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ON MULTIVARIATE ORDER STATISTICS BY MARGINAL ORDERING OF I.N.N.I.D. RANDOM VECTORS FROM DISCRETE PARENTS

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Abstract: In this study, the joint probability function (p.f.) of multivariate order statistics by marginal ordering of independent and not necessarily identically distributed (i.n.n.i.d.) random vectors from discrete parents is examined. Furthermore, the continuous cases are derived from the discrete cases.

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1. Introduction and Background

Reiss [5] considered the joint probability density function (p.d.f.) of any k order statistics of independent and identically distributed (i.i.d.) random variables with a continuous distribution function (d.f.). He also considered p.d.f. of bivariate order statistics by marginal ordering of i.i.d. random vectors with a continuous d.f. by means of multinomial probabilities of appropriate "cell frequency vectors", defining multivariate order statistics by marginal ordering of i.i.d. random vectors with a continuous d.f.

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Guilbaud [3] expressed probability of the functions of independent and not necessarily identically distributed (i.n.n.i.d.) random vectors as a linear combination of probabilities of the functions of i.i.d. random vectors and thus also for order statistics of random variables.

Vaughan and Venables [7] and Samuel and Thomas [6] denoted the joint p.d.f. of order statistics of i.n.n.i.d. random variables with continuous distribution functions (d.f.'s) by means of permanents.

Gaoxing and Lee [2] obtained the joint probability function (p.f.) of any k order statistics from a general discrete parent by "tie-runs".

David [1] considered the fundamental distribution theory of order statistics.

Khatri [4] examined marginal p.f. of a single order statistic and the joint p.f. for any two order statistics of i.i.d. random variables from a discrete parent.

In this study, the joint p.f. of multivariate order statistics by marginal ordering of i.n.n.i.d. random vectors from discrete parents is given. Moreover, transition from discrete cases to the continuous cases is discussed.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

Consider $\mathbf{x} = (x^{(1)}, x^{(2)}, ..., x^{(d)})$ and $\mathbf{y} = (y^{(1)}, y^{(2)}, ..., y^{(d)})$, then it can be written as $\mathbf{x} \leq \mathbf{y}$ if $x^{(j)} \leq y^{(j)}$, $j = 1, 2, ..., d$.

Let $\xi_i = \left(\xi_i^{(1)}, \xi_i^{(2)}, ..., \xi_i^{(d)}\right)$, $i = 1, 2, ..., n$ be n i.n.n.i.d. random vectors from discrete parents and suppose that the components of ξ_i are independent. The expression

$$X_{r:n}^{(j)} = Z_{r:n}\left(\xi_1^{(j)}, \xi_2^{(j)}, ..., \xi_n^{(j)}\right), \qquad r = 1, 2, ..., n \tag{1.1}$$

is stated as the r-th order statistic of the j-th components of $\xi_1, \xi_2, \ldots, \xi_n$. From (1.1), the ordered values of the j-th components of $\xi_1, \xi_2, \ldots, \xi_n$ are expressed as

$$X_{1:n}^{(j)} \leq X_{2:n}^{(j)} \leq \ldots \leq X_{n:n}^{(j)}. \tag{1.2}$$

From (1.2), $\mathbf{X}_{r:n} = \left(X_{r:n}^{(1)}, X_{r:n}^{(2)}, \ldots, X_{r:n}^{(d)}\right)$ can be written [5].

If $\mathbf{a}_1, \mathbf{a}_2, ...$ are defined as column vectors, then the matrix obtained by taking i_1 copies of $\mathbf{a}_1, i_2$ copies of $\mathbf{a}_2, ...$ can be denoted as

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots \end{bmatrix}$$
$$\quad i_1 \quad i_2 \quad \ldots$$

and per $\mathbf{A}$ denotes the permanent of a square matrix $\mathbf{A}$, which is defined as similar to determinants except that all terms in the expansion have a positive sign [6].

2. P.f. of Multivariate Order Statistics by Marginal Ordering of I.N.N.I.D. Random Vectors from Discrete Parents

Now, the joint p.f. of $X^{(j)}_{r_1:n}, X^{(j)}_{r_2:n}, \ldots, X^{(j)}_{r_p:n}$ and $\mathbf{X}_{r_1:n}, \mathbf{X}_{r_2:n}, \ldots, \mathbf{X}_{r_p:n}$, $p = 1, 2, ..., n$ will be given in Theorem 2.1 and Theorem 2.2, respectively.

Theorem 2.1. *Let f_i and F_i be p.f. and d.f. of ξ_i, and $f_i^{(j)}$ and $F_i^{(j)}$ be p.f. and d.f. of $\xi_i^{(j)}$, respectively. Then, the joint p.f. of $X^{(j)}_{r_1:n}, X^{(j)}_{r_2:n}, \ldots, X^{(j)}_{r_p:n}$ is*

$$f^{(j)}_{r_1,r_2,\ldots,r_p:n}\left(x_1^{(j)}, x_2^{(j)}, ..., x_p^{(j)}\right) = \sum C^{-1}\mathrm{per}\mathbf{A} \tag{2.1}$$

for $x_1^{(j)} \le x_2^{(j)} \le ... \le x_p^{(j)}$ and $= 0$, otherwise, where $0 = r_0 < r_1 < ... < r_p < r_{p+1} = n+1$,

$\mathbf{A} =$

$$\left[\ \underset{r_1-k_1-1}{\mathbf{F}^{(j)}\left(x_1^{(j)}-1\right)}\ \underset{k_1+m_1+1}{\mathbf{f}^{(j)}(x_1^{(j)})}\ \ \underset{r_2-r_1-m_1-k_2-1}{\mathbf{F}^{(j)}\left(x_2^{(j)}-1\right)-\mathbf{F}^{(j)}\left(x_1^{(j)}\right)}\ \ \underset{k_2+m_2+1}{\mathbf{f}^{(j)}(x_2^{(j)})}\ \ldots\ \underset{n-r_p-m_p}{\mathbf{1}-\mathbf{F}^{(j)}\left(x_p^{(j)}\right)}\ \right],$$

$k_0 = m_0 = k_{p+1} = 0$, $C = \prod_{l=0}^{p}\left[(k_l+m_l+1)!\,(r_{l+1}-r_l-m_l-k_{l+1}-1)!\right]$, $F_i^{(j)}\left(x_l^{(j)}-1\right) = 0$ for $x_l^{(j)} = 0$, and $\mathbf{1} = (1\ 1\ ...\ 1)'$, $\mathbf{F}^{(j)}\left(x_l^{(j)}\right) = (F_1^{(j)}\left(x_l^{(j)}\right)\ F_2^{(j)}\left(x_l^{(j)}\right) \ldots F_n^{(j)}\left(x_l^{(j)}\right))'$ are column vectors with n components for $l = 0, 1, ..., p$ and

$$\mathbf{f}^{(j)}\left(x_l^{(j)}\right) = (\ f_1^{(j)}\left(x_l^{(j)}\right)\ \ f_2^{(j)}\left(x_l^{(j)}\right)\ \ \ldots\ \ f_n^{(j)}\left(x_l^{(j)}\right))'$$

is column vector with n components for $l = 1, 2, ..., p$ and $r_{l+1} - r_l - m_l - k_{l+1} - 1 \ge 0$, $l = 0, 1, ..., p$,

$$\sum = \sum_{k_1=0}^{r_1-1}\ \sum_{m_1=0}^{r_2-r_1-1}\ \sum_{k_2=0}^{r_2-r_1-1}\ \sum_{m_2=0}^{r_3-r_2-1}\ \sum_{k_3=0}^{r_3-r_2-1} \cdots \sum_{m_p=0}^{n-r_p}$$

($F_i^{(j)}\left(x_0^{(j)}\right) = 0$, $F_i^{(j)}\left(x_{p+1}^{(j)}-1\right) = 1$).

Proof. Since $\xi_1, \xi_2, \ldots, \xi_n$ are multivariate i.n.n.i.d. random vectors from discrete parents, the joint p.f. of $X^{(j)}_{r_1:n}, X^{(j)}_{r_2:n}, \ldots, X^{(j)}_{r_p:n}$ at $x_1^{(j)}, x_2^{(j)}, ..., x_p^{(j)}$ is obtained as follows. For this purpose, the permanent of matrix $\mathbf{A}$ of order n will be used, which has $r_{l+1} - r_l - m_l - k_{l+1} - 1$ columns of $\mathbf{F}^{(j)}\left(x_{l+1}^{(j)}-1\right) - \mathbf{F}^{(j)}\left(x_l^{(j)}\right)$ between column $r_l + m_l$ and column $r_{l+1} - k_{l+1}$ ($l = 0, 1, ..., p$), and $k_l + m_l + 1$ columns of $\mathbf{f}^{(j)}\left(x_l^{(j)}\right)$ between column $r_l - k_l - 1$ and column

$r_l + m_l + 1$ $(l = 1, 2, ..., p)$. Moreover, when $X^{(j)}_{r_1:n}, X^{(j)}_{r_2:n}, \ldots, X^{(j)}_{r_p:n}$ is considered, the permanent should be multiplied by C^{-1}. If the sum of $C^{-1}\text{per}\mathbf{A}$ is taken over k_l and m_l, where $r_{l+1} - r_l - m_l - k_{l+1} - 1 \geq 0$ $(l = 0, 1, ..., p)$, then the desired result is obtained. □

Theorem 2.2. *Let f_i and F_i be p.f. and d.f. of ξ_i, and $f_i^{(j)}$ and $F_i^{(j)}$ be p.f. and d.f. of $\xi_i^{(j)}$, respectively. Moreover, assume that the same conditions in Theorem 2.1 hold. Then, the joint p.f. of $\mathbf{X}_{r_1:n}, \mathbf{X}_{r_2:n}, \ldots, \mathbf{X}_{r_p:n}$ is*

$$f_{r_1,r_2,...,r_p:n}\left(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_p\right) = \prod_{j=1}^{d} f^{(j)}_{r_1,r_2,...,r_p:n}\left(x_1^{(j)}, x_2^{(j)}, ..., x_p^{(j)}\right), \tag{2.2}$$

for $\mathbf{x}_1 \leq \mathbf{x}_2 \leq ... \leq \mathbf{x}_p$ and $= 0$, otherwise, where $\mathbf{x}_l = \left(x_l^{(1)}, x_l^{(2)}, ..., x_l^{(d)}\right)$, $l = 1, 2, ..., p$.

Proof. We omit the proof. □

3. Discussions and Results

In (2.1) (or in (2.2) for $d = 1$), if $f_i^{(j)}\left(x_l^{(j)}\right) = f^{(j)}\left(x_l^{(j)}\right)$, then the joint p.f. of order statistics of i.i.d. random variables from discrete parent is obtained. This result gives an alternative representation of (2.1) in [2], and the special cases of this result for $p = 1$ and $p = 2$ give (2) and (6) in [4], respectively.

In (2.1) (or in (2.2) for $d = 1$), if discrete case approaches to continuous case, i.e., if $k_l = m_l = 0$ and $F_i^{(j)}\left(x_l^{(j)} - 1\right) = F_i^{(j)}\left(x_l^{(j)}\right)$, then the joint p.d.f. of order statistics of i.n.n.i.d. random variables with continuous d.f.'s is obtained. This result gives the opened form of the expression which is constructed by taking order statistics instead of T in (2.2) in [3], and the general case in [7].

In (2.1) (or in (2.2) for $d = 1$), if discrete case approaches to continuous case, i.e., if $k_l = m_l = 0$ and $F_i^{(j)}\left(x_l^{(j)} - 1\right) = F_i^{(j)}\left(x_l^{(j)}\right) = F^{(j)}\left(x_l^{(j)}\right)$, then the joint p.d.f. of order statistics of i.i.d. random variables with continuous d.f. is obtained. This result gives (2.2.3) in [1] and Theorem 1.4.5 in [5].

In (2.2), if $f_i\left(\mathbf{x}_l\right) = f(\mathbf{x}_l)$, then the joint p.f. of multivariate order statistics by marginal ordering of i.i.d. random vectors from discrete parent is obtained.

In (2.2), if discrete case approaches to continuous case, i.e., if $k_l = m_l = 0$ and $F_i\left(\mathbf{x}_l - \mathbf{1}\right) = F_i\left(\mathbf{x}_l\right)$, then the joint p.d.f. of multivariate order statistics by marginal ordering of i.n.n.i.d. random vectors with continuous d.f.'s is obtained.

In (2.2), if discrete case approaches to continuous case, i.e., if $k_l = m_l = 0$ and $F_i\left(\mathbf{x}_l - \mathbf{1}\right) = F_i\left(\mathbf{x}_l\right) = F\left(\mathbf{x}_l\right)$, then the joint p.d.f. of multivariate order statistics by marginal ordering of i.i.d. random vectors with continuous d.f. is obtained. Whenever the components of vectors are independent in Lemma 2.2.2 in [5], this result gives the lemma for $d = 2$ and $p = 1$.

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