**International Journal of Pure and Applied Mathematics**
**Volume 13 No. 2 2004, 137-142**

# INTERSECTION PRESERVING AND GLOBAL EXPANSIONS OF IDEALS IN $BCK$-ALGEBRAS

Young Bae Jun[1], Hee Sik Kim[2 §]

[1]Department of Mathematical Education
Gyeongsang National University
Chinju 660-701, KOREA
e-mail: ybjun@nongae.gsnu.ac.kr

[2]Department of Mathematics
Han-Yang University
Seoul 133-791, KOREA
e-mail: heekim@hanyang.ac.kr

**Abstract:** We introduce the notions of intersection preserving and global expansion of ideals in commutative $BCK$-algebras. We show that the homomorphic image and the inverse image of $\sigma$-primary ideal are the also $\sigma$-primary.

**AMS Subject Classification:** 06F35, 03G25
**Key Words:** intersection preserving, global expansion of ideals, $\sigma$-primary ideal, $BCK$-algebras

## 1. Introduction

The notion of $BCK$-algebras was proposed by Imai and Iséki in 1966. In the same year, Iséki introduced the notion of $BCI$-algebras which is a generalization of $BCK$-algebras. For the general development of $BCK/BCI$-algebras, the ideal theory plays an important role. The first author [2] introduced the

Received: January 2, 2004


§Correspondence author

notion of expansions of subalgebras and ideals in $BCK/BCI$-algebras, and the notion of $\sigma$-primary ideals in $BCK$-algebras. He also defined the notion of residual division, and investigated related properties. As a continuation of [2], in this paper, we introduce the notion of intersection preserving and global expansions of ideals in commutative $BCK$-algebras, and then we show that the homomorphic image and inverse image of $\sigma$-primary ideal are also $\sigma$-primary.

## 2. Preliminaries

We give herein the basic notions on $BCK$-algebras. For further information, we refer the reader to the book [3]. By a *BCK-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

- $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
- $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
- $(\forall x \in X)\,(x * x = 0)$,
- $(\forall x \in X)\,(0 * x = 0)$,
- $(\forall x, y \in X)\,(x * y = y * x = 0 \;\Rightarrow\; x = y)$.

A $BCK$-algebra $X$ is said to be *commutative* if it satisfies the equality:

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x)).$$

A nonempty subset $A$ of a $BCK$-algebra $X$ is called an *ideal* of $X$ if it satisfies:

- $0 \in A$,
- $(\forall x \in X)(\forall y \in A)\,(x * y \in A \;\Rightarrow\; x \in A)$.

Denote by $\mathbb{I}(X)$ the set of all ideals of a $BCK$-algebra $X$. Note that if $x$ is an element of an ideal $I$ of a $BCK$-algebra $X$ and $y \leq x$, then $y \in I$. A proper ideal $P$ of a commutative $BCK$-algebra $X$ is said to be *prime* if it satisfies:

$$(\forall x, y \in X)\,(x \wedge y \in P \;\Rightarrow\; x \in P \text{ or } y \in P),$$

where $x \wedge y = y * (y * x)$. Equivalently, $P$ is prime if and only if it satisfies:

$$(\forall I, J \in \mathbb{I}(X))\,(I \cap J \subseteq P \;\Rightarrow\; I \subseteq P \text{ or } J \subseteq P).$$

Maximal ideals of a $BCK$-algebra have the usual meaning. Every maximal ideal of a commutative $BCK$-algebra is prime (see [1, Proposition 3.5]).

## 3. Intersection preserving and global expansions

In what follows let $X$ denote a commutative $BCK$-algebra unless otherwise specified.

**Definition 3.1.** (see [2]) An *expansion of ideals* (in $X$) is defined to be a function $\sigma : \mathbb{I}(X) \to \mathbb{I}(X)$ such that

(e1) $(\forall G \in \mathbb{I}(X))\,(G \subseteq \sigma(G))$.

(e2) $(\forall G, H \in \mathbb{I}(X))\,(G \subseteq H \Rightarrow \sigma(G) \subseteq \sigma(H))$.

**Definition 3.2.** (see [2]) Let $\sigma$ be an expansion of ideals. Then an ideal $G$ of $X$ is said to be *$\sigma$-primary* if it satisfies:

$$(\forall a, b \in X)\,(a \wedge b \in G,\, a \notin G \Rightarrow b \in \sigma(G)).$$

Note that every prime ideal of $X$ is $\sigma$-primary (see [2]).

**Definition 3.3.** Let $\sigma$ be an expansion of ideals. Then:

(i) $\sigma$ is said to be *intersection preserving* if it satisfies:

$$(\forall I, J \in \mathbb{I}(X))\,(\sigma(I \cap J) = \sigma(I) \cap \sigma(J)),$$

(ii) $\sigma$ is said to be *global* if for each homomorphism $f : Y \to X$ of $BCK$-algebras, the following holds:

$$(\forall I \in \mathbb{I}(X))\,(\sigma(f^{-1}(I)) = f^{-1}(\sigma(I))).$$

**Example 3.4.** (1) The expansion of ideals $\sigma_0 : \mathbb{I}(X) \to \mathbb{I}(X)$ defined by $\sigma_0(G) = G$ for all $G \in \mathbb{I}(X)$ is both intersection preserving and global.

(2) For each $I \in \mathbb{I}(X)$ and $a \in X$, the set

$$a^{-1}I := \{x \in X \mid a \wedge x \in I\}$$

is an ideal of $X$ containing $I$, and if $I$ and $J$ are ideals of $X$ such that $I \subseteq J$ then $a^{-1}I \subseteq a^{-1}J$ (see [1]). Hence the function

$$\sigma_a : \mathbb{I}(X) \to \mathbb{I}(X),\; I \mapsto a^{-1}I$$

is an expansion of ideals. Moreover, for every $I, J \in \mathbb{I}(X)$ we have

$$\sigma_a(I \cap J) = a^{-1}(I \cap J) = a^{-1}I \cap a^{-1}J = \sigma_a(I) \cap \sigma_a(J).$$

Thus $\sigma_a$ is intersection preserving.

For each ideal $I$ of $X$, let

$$\mathfrak{M}(I) = \bigcap\{M \mid I \subseteq M,\ M \text{ is a maximal ideal of } X\}.$$

Then $\mathfrak{M}$ is an expansion of ideals in $X$.

**Theorem 3.5.** *$\mathfrak{M}$ is intersection preserving.*

*Proof.* For every $I, J \in \mathbb{I}(X)$, let

$$\mathcal{H}_1 = \{H \mid I \cap J \subseteq H,\ H \text{ is a maximal ideal of } X\},$$

$$\mathcal{H}_2 = \{H \mid I \subseteq H \text{ or } J \subseteq H, H \text{ is a maximal ideal of } X\}.$$

Then $\bigcap \mathcal{H}_1 = \mathfrak{M}(I \cap J)$ and $\bigcap \mathcal{H}_2 = \mathfrak{M}(I) \cap \mathfrak{M}(J)$. It is clear that $\mathcal{H}_2 \subseteq \mathcal{H}_1$. If $H \in \mathcal{H}_1$ then $I \cap J \subseteq H$. But $H$ is maximal, and so $H$ is prime. It follows that $I \subseteq H$ or $J \subseteq H$, that is, $H \in \mathcal{H}_2$. Hence $\mathcal{H}_1 = \mathcal{H}_2$ and $\mathfrak{M}(I \cap J) = \mathfrak{M}(I) \cap \mathfrak{M}(J)$. This completes the proof. □

We have herein the following question: Is $\mathfrak{M}$ global? But we conjecture that $\mathfrak{M}$ is not global. So, finding an example is our task in future.

**Theorem 3.6.** *Let $\sigma$ be an expansion of ideals which is intersection preserving. If $Q_1, Q_2, \cdots, Q_n$ are $\sigma$-primary ideals of $X$ and $P = \sigma(Q_i)$ for all $i$, then $Q = \bigcap_{i=1}^{n} Q_i$ is $\sigma$-primary.*

*Proof.* Let $a, b \in X$ be such that $a \wedge b \in Q$ and $a \notin Q$. Then $a \notin Q_k$ for some $k \in \{1, 2, \cdots, n\}$. But $a \wedge b \in Q \subseteq Q_k$ and $Q_k$ is $\sigma$-primary, which imply that $b \in \sigma(Q_k)$. Now

$$\sigma(Q) = \sigma(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n} \sigma(Q_i) = P = \sigma(Q_k),$$

and so $b \in \sigma(Q)$. Therefore $Q$ is $\sigma$-primary. □

Let $f : Y \to X$ be a homomorphism of $BCK$-algebras. Note that if $I$ is an ideal of $X$, then $f^{-1}(I)$ is an ideal of $Y$, and that if $f$ is surjective and $J$ is an ideal of $Y$ then $f(J)$ is an ideal of $X$.

**Theorem 3.7.** *Let $\sigma$ be a global expansion of ideals and let $f : Y \to X$ be a homomorphism of commutative $BCK$-algebras. If $I$ is a $\sigma$-primary ideal of $X$, then $f^{-1}(I)$ is a $\sigma$-primary ideal of $Y$.*

*Proof.* Let $a, b \in Y$ be such that $a \wedge b \in f^{-1}(I)$ and $a \notin f^{-1}(I)$. Then

$$f(a) \wedge f(b) = f(a \wedge b) \in f(f^{-1}(I)) \subseteq I$$

and $f(a) \notin I$. Since $I$ is $\sigma$-primary, it follows that $f(b) \in \sigma(I)$ so that $b \in f^{-1}(\sigma(I)) = \sigma(f^{-1}(I))$. Hence $f^{-1}(I)$ is $\sigma$-primary. □

**Lemma 3.8.** *Let $f : Y \to X$ be a homomorphism of $BCK$-algebras. If $J$ is an ideal of $Y$ containing the kernel of $f$, then $f^{-1}(f(J)) = J$.*

*Proof.* Clearly $J \subseteq f^{-1}(f(J))$. Now let $y \in f^{-1}(f(J))$. Then $f(y) \in f(J)$, and so there exists $x \in J$ such that $f(y) = f(x)$. Hence $f(y*x) = f(y)*f(x) = 0$, which implies that $y * x \in \mathrm{Ker} f \subseteq J$. Since $J$ is an ideal containing $x$, it follows that $y \in J$ so that $f^{-1}(f(J)) \subseteq J$. Therefore $f^{-1}(f(J)) = J$. □

**Theorem 3.9.** *Let $f : Y \to X$ be a surjective homomorphism of commutative $BCK$-algebras and let $J$ be an ideal of $Y$ containing* $\mathrm{Ker} f$. *Then $J$ is $\sigma$-primary if and only if $f(J)$ is a $\sigma$-primary ideal of $X$, where $\sigma$ is a global expansion of ideals.*

*Proof.* Sufficiency follows from Theorem 3.7 and Lemma 3.8. Suppose that $J$ is $\sigma$-primary. Let $a, b \in X$ be such that $a \wedge b \in f(J)$ and $a \notin f(J)$. Since $f$ is surjective, we have $f(x) = a$ and $f(y) = b$ for some $x, y \in Y$. Then

$$f(x \wedge y) = f(x) \wedge f(y) = a \wedge b \in f(J),$$

which implies $x \wedge y \in f^{-1}(f(J)) = J$. Now $f(x) = a \notin f(J)$ implies $x \notin J$. Since $J$ is $\sigma$-primary, it follows that $y \in \sigma(J)$ so that $b = f(y) \in f(\sigma(J))$. Using Lemma 3.8 and the fact that $\sigma$ is global, we get

$$\sigma(J) = \sigma(f^{-1}(f(J))) = f^{-1}(\sigma(f(J))),$$

and so $f(\sigma(J)) = f(f^{-1}(\sigma(f(J)))) = \sigma(f(J))$ by the surjectivity of $f$. Therefore $f(J)$ is $\sigma$-primary. This completes the proof. □

## References


[1] M. Aslam, A.B. Thaheem, On ideals of $BCK$-algebras, *Demonstratio Math.*, **27**, No. 3-4 (1994), 635–640.

[2] Y.B. Jun, Expansions of subalgebras and ideals in $BCK/BCI$-algebras, *Sci. Math. Jpn.*, Submitted.

[3] J. Meng, Y.B. Jun, *BCK-Algebras,* Kyungmoon Sa Co., Korea (1994).