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REMARK ON EXISTENCE RESULT FOR SECOND ORDER EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract: Sufficient conditions for existence of the semilinear evolution equation in Banach spaces are obtained. An application to partial wave equations is given.

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1. Introduction

Throughout the past few years several authors have been investigated the existence and uniqueness of mild and strong solutions to abstract initial value problems with nonlocal initial conditions. Such works, concerning abstract nonlocal semilinear initial value problems was initiated by Byszewski [7]. In

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this paper Byszewski proved the existence of classical, strong and mild solutions using the Banach Fixed Point Theorem and the semigroup theory. We refer the reader to a complementary literature respect first order differential equations with nonlocal conditions [7], [8], [9], [3], [4], [2], [13], [16], [1], [11].

In this paper we study the global existence of solutions for second order initial value problem for semilinear differential equations, with nonlocal conditions, of the form

$$x''(t) = Ax(t) + f(t, x(t)), \ t \in I = [0, T], \tag{1}$$

$$x(0) = x_0 + g(x), \tag{2}$$

$$x'(0) = y_0 + h(x), \tag{3}$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ in a Banach space X, $f : I \times X \to X$, $g, h : C(I, X) \to X$ are given functions.

Some second order partial differential equations with nonlocal conditions modelled using the cosine function theory has been considered in the literature, see for example [5]-[13]. It is relevant to observe that the problems in these papers are studied by using the Schaefer (see, [5], [15], [14]) and the Banach Fixed Point Theorems (see, [11], [13]). The results of papers [5], [15], [14] are obtained under the assumption that the cosine function $C(t), t \in R$, generated by A, is compact for every $t > 0$. But if $C(t), t > 0$ is compact then the dimension of the state space X is finite, see Travis and Webb [18] for details. So the problems studied in these papers do not cover partial differential equations. This paper not only extends the above results, but it also completes them.

This work may be viewed as an attempt to develop a general existence theory for abstract nonlocal problems of the form (1), (2) under the compactness assumption on $S(t)$. Our basic tools are the theory of cosine functions and Krasnoselski-Schaefer type theorem. Also, this study is important from the viewpoint of applications since it covers wave equation with nonlocal conditions.

The outline of this paper is as follows. In Section 2, we recall some facts about cosine functions and fixed point theory. The main existence result is stated and proved in Section 3. Section 4 is devoted to the discussion of some examples.

2. Preliminaries

In this section we recall certain results that are necessary in the sequel.

Definition 1. A C_0-cosine operator function is defined as a one parameter family of operators $\{C(t) : t \in R\}$, $C(t) \in L(X)$, $t \in R$, having the following properties:

- $C(t+s) + C(t-s) = 2C(t)C(s)$ for any $t, s \in R$;
- $C(0) = I$ is the identity on X;
- $s - \lim_{h \to 0} C(h)x = x$ for any $x \in X$.

With a C_0-cosine operator function $C(\cdot)$, we associate the C_0-sine operator function

$$S(t)x := \int_0^t C(s)x ds, \qquad x \in X, t \in R,$$

and the lineals

$$E^k := \left\{ x \in X : C(\cdot)x \in C^k(R, X) \right\}.$$

Definition 2. A linear operator A with the domain $D(A)$ consisting of all x, for which there exists the limit

$$Ax := s - \lim_{h \to 0^+} 2\frac{C(h) - I}{h^2}x,$$

is called an infinitesimal generator of a C_0-cosine operator function $C(\cdot)$.

It is known that A is closed and densely defined in X, see [18]. We will assume that the adjoint operator A^* is densely defined on X^*. Also, it is known that the sets $\{C(t) : t \in [0,T]\}$, $\{S(t) : t \in [0,T]\}$ are bounded in $L(X)$.

Proposition 1. *For any C_0-cosine operator function $C(\cdot)$, there exists constants $M \geq 1$ and $\omega \geq 0$ such that for all $t \in R$, we have the estimate*

$$\|C(t)\| \leq M \cosh(\omega t), \quad t \in R, \tag{4}$$

where $\cosh(\omega t) := \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$ is the hyperbolic cosine. Infimum of the numbers ω from (4) is called the type of a C_0-cosine operator function and is denoted by $\omega_C(A)$.

Proposition 2. (see [18]) *The C_0-sine operator function $S(\cdot)$ is continuous in the uniform operator topology.*

- *For any $x \in X$, the following relations hold:*

$$s - \lim_{t \to 0} t^{-1}S(t)x = x \quad \text{and} \quad s - \lim_{t \to 0} 2t^{-2}\int_0^t S(r)xdr = x.$$

- *If an operator* $C(t)$ *is compact for each* $t \in (\alpha, \beta)$ *and for certain* $\alpha < \beta$, *then* $\dim X < \infty$.

Proposition 3. (see [18]) *The following conditions are equivalent:*

- *a* C_0*-sine operator function* $S(\cdot)$ *is compact:*
- *the resolvent* $\left(\lambda^2 I - A\right)^{-1}$ *is compact for any* λ *with* $\mathrm{Re}\lambda > \omega_C(A)$.

Definition 3. *A function* $x(\cdot) \in C([0,T], X)$ *is a mild solution of the* $(1)-(3)$ *if* $x(0) = x_0 + g(x)$ *and the integral equation below is verified*

$$x(t) = C(t)(x_0 + g(x)) + S(t)(y_0 + h(x)) + \int_0^t S(t-s) f(s, x(s))\, ds\,, \quad (5)$$

for $t \in [0,T]$.

We need the following Krasnoselski-Schaefer type fixed point theorem to prove our main result.

Theorem 1. (see [6]) *Let* Φ_1, Φ_2 *be two operators satisfying:*

(a) Φ_1 *is contraction, and*

(b) Φ_2 *is completely continuous.*

Then either:

(i) *the operator equation* $\Phi_1 x + \Phi_2 x = x$ *has a solution, or*

(ii) *the set* $\mathcal{E} = \left\{u \in X : \lambda\Phi_1\left(\frac{u}{\lambda}\right) + \lambda\Phi_2 u = u\right\}$ *is unbounded for* $\lambda \in (0,1)$.

3. The Existence Result

We assume the following hypothesis:

(A1) A is the infinitesimal generator of of a C_0-cosine operator function and the associated C_0-sine operator function $S(\cdot)$ is compact and $M = \sup\{\|C(t)\| : 0 \le t \le T\}$

(A2) For each $t \in [0,T]$, the function $f(t,\cdot) : X \to X$ is continuous and for each $y \in X$, the function $f(\cdot, y) : [0,T] \to X$ is strongly measurable.

(A3) For every positive integer k, there exists $\alpha_k(\cdot) \in L_1[0,T]$ such that for a.e. $t \in [0,T]$
$$\sup_{\|y\| \le k} \|f(t, y(t))\| \le \alpha_k(t).$$

(A4) There exists a continuous function $p : [0,T] \to R$ such that
$$\|f(t,y)\| \le p(t)\,\omega(\|y\|), \quad t \in [0,T], y \in X,$$
where $\omega : (0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(A5) The functions $g, h : C([0,T], X) \to X$ are continuous, h is uniformly continuous and such that
$$\|g(y)\| \le \int_0^T p(s)\|y(s)\|\,ds, \|h(y)\| \le \int_0^T p(s)\,\omega(\|y(s)\|)\,ds,$$
$$\|g(x) - g(y)\| \le L \int_0^T \|x(s) - y(s)\|\,ds\,,$$
for each $x, y \in C([0,T], X)$.

(A6) $G(\infty) = \infty$, where $G(u) - G(u_0) = \int_{u_0}^{u} \frac{ds}{s+\omega(s)}$. The function
$$H(s) = G(2s - c) - G(s) - M_1 \int_0^T p(s)\,ds$$
is increasing for $2s > c$, $c = M_1(\|x_0\| + \|y_0\|)$, $M_1 = M\max\{1, T\}$ and the functional equation $H(s) = 0$ has a solution.

(A7) $\max_{0 \le t \le T} \|C(t)\|\,L < 1$.

Theorem 2. *Assume that the assumptions (A1)-(A7) are satisfied. Then there exists a mild solution to (1), (2).*

Proof. The proof will be splited into several steps. Now we define operators Φ_1, Φ_2 on
$$B_k = \{z(\cdot) \in C([0,T], X) : \|z\| \le k\}$$
by
$$(\Phi_1 z)(t) = C(t)(x_0 + g(z))$$
and
$$(\Phi_2 z)(t) = S(t)(y_0 + h(z)) + \int_0^t S(t-s) f(s, z(s))\,ds.$$

We will show that Φ_1 verifies a contraction condition while Φ_2 is a completely continuous operator.

Step 1. Φ_1 *is contraction.* This step is obvious.

Step 2. Φ_2 *is completely continuous.* Let

$$B_k = \{y(\cdot) \in C([0,T], X) : \|y\| \leq k\}.$$

We first show that Φ_2 maps B_k into an equicontinuous family. Let $z \in B_k$ and $t_1, t_2 \in (0,T]$. Then if $t_1 < t_2$, and $\varepsilon > 0$ is small enough

$$\begin{aligned}
&\|(\Phi_2 z)(t_1) - (\Phi_2 z)(t_2)\| \leq \|S(t_1) - S(t_2)\| (\|y_0\| + \|h(z)\|) \\
&+ \int_0^{t_1-\varepsilon} \|S(t_1 - r) - S(t_2 - r)\| \|f(r, z(r))\| dr \\
&+ \int_{t_1-\varepsilon}^{t_1} \|S(t_1 - r) - S(t_2 - r)\| \|f(r, z(r))\| dr \\
&+ \int_{t_1}^{t_2} \|S(t_2 - r)\| \|f(r, z(r))\| dr \\
&\leq \|S(t_1) - S(t_2)\| (\|y_0\| + \|h(z)\|) \\
&+ \int_0^{t_1-\varepsilon} \|S(t_1 - r) - S(t_2 - r)\| \alpha_k(r) dr \\
&+ \int_{t_1-\varepsilon}^{t_1} \|S(t_1 - r) - S(t_2 - r)\| \alpha_k(r) dr \\
&+ \int_{t_1}^{t_2} \|S(t_2 - r)\| \alpha_k(r) dr.
\end{aligned} \tag{6}$$

It is known that the compactness of $S(t), t > 0$ implies the continuity in the uniform operator topology. Therefore by (6) the right hand side tends to zero as $t_2 - t_1 \to 0$, which means that $\Phi_2 z$ is continuous from the right at t_1. In the same manner one can prove that $\Phi_2 z$ is continuous from the left. On the other hand the compactness $h(B_k)$ implies equicontinuity of $\Phi_2 z$ at $t = 0$. Thus Φ_2 maps B_k into an equicontinuous family of functions. It is easy to see that the family $\Phi_2 B_k$ is uniformly bounded. Next we show that $\overline{\Phi_2 B_k}$ is compact. Since we have shown $\Phi_2 B_k$ is an equicontinuous collection, it suffices by the Arzela-Ascoli Theorem to show that Φ_2 maps B_k into a relatively compact set in X. To this end, let $0 < t \leq T$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $z(\cdot) \in B_k$, we define

$$(\Phi_2^\varepsilon z)(t) = S(t)(y_0 - h(z)) + \int_0^{t-\varepsilon} S(t-s) f(s, z(s)) ds.$$

Since $S(t)$ is a compact semigroup, the set $V_\varepsilon(t) = \{(\Phi_2^\varepsilon z)(t){:}y(\cdot) \in B_k\}$ is relatively compact in X for every ε, $0 < \varepsilon < t$. Moreover, for every $y(\cdot) \in B_k$, we have

$$\begin{aligned}\|(\Phi_2 z)(t) - (\Phi_2^\varepsilon z)(t)\| &\leq \int_{t-\varepsilon}^{t} \|S(t-s)\| \|f(s,z(s))\| \, ds \\ &\leq \int_{t-\varepsilon}^{t} \|S(t-s)\| \alpha_k(s) \, dr.\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set

$$\{(\Phi_2 z)(t) : y(\cdot) \in B_k\}.$$

Hence, the set $\{(\Phi_2 z)(t) : z(\cdot) \in B_k\}$ is relatively compact in X. Now by the Arzela-Ascoli Theorem $\overline{\Phi_2 B_k}$ is compact. It remains to show that the operator Φ_2 is continuous. Let $\{z_n(\cdot)\} \subset C([0,T],X)$ with $z_n \to z$ in $C([0,T],X)$. Then there exists an integer $r > 0$ such that $\|z_n(t)\| \leq r$ for all n and $t \in [0,T]$, so $z_n \in B_r$ and $z \in B_r$. By

$$f(s, z_n(s)) \to f(s, z(s)) \,,$$

for each $s \in [0,T]$ and since

$$\|f(s, z_n(s)) - f(s, z(s))\| \leq 2\alpha_r(s) \,,$$

we have by inequality below

$$\begin{aligned}\|\Phi_2 z_n - \Phi_2 z\| &= \sup_{0 \leq t \leq T} \|(\Phi_2 z_n)(t) - (\Phi_2 z)(t)\| \\ &\leq M_1 \|h(z_n) - h(z)\| + M_1 \int_0^T \|f(s, z_n(s)) - f(s, z(s))\| \, ds \,,\end{aligned}$$

and the dominated convergence theorem that the right hand side of the letter inequality approaches 0 as $n \to \infty$. Thus, Φ_2 is continuous. This completes the proof that Φ_2 is completely continuous on $C([0,T],X)$.

Step 3. The set

$$\mathcal{E}(\Phi) := \left\{ z(\cdot) : \lambda \Phi_1\left(\frac{z}{\lambda}\right) + \lambda \Phi_2 z = z \right\}$$

is bounded for $\lambda \in (0,1)$.

Let $z(\cdot) \in \mathcal{E}(\Phi)$. Then $\lambda\Phi_1\left(\frac{z}{\lambda}\right) + \lambda\Phi_2 z = z$ for some $\lambda \in (0,1)$ and.

$$\begin{aligned} \|z(t)\| = \lambda \left\| \Phi_1\left(\frac{z}{\lambda}\right) + \Phi_2 z \right\| &\leq M_1(\|x_0\| + \|y_0\|) \\ &+ M_1 \int_0^T p(s)(\|z(s)\| + \omega(\|z(s)\|))\, ds \\ &+ M_1 \int_0^t p(s)(\|z(s)\| + \omega(\|z(s)\|))\, ds. \end{aligned} \tag{7}$$

Denoting the right hand side of (7) by $q(t)$, we get

$$\begin{aligned} c &= M_1(\|x_0\| + \|y_0\|) \\ q(0) &= c + M_1 \int_0^T p(s)(\|z(s)\| + \omega(\|z(s)\|))\, ds, \\ q'(t) &= M_1 p(t)(\|z(t)\| + \omega(\|z(t)\|)) \leq M_1 p(t)(q(t) + \omega(q(t))). \end{aligned}$$

Note that, in view of the assumption (A6), the range of G is $[0,\infty)$. Hence, by Bihari type inequality, see for example [12] Theorem 1.3.1, we obtain

$$q(t) \leq G^{-1}\left[G(q(0)) + M_1 \int_0^t p(s)\, ds\right], \ 0 \leq t \leq T. \tag{8}$$

Now, we observe that

$$\begin{aligned} 2q(0) - c &= c + 2M_1 \int_0^T p(s)(\|z(s)\| + \omega(\|z(s)\|))\, ds \\ &= q(T) \leq G^{-1}\left[G(q(0)) + M_1 \int_0^T p(s)\, ds\right]. \end{aligned}$$

As a result, we get

$$G(2q(0) - c) - G(q(0)) \leq M_1 \int_0^T p(s)\, ds.$$

By the assumption (A6) it follows that the equation

$$H(s) = G(2s - c) - G(s) - M_1 \int_0^T p(s)\, ds = 0$$

has a solution c_0 such that $q(0) \leq c_0$. Hence, by (8), it follows that the estimate

$$\|z(t)\| \leq G^{-1}\left[G(c_0) + M_1 \int_0^T p(s)\, ds\right], \ 0 \leq t \leq T$$

holds. So there is a constant $K>0$ such that $\|z(t)\| \leq K$, where K depends on M_1, p, T, ω.

Consequently by Theorem 1, the operator Φ has a fixed point in $C([0,T], X)$. So the equation (1)-(3) has a mild solution. The theorem is proved. □

4. Wave Equation with Nonlocal Conditions

In this section we illustrate the result of the paper with the wave equation. On the space $X = L_2(0,\pi)$ we consider the operator $Af(\theta) = f''(\theta)$ with the domain $D(A) = \{f(\cdot) \in H_2(0,\pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $C(t), t \in R$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in N$, with the corresponding normalized eigenvectors $z_n(\theta) := (2/\pi)^{1/2} \sin n\theta$ and the following conditions hold:

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 \langle \varphi, z_n \rangle z_n$.

(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) \langle \varphi, z_n \rangle z_n$. Moreover, from these expression, it follows that $S(t)\varphi = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nt) \langle \varphi, z_n \rangle z_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0,\pi]$.

Now, we consider the boundary-value problem with nonlocal conditions

$$\begin{aligned}
\frac{\partial^2 w(t,\theta)}{\partial t^2} &= \frac{\partial^2 w(t,\theta)}{\partial \theta^2} + F(t,\theta,w(t,\theta)),\ t \in I = [0,\pi], \\
w(t,0) &= w(t,\pi) = 0,\ t \in I, \\
w(0,\theta) &= x_0(\theta) + \int_0^T \alpha(s) f_1(s, w(s,\theta))\, ds,\ \theta \in I, \\
\frac{\partial w(0,\theta)}{\partial t} &= y_0(\theta) + \int_0^T \beta(s) f_2(s, w(s,\theta))\, ds,\ \theta \in I, \qquad (9)
\end{aligned}$$

where $x_0, y_0 \in X$, $\alpha \in L_2(I)$, $\beta \in L_2(I)$ $F : I \times I \times R \to R$ and $f_i : I \times R \to R$, $i = 1,2$ are continuous. The nonlocal differential problem (9) can be modelled as the abstract nonlocal Cauchy problem

$$\begin{aligned}
u''(t) &= Au(t) + f(t,u(t)),\ t \in I, \\
u(0) &= x_0 + g(u), \\
u'(0) &= y_0 + h(u),
\end{aligned}$$

where $f(t,x)(\theta) = F(t,\theta,x(\theta))$, $x \in X$, and $g,h : C(I,X) \to X$ are defined by

$$g(u)(\theta) = \int_0^T \alpha(s) f_1(s,u(s,\theta))\,ds,$$

$$h(u)(\theta) = \int_0^T \beta(s) f_2(s,u(s,\theta))\,ds, \quad u \in C(I,X).$$

Consequently, the following existence result is a consequence of Theorem 2.

Theorem 3. *Assume that the conditions of Theorem 2 are satisfied. Then there exists a mild solution to the wave equation (9).*

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