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ON A CLASS OF MONGE-AMPÉRE BOUNDARY VALUE PROBLEMS AND ESTIMATES OF BOUNDS

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Abstract: The purpose of this paper is to generalize the result of Ma concerning the Monge-Ampère equation written in its general form as $\det u_{ij} = g(|\nabla u|^2)h(u)$, where u_{ij} denote the Hessian of u, and g, h are positive functions. With a Robin and Dirichlet boundary conditions he obtained in $\mathbb{R}^2$ an estimate for the mean curvature of $\partial\Omega$ of Ω, where Ω is a bounded convex domain. Our goal is to extend the Dirichlet case in $\mathbb{R}^N$ and to investigate the nonlinear boundary condition $\frac{\partial u}{\partial n} + \alpha(u) = 0$ on $\partial\Omega$, where α is subject to some appropriate conditions. This extension is due to the generalization of the maximum principle in $\mathbb{R}^N$.

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1. Introduction

This paper deals with a class of Monge-Ampère boundary problems subject to the nonlinear and Dirichlet boundary conditions. In [5], Ma considered the Monge-Ampère equation

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$$\det u_{ij} = c \text{ in } \Omega, \tag{1}$$

where Ω is a bounded convex domain sufficiently regular in the plane, subject to the following Robin boundary condition

$$\frac{\partial u}{\partial n} + \alpha\, u = 0 \text{ on } \partial\Omega, \tag{2}$$

where n is the outward normal vector to the boundary $\partial\Omega$ of Ω and α is a positive constant. Ma [5] established in the plane that if the solution u of (1), (2) is of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$ and convex then

$$\max_{x\in\bar{\Omega}} |\nabla\, u|^2 \;\leq\; \frac{c}{{K_0}^2} + \frac{2c}{\alpha K_0} \text{ in } \bar{\Omega}, \tag{3}$$

$$-\frac{c^{\frac{1}{2}}}{2{K_0}^2} - \frac{c^{\frac{1}{2}}}{\alpha K_0} \;\leq\; u \leq\; 0 \text{ in } \bar{\Omega}, \tag{4}$$

where $K_0 := \min_{x\in\partial\Omega} K(x)$.

In the Dirichlet case, Monge-Ampère equations have been investigated in [6,8] and for the other different boundary conditions Urbas showed a very simpler technique which works in the N dimensional case (see [10]). Our goal is to extend the maximum principle in $\mathbb{R}^N$ and to generalize the result of Ma [5] for the Monge-Ampère equations taken in its general form as

$$\det u_{ij} = h(u) g(\nabla\, u|^2). \tag{5}$$

In Section 2, we form a convenient differential inequality in Φ, where

$$\Phi := g(|\nabla\, u|^2) - h(u), \tag{6}$$

and basing on the second maximum principle of E. Hopf [2], we conclude that the combination Φ attains its maximum principle on the boundary $\partial\Omega$ of Ω. This will be established by choosing appropriately g, h and using the famous arithmetic-geometric inequality defined in this case by

$$\Delta\, u \;\geq\; N(gh)^{\frac{1}{N}}. \tag{7}$$

Section 3 indicates some applications of the derived maximum principle. We consider the Monge-Ampère equations (5) with a nonlinear boundary condition of the following form

$$\frac{\partial u}{\partial n} + \alpha(u) = 0 \quad \text{on } \partial\Omega, \tag{8}$$

where the function α and its derivative α' are subject to some conditions. This generalizes an earlier result of Ma [5] for which the estimation of the solution u and its gradient $|\nabla u|$ are extended for such α in dimension 2. In Section 4, we are interested to the Dirichlet case in $\mathbb{R}^N$ and we obtain point-wise lower bounds for $|\nabla u|$ in terms of the geometry of Ω. Such bound have already been obtained in $\mathbb{R}^2$ by Ma [5] for some particular cases of g and h in $\mathbb{R}^2$, when $g(u)h(|\nabla u|) := \text{const}$. Without using maximum principles of parabolic differential equations, the result of Ma [5] was recently generalized by Urbas using a very simple proof which is also valid for different boundary conditions (see [10]).

Finally, we mention that in [7] the generalization of the problem (2), (5) in $\mathbb{R}^N$ was investigated without using the normal coordinates of the Monge-Ampère equations which is here the key of the proof.

2. On a Maximum Principle

In what follows, we shall assume that the solution u of the Monge-Ampère equations defined by (5) is at least of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$ in a bounded domain Ω as described in Section 1. In this section, we will show that the maximum principle of the following combination Φ defined by

$$\Phi := g(u_{,i}u_{,i}) + h(u) \qquad \text{in } \bar{\Omega}, \tag{9}$$

is attained on the boundary $\partial\Omega$, where the functions f and g are subject to some conditions.

For the differential equation of the form

$$\Delta u + f(u) = 0 \text{ in } \bar{\Omega},$$

the corresponding function constructed for this type of equation depends essentially on the dimension N and the imposed boundary conditions for which in general the treatment in $\mathbb{R}^2$ differs from that of $\mathbb{R}^N$, where $N \geq 3$. This is due to some differential equalities which are valid only in $\mathbb{R}^2$, as

$$|\nabla u|^2 u_{,ij}u_{,ij} = |\nabla u|^2(\Delta u)^2 + u_{,i}u_{,ik}u_{,j}u_{,jk} - 2(\Delta u)u_{,i}u_{,j}u_{,ij}.$$

It is already known that the combination Φ defined by (9) attains its maximum principle at three different places for an arbitrary g and f (see R. Sperb [8]). Assuming that Φ in nonconstant, the corresponding maximum is attained on the boundary $\partial\Omega$ as a first possibility, at a critical point as a second possibility and finally at an interior point of the domain Ω. In our context, we choose g and f such that the elliptic differential inequality formed is strictly positive.

Theorem 1. *Let u be a strictly convex solution of (5) and Φ the combination defined by (9), then*

$$\frac{1}{2}u^{ij}\Phi_{,ij} + ... \quad = \quad g'(|\nabla u|^2)(-\frac{h'}{g} + \frac{h'}{gh} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}) \\ + \quad 2g'\Delta u + Nh', \tag{10}$$

where the dots stand for terms of the form $V_{,k}\Phi_{,k}$ with specific vector fields $V_{,k}$ which are bounded except at critical points of u.

To start the proof of Theorem 1, we construct an appropriate differential inequality for Φ except at a critical value of the solution u. Let Φ defined by (9) then

$$\Phi_{,i} = 2u_{,ik}u_{,k}g' + u_{,i}h', \tag{11}$$

$$\Phi_{,ij} \quad = \quad 2g'(u_{,ijk}u_{,k} + u_{,jk}u_{,ik}) + u_{,ij}h' \\ + \quad 4(u_{,jl}u_{,l}u_{,ik}u_{,k})g'' + u_{,i}u_{,j}h''. \tag{12}$$

Let u_{ij} be the inverse of the Hessian matrix $H := u^{ij}$. As u is strictly convex solution of (5), the matrix u^{ij} is positive definite and consequently by computing

$$u^{ij}\Phi_{,ij} \quad = \quad 2g'(u^{ij}u_{,ijk}u_{,k} + u^{ij}u_{,ij}h' + u^{ij}u_{,jk}u_{,ik}) \\ + \quad 4(u^{ij}u_{,jl}u_{,l}u_{,ik}u_{,k})g'' + u^{ij}u_{,i}u_{,j}h'', \tag{13}$$

we claim that $u^{ij}\Phi_{,ij}$ is strictly positive in $\bar{\Omega}$. We mention that the following identities $u^{ij}u_{,ij}u_{,jl} = \Delta u$, $u^{ij}u_{,ij} = N$, $u^{ij}u_{,il}u_{,l}u_{,jk}u_{,k} = u_{,kl}u_{,k}u_{,l}$ and $(gh)[u^{ij}u_{,ijk}u_{,k}] = (gh)_{,j}u_{,j}$, are valid in $\mathbb{R}^N$. Then we are able to prove that Φ satisfies an appropriate differential inequality, we compute

$$u^{ij}\Phi_{,i}u_{,j} = u^{ij}\{2u_{,j}u_{,ik}u_{,k}g' + u_{,i}u_{,j}h'\}, \tag{14}$$

$$u_i\Phi_{,i} = 2g'u_{,i}u_{,ik}u_{,k} + u_{,i}u_{,i}h'. \tag{15}$$

From (14) and (15), we obtain

$$-u^{ij}u_{,i}u_{,j}h' + u^{ij}\Phi_{,i}u_{,j} \quad = \quad 2g'u^{ij}u_{,j}u_{,ik}u_{,k} \\ = \quad 2u_{,i}u_{,i}g', \tag{16}$$

$$2u_{,ij}u_{,j}u_{,i}g' - u_{,i}\Phi_{,i} = -u_{,i}u_{,i}h'. \tag{17}$$

Hence by (16) and (17), we deduce the following inequality

$$u^{ij}\Phi_{,ij} + ... \quad = \quad 2g'(|\nabla u|^2)(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}) \\ + \quad 2g'\Delta u + Nh'. \tag{18}$$

Now through the arithmetic-geometric inequality (7) (or simply since g' is positive we can use $\Delta u > 0$) in $\mathbb{R}^N$, we get

$$\begin{aligned} u^{ij}\Phi_{,ij} + ... \quad &\geq \quad 2g'(|\nabla u|^2)(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}) \\ &+ \quad 2g'N(gh)^{\frac{1}{N}} + Nh, \end{aligned} \tag{19}$$

where g, h, g' and h' satisfy the following conditions

$$g' > 0, \; h' > 0, \tag{20}$$

and

$$(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}) \geq 0. \tag{21}$$

Then the maximum of Φ is attained on the boundary $\partial\Omega$ of Ω at some point P. If inequalities (20) are reversed and (21) unchanged, we then conclude that the minimum value of Φ occurs on the boundary $\partial\Omega$ or at a critical point of u.

We have established the following result.

Theorem 2. *If u is the solution of* (5) *and f, g are given functions satisfying* (20) *and* (21) *then the combination Φ defined by* (9) *attains its maximum on the boundary $\partial\Omega$ at some point P.*

To illustrate Theorem 2, we give the following example.

Example. Let g and f two positive functions defined by $g(s) := \ln(s+e),\; s := |\nabla u|^2$ and $h(t) := \exp(t),\; t := u$. We compute the first and second derivatives of each function: $g'(s) = \frac{1}{(s+e)} > 0$, $g''(s) = -\frac{1}{(s+e)^2}$ and $h'(t) = h''(t) = h(t)$. We can now observe in view of $-\frac{h}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'} > 0$ the condition (21) is satisfied.

As an application, we will give in the next section some bounds for the solution u and its gradient $|\nabla u|$.

3. Estimates of the Solution u and its Gradient $|\nabla u|$

In [5] Ma established the following theorem.

Theorem 3. *Under the above hypothesis on the domain Ω and a constant c, if the solution $u \in C^2(\bar{\Omega}) \cap C^3(\Omega)$ is a strictly convex solution for the boundary value problems* $(1)-(2)$, *then we have the following estimates*

$$\max_{x\in\bar{\Omega}} |\nabla u|^2 \leq \frac{c}{{K_0}^2} \; in \; \bar{\Omega}, \tag{22}$$

$$-\frac{c^{\frac{1}{2}}}{2{K_0}^2}-\frac{c^{\frac{1}{2}}}{\alpha K_0}\;\leq\;u\leq\;0 \qquad \textit{in}\ \bar{\Omega}, \tag{23}$$

where $K_0=\min_{x\in\Omega}K(x)$, $K(x)$ *is the curvature of the boundary* $\partial\Omega$ *of* Ω.

In this section, we consider Φ defined as follows

$$\Phi := g(|\nabla u|^2)-h(u), \tag{24}$$

with

$$h'<0, \quad g'>0. \tag{25}$$

We generalize this result for more general Monge-Ampère equations (5) with nonlinear boundary condition (8). We shall deduce that the combination Φ defined by (24) attains its maximum principle on the boundary $\partial\Omega$ of Ω, where Ω is a strictly convex bounded domain in $\mathbb{R}^N$, $N\;\geq\;2$. This result of maximum principle establish well in $\mathbb{R}^2$ the given isoperimetric bounds in [5]. As an application of this maximum principle, Ma [5] have given various bounds for the solution u and its gradient $|\nabla u|$ concerning the Monge-Ampére equations (1) subject to the Robin boundary condition (2).

The purpose of this paper is to investigate a class of boundary value problems for Monge-Ampère equations of type (5) with non-linear boundary condition (8) in $\mathbb{R}^2$. In the next theorem, we establish the explicitly bounds for the solution u and its gradient $|\nabla u|$, where the function α with its derivative α' appearing in (8) are subject to the following conditions

$$\alpha\,(0)=0, \qquad \alpha'\;>\;0 \quad \text{in}\ \bar{\Omega}, \tag{26}$$

where α is of class C^2 in $\bar{\Omega}$.

The tools used in the proof is the application of the maximum principle and the expression of the equation (5) rewritten in normal coordinates as

$$u_{nn}(u_{ss}+Ku_n)=gh+[(u_s)_n]^2. \tag{27}$$

In fact, this maximum principle can be deduced as in Section 2: in place of (3) we get

$$\begin{aligned} u^{ij}\Phi_{,ij}+... \quad &= \quad 2g'(\frac{h'}{g}+\frac{h'}{h}+\frac{h'g''}{(g')^2}-\frac{h''}{h'})(|\nabla u|^2) \\ &+ \quad 2g'\Delta\,u-Nh'. \end{aligned} \tag{28}$$

So the obvious conditions (20) and (21) are

$$g'>0,\ h'<0,\ \text{and}\ \frac{h'}{g}+\frac{h'}{h}+\frac{h'g''}{(g')^2}-\frac{h''}{h'}\;\geq\;0. \tag{29}$$

Theorem 4. *Let u be a strictly convex solution of Monge-Ampère equations (5) of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$ subject to (8) and let Ω be a bounded domain, regular and convex in $\mathbb{R}^2$. Then we have*

$$\max_{x\in\bar{\Omega}} g(|\nabla u|^2) \le g(\frac{D^2}{K_0^2}) + h(\frac{E}{K_0\alpha'(\xi)}), \tag{30}$$

$$\frac{g(\frac{D^2}{K_0^2}) + h(\frac{E}{K_0\alpha'(\xi)})}{h'(\xi)} < u < 0. \tag{31}$$

The two functions D and E arising in (31) are defined by

$$D := \frac{2g(c)g'(c)h(c)}{h'(c)}, \quad E := \frac{2g(c)g'(c)}{h'(c)},$$

where c denotes a positive constant.

To start the proof of Theorem 4, we use the fact that Φ attains its maximum on the boundary $\partial\Omega$ of Ω as indicated above. We employ the normal coordinates of the equation (27) in $\mathbb{R}^2$ and we compute the normal and tangential derivatives of Φ in order to obtain the desired estimates. If Φ attains its maximum principle on the boundary $\partial\Omega$ of Ω, at some point P, we must obtain at this point

$$\frac{\partial\Phi}{\partial n} \ge 0, \tag{32}$$

and

$$\frac{\partial\Phi}{\partial s} = 0. \tag{33}$$

So we compute explicitly the normal and tangential derivatives of Φ at P and we find

$$\begin{aligned}\frac{\partial\Phi}{\partial n} = g'\frac{\partial}{\partial n}(|\nabla u|^2) - \frac{\partial u}{\partial n}h' &= g'\frac{\partial}{\partial n}({u_n}^2 + {u_s}^2) - (u_n)h' \\ &= 2g'(u_n u_{nn} + u_s(u_s)_n) - (u_n)h' \ge 0, \end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial s} = g'\frac{\partial}{\partial s}(|\nabla u|^2) - \frac{\partial u}{\partial s}h' &= g'\frac{\partial}{\partial s}({u_n}^2 + {u_s}^2) - (u_s)h' \\ &= 2g'(u_n u_{ns} + u_s u_{ss}) - (u_s)h'. \end{aligned} \tag{35}$$

By the strict convexity of the solution u and the fact that u is subharmonic on the boundary $\partial\Omega$ of Ω, it follows from (27) that

$$u_{ss} + K u_n > 0 \text{ on } \bar{\Omega}. \tag{36}$$

It is very well known in $\mathbb{R}^2$ that the following identity is valid at the boundary point P

$$(u_s)_n = (u_n)_s - Ku_s. \tag{37}$$

Thus, we have by (8)

$$\begin{aligned} (u_s)_n &= (-\alpha(u))_s - Ku_s \\ &= (-u_s\alpha'(u)) - Ku_s \\ &= -(\alpha'(u) + K)u_s. \end{aligned} \tag{38}$$

By differentiating (24) with respect to s, the tangential derivative of the combination Φ takes the following form

$$\frac{\partial \Phi}{\partial s} = (2g'(\alpha(u)\alpha'(u) + u_{ss}) - h')u_s = 0, \tag{39}$$

from which we have to investigate separately the two following possibilities

$$u_s = 0, \tag{40}$$

or

$$2g'(\alpha(u)\alpha'(u) + u_{ss}) = h'. \tag{41}$$

We assume that (40) does not hold and (41) occurred at some point P on the boundary $\partial\Omega$, we obtain

$$u_{ss} = \frac{h'}{2g'} - \alpha(u)\alpha'(u), \tag{42}$$

which is negative by (26) and (29). We reach then a contradiction with the condition (36). In fact, the two conditions (26) and (36) together lead u_{ss} to be positive. We observe that the second normal derivative u_{nn} of u on the boundary $\partial\Omega$ has a constant sign

$$u_{nn} > 0. \tag{43}$$

Finally, employing (34), (43) and the maximum principle of E. Hopf [2], we obtain

$${u_s}^2(\alpha'(u) + K) < 0. \tag{44}$$

We arrive now to a contradiction since the function α' and the mean curvature K of the boundary $\partial\Omega$ of Ω are supposed in the prequel positive. Consequently

$u_s = 0$ at P, and therefore we deduce that $u = $ const. $= c$ at the same point P. Evaluating now the combination Φ defined by (24), we find

$$\Phi := g(\alpha^2(u)) - h(u). \tag{45}$$

By the second maximum principle of E. Hopf [2] we obtain $\alpha(u) < 0$ at P. Using again the Mean Value Theorem and (26) we conclude that $u < 0$ on the boundary $\partial\Omega$ of Ω and applying the maximum principle of E. Hopf [3] we conclude that

$$u < 0 \quad \text{in } \bar{\Omega}. \tag{46}$$

This implies that the maximum of the combination Φ on the boundary $\partial\Omega$ must be at a point where u itself is a minimum. Thus $u(P) = \min_{\partial\Omega} u$, and

$$u_{ss}(P) \geq 0. \tag{47}$$

Inserting the above inequalities (7), (31),(33) and (34) in the normal derivative of Φ we found

$$\frac{\partial \Phi}{\partial n} = 2g'\{-\alpha(u)u_{nn} - {u_s}^2(\alpha'(u) + K)\} + h'\alpha(u) \geq 0. \tag{48}$$

Using the Monge-Ampère equations (29) with the normal derivative of Φ (48), we close to

$$\frac{g(P)h(P)}{u_{ss}(P) + K(P)u_n(P)} \geq \frac{h'(P)}{2g'(P)}, \tag{49}$$

from which we can deduce that

$$-\alpha(u)K(P) \leq -\frac{2g(P)g'(P)h}{h'(P)}. \tag{50}$$

The non-linear term $\alpha(u)$ in (50) can be estimated by

$$\alpha(u)(P) \geq \frac{2g(P)g'(P)h}{h'(P)K(P)}. \tag{51}$$

Employing now (25) and (51), we get

$$h \geq \frac{h'K\alpha'(\xi)u}{2gg'}. \tag{52}$$

Finally, the solution u can be expressed as follows

$$u \geq \frac{2hgg'}{h'K\alpha'(\xi)}. \tag{53}$$

In order to obtain our desired inequality we make use of (25) to get

$$h(u) \le h(\frac{2hgg'}{h'K\alpha'(\xi)}), \tag{54}$$

which implies that

$$\begin{aligned} \max_{\bar\Omega} \Phi : &= g(\alpha^2(u))(P) - h(u)(P) \\ &\le g(\frac{D^2}{K_0^2}) + h(\frac{E}{K_0\alpha'(\xi)}), \end{aligned} \tag{55}$$

where $u < \xi < 0$. The term $\alpha'(\xi)$ appearing in (55) is a consequence of the mean value theorem where α is assuming to be at least of class C^2 on $\bar\Omega$. The first condition (30) is now achieved.

In the particular case when the positive function h is constant and equals to one and the function g is a positive constant, we obtain the new results due to Ma [5] with $\alpha(u) = \alpha u$ on the boundary condition $\partial\Omega$ of Ω.

For the second estimate (31), we use the following fact

$$\begin{aligned} -h(u_{\min})(P) &\le \max_{\bar\Omega} g(|\nabla u|^2) - h(u) \\ &\le g([\frac{2(gg'h)(P)}{h'(P)K_0(P)}]^2) + \frac{(2gg'h)(P)K_0(P)}{2h'(P)K_0(P)\alpha'(\xi)(P)}. \end{aligned} \tag{56}$$

The use of (40), (42), (45), (47) and (54) conducts us to

$$\frac{2(gg'h)}{K_0(P)h'(P)} \le u < 0 \quad \text{on } \partial\Omega, \tag{57}$$

and

$$\frac{g([\frac{2(gg'h)(P)}{h'(P)K_0(P)}]^2) + h(\frac{(2gg'h)(P)}{h'(P)K_0(P)\alpha'(\xi)(P)})}{h'(\bar\xi)} < u < 0, \tag{58}$$

where $\bar\xi$ is a real number between u and 0. In view of (40) the assertion (31) is proved. □

Example. Let u be a classical solution strictly convex of the following problem

$$\det u_{ij} = -cu|\nabla u|^2 \quad \text{in } \Omega, \tag{59}$$

where c is a positive constant and Ω is a bounded domain regular convex in $\mathbb{R}^N$, $N \ge 2$, subject to the Robin boundary condition

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \partial\Omega. \tag{60}$$

The following two relations are satisfied

$$\max_{\bar{\Omega}} |\nabla u|^2 \leq \frac{4c}{K_0^2} - \frac{2}{K_0 \alpha'}, \tag{61}$$

$$-\frac{4c}{K_0^2} - \frac{2}{K_0 \alpha'} < u < 0. \tag{62}$$

4. Various Bounds for the Dirichlet Problem in $\mathbb{R}^N$

Concerning the various bounds for u and its gradient ∇u, we consider Monge-Ampére equations (5) subject to the Dirichlet condition $u = 0$. Ma investigated in $\mathbb{R}^2$ a lower bound for both of the following problems

$$\det u_{,ij} = c \qquad \text{in } \Omega, \tag{63}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{64}$$

and

$$\det u_{,ij} = f(x) \qquad \text{in } \Omega, \tag{65}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{66}$$

where $0 < f(x) \leq c$ and u is a strictly convex solution. In this section, we investigate the general case. In fact, we give a lower bound for u and its gradient ∇u in terms of the geometry of Ω, for any functions g and h subject to (20) and (21). By applying Theorem 2 to the combination Φ defined by (24), we obtain at P

$$\Phi = g(u_n(P)^2) - h(0), \tag{67}$$

and consequently

$$-h(u) \leq [g(u_n(P)^2) - h(0)] + g(|\nabla u|^2). \tag{68}$$

At a critical point of u, this last inequality (68) will be reduced to

$$0 < -h(u_{\min}) \leq g(u_n(P)^2) - h(0), \tag{69}$$

which gives

$$-h(u_{\min}) + h(0) \leq g(u_n(P)^2). \tag{70}$$

In view of (64), the differential equation (5) can be rewritten at P as

$$K(P) u_n(P)^{N-1} u_{nn}(P) = g(u_n(P)^2) h(0), \tag{71}$$

where K is defined by

$$K := \prod_{1}^{N-1} K_i. \tag{72}$$

By (71) we can deduce the second normal derivative of u as follows

$$u_{nn}(P) = \frac{g(u_n(P)^2)h(0)}{K(P)u_n(P)^{N-1}}. \tag{73}$$

Now we need to use the following inequality valid in $\bar{\Omega}$ for $N \geq 2$

$$Nu_{,ki}u_{,ki} \geq (\Delta u)^2, \tag{74}$$

which is due to Payne (see Sperb [8]).

Evaluating (74) on the boundary $\partial\Omega$, we obtain

$$Nu_{nn}u_{nn} \geq (\Delta u|_{\partial\Omega})^2. \tag{75}$$

Using (7) and (75), we deduce that

$$u_{nn} \geq \sqrt{N}(gh)|_{\partial\Omega}^{\frac{1}{N}}. \tag{76}$$

Combining (69) and (76), we conclude that

$$\frac{g(u_n(P)^2)h(0)}{K(P)u_n(P)^{N-1}} \geq \sqrt{N}[g(u_n(P)^2)h(0)]^{\frac{1}{N}}. \tag{77}$$

Hence by (77), we get

$$K \leq \frac{g(u_n(P)^2)h(0)^{1-\frac{1}{N}}}{\sqrt{N}|\nabla u|^{\frac{N-1}{2}}}. \tag{78}$$

We then have proved the following theorem.

Theorem 5. *We assume that u is a classical solution of* (24) − (25) *subject to the Dirichlet condition* (64), *then we have the following estimates*

$$-h(u_{\min}) + h(0) \leq g(u_n(P)^2), \tag{79}$$

$$K \leq \frac{g(u_n(P)^2)h(0)^{1-\frac{1}{N}}}{\sqrt{N}|\nabla u|^{\frac{N-1}{2}}}. \tag{80}$$

Theorem 1.1 cited in Ma [5] is a particular case of Theorem 5 and the bound obtained in this note is valid for the N-dimensional space.

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