/Q'(1)$ and

$$Q(z) = \frac{1}{\alpha}\lim_{n\to\infty} K_{t_n} = \frac{1}{\alpha}\lim_{n\to\infty} \exp\left(t_n\beta\right)\,\left(1 - F_{t_n}(z)\right).$$

Further, by (4.4) and (2.1) we get

$$Q(z) = -Q'(z)(1-z)\left[\frac{(1-z)\mathfrak{p}(z)}{\beta} + \frac{1+z}{2}\right].$$

Combining these equalities with (4.3) we just calculate

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{\alpha} K_t(z) - Q(z) &= \lim_{t\to\infty}\left[\frac{1}{\alpha} e^{t\beta}\,\left(1 - F_t(z)\right) - e^{t\beta} Q(F_t(z))\right]\\ &= \lim_{t\to\infty} e^{t\beta}\left(1 - F_t(z)\right)\\ \times\left[\frac{1}{\alpha} + Q'(F_t(z))\right.&\left.\left(\frac{(1-F_t(z))\mathfrak{p}(F_t(z))}{\beta} + \frac{1+F_t(z)}{2}\right)\right]\\ &= \lim_{t\to\infty} K_t(z)\left(\frac{1}{\alpha} + Q'(F_t(z))\right) = 0,\end{aligned}$$

because of $Q'(1) = -\frac{1}{\alpha}$ and $\{K_t\}_{t\geq 0}$ is a compact family.

So, we have proved (I), (II) and (III c).

To proceed we first note that by (2.1)

$$\frac{\beta(1-z) + f(z)}{(1-z)f(z)} = \frac{\mathfrak{p}(z) - \frac{\beta}{2}}{(1-z)\mathfrak{p}(z) + \frac{\beta}{2}(1+z)}.$$

Since the denominator in this fraction does not vanish in Δ, and tends to $\beta \neq 0$ as z goes to 1 non-tangentially, we conclude that conditions (a) and (b) are equivalent.

Since

$$\beta Q(z) = Q'(z)f(z),$$

we have

$$\frac{\beta(1-u)+f(u)}{(1-u)f(u)} = \frac{Q'(z)}{Q(z)} + \frac{1}{1-z}\,,$$

and thus

$$\int_0^z \frac{\beta(1-u)+f(u)}{(1-u)f(u)}du = \log\left(\frac{Q(z)}{1-z}\right)\,. \tag{4.11}$$

Therefore, condition (a) holds if and only if the function Q is conformal at the boundary point $z=1$, i.e., condition (c) holds. The proof of the theorem is complete. □

Remark 1. Another proof that $K \not\equiv 0$ if and only if Q is conformal at the boundary point $z=1$ was given in [20].

Using this fact one can prove the equivalence of assertions (a) and (c) of the theorem by the following simple argument.

Using the identity

$$\frac{\beta}{f(z)} = \left(\frac{\beta(z-1)-f(z)}{(z-1)f(z)}\right) - \frac{1}{1-z}\,,$$

one can solve the Cauchy problem (1.3) as follows:

$$\beta t = \log(1-z) - \log(1-F_t(z)) - \int_z^{F_t(z)} \left(\frac{\beta(1-u)+f(u)}{(1-u)f(u)}\right)du, \tag{4.12}$$

and, consequently,

$$\log\Big(\exp(\beta t)\,(1-F_t(0))\Big) = -\int_0^{F_t(0)} \left(\frac{\beta(1-u)+f(u)}{(1-u)f(u)}\right)du. \tag{4.13}$$

It is known that for any point $z \in \Delta$ the curve $w = F_t(z)$, $t \geq 0$, is non-tangential. Therefore, if condition (a) holds, relation (4.13) implies that $K(0) \neq 0$. By the proof of the first part of the theorem, K is a univalent function with $\operatorname{Re} K(z) > 0$.

Conversely. Suppose that K is a univalent function. It follows by (4.11) and (4.13) that

$$\exp(\beta t)\,(1-F_t(0)) = \frac{1-F_t(0)}{Q(F_t(0))}\,.$$

Then

$$K(0) = \lim_{t\to\infty} \frac{1 - F_t(0)}{Q(F_t(0))}\,.$$

By Proposition 4.9 [12] the angular derivative

$$Q'(1) = \angle \lim_{z\to 1} \frac{Q(z)}{z-1}$$

exists and does not equal to 0, i.e., Q is conformal at the point $z = 1$.

Remark 2. It is clear that the function $g(z) = 1 - Q(z)$ is a solution of the Yaglom equation (4.2). In addition, it is known that $g(z) = \sum\limits_{k=0}^{\infty} b_k z^k$, where $b_k = \lim\limits_{t\to\infty} P(Z(t) = k|\, Z(t) > 0)$. Therefore, our theorem can be easily reformulated in terms of the generating function g.

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