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POLYNOMIALLY CONVEX COMPACTS OF $\mathbf{C}^{\mathbf{N}}$ AND SHEAF COHOMOLOGY

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove Theorem A and Theorem B for "coherent " sheaves on polynomially convex compact subsets of $\mathbf{C}^{\mathbf{N}}$.

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Here we study the following concept for compact subsets of certain topological vector spaces.

Definition 1. Let X be a complex space (even infinite-dimensional), $K \subset X$ a compact subset, $\mathcal{F}$ an $\mathcal{O}_X$-sheaf and $\mathcal{G}$ an $\mathcal{O}_K$-sheaf. We will say that $\mathcal{F}$ is coherent if for every $P \in X$ there is an open neighborhood U of P in X, positive integers a, b and a morphism $f : \mathcal{O}_U^{\oplus a} \to \mathcal{O}_U^{\oplus b}$ of $\mathcal{O}_U$-sheaves such that $\mathcal{F}|U \cong \mathrm{Coker}(f)$ (as $\mathcal{O}_U$-sheaves). We will say that $\mathcal{G}$ is coherent if there is an open neighborhood U of K such that $\mathcal{G}$ extends as an $\mathcal{O}_U$-sheaf to a coherent sheaf on U.

This definition was inspired by the finite dimensional case and by the definition of coherent ring ([2], Appendix 2).

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Theorem 1. *Let $K \subset \mathbf{C}^{\mathbf{N}}$ be a compact subset with a fundamental system of neighborhoods formed by holomorphically convex open subsets of V and $\mathcal{G}$ a coherent sheaf on K. Then there are integers $s \geq 2$, $n_i > 0$, $1 \leq i \leq s$, morphisms $M_j : \mathcal{O}_K^{\oplus n_j} \to \mathcal{O}_K^{\oplus n_{j+1}}$, $1 \leq j \leq s-1$ (i.e. an $n_j \times n_{j+1}$ matrix with holomorphic functions on K as entries), such that M_1 is injective, $\mathrm{Im}(M_i) = \mathrm{Ker}(M_{i+1})$ for $1 \leq i \leq s-2$ and $\mathrm{Coker}(M_{s-1}) \cong \mathcal{G}$.*

Theorem 2. *Let $K \subset \mathbf{C}^{\mathbf{N}}$ be a polynomially convex compact subset and $\mathcal{G}$ a coherent $\mathcal{O}_K$-sheaf. Then $H^i(K, \mathcal{G}) = 0$ for every $i > 0$.*

Proof of Theorem 1. For any integer $n \geq 1$, let $\pi_n : \mathbf{C}^{\mathbf{N}} \to \mathbf{C}^n$ be the projection on the first n factors. Take an open neighborhood U of K on which $\mathcal{G}$ extends and it is coherent. We may assume that U is holomorphically convex. Working on each connected component of U we are reduced to the case in which U is connected. By [1], Theorem 3, there is an integer $n \geq 1$ and a domain of holomorphy $W \subset \mathbf{C}^n$ such that $U = \pi_n^{-1}(W)$. □

Proof of Theorem 2. Since K is polynomially convex, it has a fundamental system of neighborhhods formed by holomorphically convex open subsets of $\mathbf{C}^{\mathbf{N}}$. By Theorem 1 there are integers $s \geq 2$, $n_i > 0$, $1 \leq i \leq s$, morphisms $M_j : \mathcal{O}_K^{\oplus n_j} \to \mathcal{O}_K^{\oplus n_{j+1}}$, $1 \leq j \leq s-1$, such that M_1 is injective, $\mathrm{Im}(M_i) = \mathrm{Ker}(M_{i+1}$ for $1 \leq i \leq s-2$ and $\mathrm{Coker}(M_{s-1}) \cong \mathcal{G}$. By [3], Theorem 1, we have $H^i(K, \mathcal{O}_K) = 0$ for every $i > 0$. Since M_1 is injective, we have $\mathrm{Im}(M_1) \cong \mathcal{O}_K^{\oplus n_1}$ and hence $H^i(K, \mathrm{Im}(M_1)) = 0$ for every $i > 0$. From the exact sequences

$$0 \to \mathrm{Im}(M_j) \to \mathcal{O}_K^{\oplus n_{j+1}} \to \mathrm{Im}(M_{j+1}) \to 0\,, \tag{1}$$

for $1 \leq j \leq s-2$ and induction on j we obtain $H^i(K, \mathrm{Im}(M_h)) = 0$ for all $i > 0$, $1 \leq h \leq s-1$. From the exact sequence

$$0 \to \mathrm{Im}(M_{s-1}) \to \mathcal{O}_K^{\oplus n_s} \to \mathrm{Coker}(M_{s-1}) \to 0\,, \tag{2}$$

and the isomorphis, $\mathrm{Coker}(M_{s-1}) \cong \mathcal{G}$, we obtain $H^i(K, \mathcal{G}) = 0$ for every $i > 0$. □

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