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ON THE INJECTIVITY OF THE SYMMETRIC MULTIPLICATION MAP FOR LINE BUNDLES ON SMOOTH PROJECTIVE CURVES, II

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Abstract: Let X be a smooth projective curve and $L \in \mathrm{Pic}(X)$. Here we study the injectivity the symmetric multiplication map $\mu_L : S^2(H^0(X,L)) \to H^0(X, L^{\otimes 2})$ for certain double coverings of high genus curves.

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1. The Statements

This note is a continuation of [1]. Let X be a smooth and connected projective curve and $L \in \mathrm{Pic}(X)$. Let $\mu_L : S^2(H^0(X,L)) \to H^0(X, L^{\otimes 2})$ be the symmetrized multiplication map. Here (stimulated by [5]) we study the injectivity of the linear map μ_L. If $h^0(X,L) \leq 2$, then μ_L is obviously injective. If L is base point free the map μ_L is injective if and only if $h_L(X)$ is not contained in a quadric hypersurface of $\mathbf{P}(H^0(X,L)^*)$, where $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ is the morphism associated to the complete linear system $|L|$. The case of general curves of genus g was considered in [5]. Using her proofs in Section 2 we will prove the following results.

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Theorem 1. *Let $g \geq 7$ be an odd integer and $f : X \to Y$ a general double covering with X smooth curve of genus g and Y general smooth curve of genus $(g+1)/2$. The map μ_L is injective for every $L \in \mathrm{Pic}(X)$ with $\deg(L) \leq g+1$.*

Theorem 2. *Let $g \geq 8$ be an even integer and $f : X \to Y$ a general double covering with X smooth curve of genus g and Y general smooth curve of genus $g/2$. The map μ_L is injective for every $L \in \mathrm{Pic}(X)$ with $\deg(L) \leq g+1$.*

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$.

2. The Proofs

For every projective curve X and any $L \in \mathrm{Pic}(X)$, let $S^k(H^0(X,L)) \to H^0(X, L^{\otimes k})$ be the symmetrized multiplication map. We give the following easy results which quite often show that $\mu_{L,k}$ is not injective.

Remark 1. Let X be a smooth projective curve and $L, M \in \mathrm{Pic}(X)$ such that $M \cong L(-D)$ for some effective divisor D. The multiplication by the equation of D (resp. kD) induces an inclusion $H^0(X,M) \to H^0(X,L)$ (resp. $H^0(X, M^{\otimes k}) \to H^0(X, L^{\otimes k})$) and these inclusions are compatible with the symmetrized multiplication map. Thus if μ_M is not injective, then μ_L is not injective. Similarly, μ_L is injective if and only if $\mu_{L(-B)}$ is injective, where B is the base locus of $H^0(X,L)$.

Lemma 1. *Let X be an integral projective curve such that there is $L \in \mathrm{Pic}^d(X)$ spanned by its global section and such that $\mu_{L,k}$ is not injective. For every $R \in \mathrm{Pic}(X)$ such that $h^0(X,R) \geq d+1$ the map $\mu_{R,k}$ is not injective.*

Proof. Fix an effective divisor D such that $\mathcal{O}_X(D) \cong L$. Since $h^0(X,R) \geq d+1$, we have $h^0(X, R(-d)) > 0$. Hence we may see $H^0(X,L)$ (resp. $H^0(X, L^{\otimes k})$ as a subspace of $H^0(X,R)$ (resp. $H^0(X, R^{\oplus k})$) and these inclusions are compatible with the multiplication map of L and R. □

Corollary 1. *Let X be a smooth k-gonal curve. For every $R \in \mathrm{Pic}(X)$ such that $h^0(X,R) \geq 2k+1$ the map μ_L is not injective.*

Proof. Fix $M \in \mathrm{Pic}^k(X)$ computing the gonality of X and set $L := M^{\otimes 2}$. Apply Remark 1 and Lemma 1 □

Proof of Theorem 1. Let E be an elliptic curve and T a stable genus g curve with g irreducible components, say E_i, $1 \leq i \leq g$, each of them isomorphic to

E and such that $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We assume that for every integer i such that $2 \leq i \leq g-1$ the two points $P_i := E_{i-1} \cap E_i$ and $Q_i := E_i \cap E_{i+1}$ are Pic-independent in the sense of [6], i.e. for all pair of integers $(a,b) \neq (0,0)$ the line bundle $\mathcal{O}_{E_i}(aP_i + bQ_i)$ is not trivial. This is exactly what was used in [5] to prove [5], Theorem 1.1, i.e. we may use T as she used the stable curve C union of g elliptic curves. Thus by her proof (which heavily use the theory of limit linear series), her theorem if true for all genus g smooth curves X and all $L \in \mathrm{Pic}^d(X)$, $d \leq g+1$, such that X is in a suitable neighborhood of any such T. Hence by an elementary part of the theory of admissible coverings (see [3]) done with as target a genus $(g+1)/2$ curve insted of a genus 0 curve, it is sufficient to find one T as above and a stable genus $(g+1)/2$ curve A such that T is an unramified covering of A. Let A be the union of $(g+1)/2$ copies of E, say E_i, $1 \leq i \leq (g-1)/2$, such that $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We assume that for all integers x such that $2 \leq x \leq g-1$, $P_x = Q_{g-x+1}$; this is sufficient to see T as a double covering of $\cup_{i=1}^{(g+1)/2} E_i$ in which for $1 \leq i \leq (g-1)/2$ the curves E_i and E_{g-i} are mapped isomorphically onto E_i (the latter curve being mapped mapping the point Q of it onto the point P of E_i and the point P of it onto the point Q of E_i), while $E_{(g+1)/2}$ is mapped two-to-one onto itself. □

Proof of Theorem 2. Let E be an elliptic curve and Y a semistable genus g curve with g irreducible components, say E_i, $1 \leq i \leq g$, each of them isomorphic to E and one smooth and rational component B. We assume that the curve T of the proof of Theorem 1 is the stable reduction of Y and that Y is obtained from T adding the smooth rational curve B which intersects $E_{g/2}$ and $E_{1+g/2}$, so that on Y the components $E_{g/2}$ and $E_{1+g/2}$ are disjoint. We assume that for every elliptic component E_i of Y containing two singular points, say P_i and Q_i, these points P_i and Q_i are Pic-independents. Then we follow the proof of Theorem 1. The only difference is to show that Y is an admissible covering of a connected nodal curve C of genus $g/2$. Take as C the subcurve $\cup_{i=1}^{g/2} E_i \cup A$ of Y. For every integer i such that $1 \leq i \leq g$ the components E_i and E_{g-i+1} are mapped isomorphically onto E_i (the map is the identity on E_i and E_{g-i+1}, but the points called Q (resp. P) on E_{g-i+1} is the point called P (resp. Q) on E_i. The restriction to B of the admissible double covering $Y \to C$ is a degree two map $B \to B$.

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