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QUASILINEAR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract: In the paper the author, using Aleksandrov-Pucci maximum principle, proves an L^∞ a priori estimate and also uniqueness for weak solution u of a Dirichlet problem associated to quasilinear strictly elliptic equations with Charatheodory coefficients. The results obtained are a first step in the study of weak solvability of boundary value problems for quasilinear elliptic equations.

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1. Introduction and Assumptions

The main goal of the present paper is to establish an a priori bound of $\|u\|_{L^\infty(\Omega)}$ for the weak solutions $u \in W^{1,q}(\Omega)$, $q > n$, of the equation:

$$\mathcal{M}u = 0 \qquad \text{almost everywhere in} \quad \Omega, \text{ where}$$

$$\mathcal{M} \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial_{x_i}} (a_{ij}(x,u) \frac{\partial}{\partial_{x_j}} u) + b(x,u). \tag{1.1}$$

The coefficients $a_{ij}(x,z)$, $b(x,z) \in \Omega \times \mathbb{R}$ are Charatheodory functions and

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also a_{ij} are uniformly elliptic and such that is true the local uniform continuity with respect to the second variable u, uniformly in x.

On the coefficients $b(x,u)$ we point out the characteristic hypotheses that it is less or equal to a function of the Lebesgue space $L^q(\Omega)$, $q > n$.

The second result is uniqueness of the solution of the Dirichlet problem associated to the above equation, then we have extended the uniqueness result obtained in [8] from the linear case to the quasilinear one.

Divergence form quasilinear elliptic equations have been studied by Ladyzenskaya and Ural'ceva in [5], where they consider the terms a_{ij} to be sufficiently smooth. Also in the book [3] the authors have supposed regularity assumptions on the coefficients.

We wish to mention the study made by Simon in [10], where he suppose that the terms having first partial derivatives are local Lipschitz continuous and also the recent paper [6], where are considered Hölder continuous coefficients for a divergence form elliptic equation to obtain L^∞ estimates for the gradient of the solution.

We point out that our study generalizes those results because in this note the coefficients a_{ij} are discontinuous, precisely belong to the class VMO, and it is possible to prove that $C^0 \subset VMO$.

The class VMO, first considered by Sarason in the note [9], has been used by many authors, we want to recall the two papers [1] and [2], where has been very well studied the linear elliptic equations in nondivergence form, and the study made in [7] for nondivergence form equations.

The *vanishing* mean oscillation functions made a subset of the set of *bounded* mean oscillation functions, known as BMO class and defined by John and Nirenberg in the note [4].

Let us now establish the assumptions we will need in the following.

Let us consider Ω be a bounded domain in $\mathbb{R}^n$, $n > 2$, $a_{ij}(x,z)$ and $b(x,z)$ be Charatheodory functions or equivalently measurable on $x \in \Omega$, $\forall z \in \mathbb{R}$, and continuous respect to z for all almost $x \in \Omega$. Moreover we assume that:

$\exists\lambda$ positive constant such that

$$a_{ij}(x,z)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{a.a.} \quad x \in \Omega, \forall z \in \mathbb{R}\,, \tag{1.2}$$

and

$$a_{ij}(x,z) = a_{ji}(x,z) \quad \text{a.a.} \quad x \in \Omega, \forall z \in \mathbb{R}\,.$$

Let a_{ij} be local uniform continuous in z, uniformly in x:

$$|a_{ij}(x,z) - a_{ij}(x,z')| \leq a(x)\mu_M(|z-z'|) \text{ a. e. } \Omega, \forall z, z' \in [-M, M], \tag{1.3}$$

where $a(x) \in L^\infty(\Omega)$, $\mu_M(t)$ is a non-decreasing function, and $\lim_{t\to 0}\mu_M(t) = 0$, $a_{ij}(x,0) \in L^\infty(\Omega)$.

We set that $a_{ij}(x,z) \in VMO$ with respect to x and are loc. unif. in z:

$$\sup_{\rho\le r} \frac{1}{|\Omega_\rho|}\int_{\Omega_\rho}\left|a_{ij}(x,z) - \frac{1}{|\Omega_\rho|}\int_{\Omega_\rho} a_{ij}(y,z)dy\right| dx = \eta_M(r)\,, \tag{1.4}$$

for all $z \in [-M, M]$ and satisfy the following condition

$$\lim_{r\to 0}\eta_M(r) = 0\,, \tag{1.5}$$

where $\Omega_\rho = \Omega \cap B_\rho$ and B_ρ is in class of the balls with radius ρ centered at the points of Ω.

As it concerns $b(x,z)$ we suppose that:

$$|b(x,z)| \le b_1(x) \quad \text{a.e. } \Omega \quad \forall z \in \mathbb{R}\,, \tag{1.6}$$

where $b_1(x) \in L^q(\Omega)$, $q > n$.

2. A Priori Estimates

Theorem 2.1. *Let $u \in W^{1,q}(\Omega)$, $q > n$ be a solution of the quasilinear equation (1.1) and (1.2) and (1.6) be true. Then there exists a constant C dependent on n and diamΩ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\,(n, |\Omega|)\cdot(\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}). \tag{2.1}$$

Proof. Let us set $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. Then using Lemma 10.8 in [1] and taking into account that $u \in W^{1,q}(\Omega)$ we have

$$\sup_{\Omega^+} u \le C(n, |\Omega|)\cdot(\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}) \quad (u^+ = \max(u(x), 0))\,. \tag{2.2}$$

If we consider the function $v = -u$ we have

$$\sup_\Omega v \le C(n, |\Omega|)\cdot(\|v^+\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})\,,$$

and then

$$\begin{aligned}-\inf_\Omega u &\le C(n, |\Omega|)\cdot(\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})\\ &\le C(n, |\Omega|)\cdot(\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)})\,,\end{aligned}$$

and

$$u(x) \geq \inf_{\Omega} u \geq -C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}).$$

The last inequality combined with (2.2) gives

$$\sup_{\Omega} |u| \leq C(n, |\Omega|) \cdot (\|u\|_{L^2(\Omega)} + \|b_1\|_{L^q(\Omega)}). \qquad \square$$

3. Uniqueness Result

Theorem 3.1. *Let a_{ij} be bounded and measurable functions independent of z, let (1.2) condition be true, $b(x, z)$ non increasing in z, a.e. in Ω. If $u, v \in W^{1,q}(\Omega)$ $q > n$, are solutions of the following Dirichlet problem*

$$\begin{cases} \mathcal{M}\tau = 0 & \text{a.e. in } \Omega\,, \\ \tau = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

then $u = v$ in $\overline{\Omega}$.

Proof. The difference $w = u - v \in W^{1,q}(\Omega)$ solves

$$D_i(a_{ij}(x)D_j w) + b(x, u) - b(x, v) = 0 \quad \text{a.e. in } \Omega\,. \tag{3.2}$$

Let us set $\Omega^+ = \{x \in \Omega : w(x) > 0\}$, we have $u(x) > v(x)$ a.e. in Ω^+, and from the hypothesis $b(x, u(x)) \leq b(x, v(x))$ a.e. in Ω^+, then

$$b(x, u(x)) - b(x, v(x)) \leq 0.$$

Thus (3.2) conducts to the following inequality

$$D_i(a_{ij}(x)D_j w) \geq 0 \quad \text{almost everywhere in } \Omega^+.$$

Applying the maximum principle for divergence form operators

$$w(x) \leq \sup_{\Omega^+} w \leq \sup_{\partial\Omega^+} w^+ = 0 \quad (w = 0 \text{ on } \partial\Omega)\,,$$

then $w \leq 0$ in $\overline{\Omega}$.

If we substitute w by $-w$ we have $w \geq 0$, that is equivalent to $u = v$ in Ω. $\square$

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