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PERIODIC SPLINE ORTHONORMAL BASES CLOSEST TO PERIODIC B-SPLINES

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Abstract: An orthonormal basis for the space of periodic cardinal splines is presented. This basis is closer to the B-spline basis than other orthonormal bases for the space, where the distance between two bases is measured as the total sum of L^2-metric between each basis function and its counterpart. Its convergence to a periodic version of cardinal function at the limit when the order tends to infinity is shown.

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1. Introduction

Cardinal splines [5, 7] are piecewise polynomials created as linear combinations of cardinal B-splines in an intention to construct a bridge connecting the piecewise linear functions and the cardinal function series [9]. Interpolation of data

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by straight line segments yields a piecewise linear function that is non-smooth and "polygonal." On the other hand, the cardinal function [9], which is also referred to [1] as Shannon sampling function [4], gives an analytic and thus "extremely smooth" interpolation. The bridge has been completed first in terms of exponential Euler splines [6], which converge to Fourier exponential functions at the limit when the order of the splines tends to infinity. Then the same bridge in terms of sampling bases was attempted by [2] and completed extensively by [8].

An orthonormal basis for the space of periodic cardinal splines was given by [3], which converges to Fourier exponential functions when the order tends to infinity. This basis can be construed as a periodic version of the exponential Euler splines. An orthonormal basis generally has an advantage that coefficients for the least-square approximation of a given function can be evaluated simply by integrating the given function multiplied by the complex conjugate of basis functions. However, the one presented in [3] is oscillating throughout the domain so that we need to perform the integration over the entire domain. The periodic B-spline basis is locally supported but it is not orthogonal in the first place.

The present paper gives an orthonormal basis that is closest to the periodic B-spline basis in an attempt to make one having good locality, where the distance between two bases is measured as the total sum of L^2-metric between each basis function and its counterpart. Although the present basis functions are not locally supported, their amplitude decays rapidly so that they can be substituted in practice by their truncation within a short interval for the lower order cases. Their convergence to a periodic version of cardinal functions at the limit when the order tends to infinity is also shown.

2. Preliminaries for the Space of Periodic Splines

The linear space of periodic splines with equispaced knots is defined on the basis of periodic B-splines in this section. Several properties are also prepared by quoting results of [3].

Definition 1. Let T and N be a real number and a natural number, respectively. Then a periodic B-spline of order m is defined as

$$\varphi_0^m(t) = \sum_{p=-\infty}^{\infty} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m \mathrm{e}^{i2\pi pt/T}, \quad m = 1, 2, 3, \cdots , \tag{1}$$

which has the period of T and the knot interval of T/N.

The following recurrence formula [3] follows from (1) and the convolution theorem in the Fourier series expansion.

Proposition 2.

$$\varphi_0^m(t) = \begin{cases} 1, & t \in (-T/2N + qT, T/2N + qT), \\ & \quad q = 0, \pm 1, \pm 2, \cdots, \\ 0, & \textit{otherwise}, \end{cases} \tag{2}$$

$$\varphi_0^m(t) = \frac{1}{T}\int_0^T \varphi_0^{m-1}(\tau)\varphi_0^1(t-\tau)\,d\tau, \quad m = 2, 3, 4, \cdots . \tag{3}$$

By (2) and (3), the periodic B-splines are expressed [3] in the form of a piecewise polynomial as follows:

Proposition 3.

$$\varphi_0^m(t) = mT^{1-m}\sum_{q=-\infty}^{\infty}\sum_{r=0}^{m}\frac{(-1)^r}{r!(m-r)!}\{t-((r-m/2)/N+q)T\}_+^{m-1}, \quad m = 1, 2, 3, \ldots, \tag{4}$$

where

$$\{t-a\}_+^{m-1} = \begin{cases} \{t-a\}^{m-1}, & t > a, \\ 0, & t \le a. \end{cases} \tag{5}$$

Examples of the periodic B-spline φ_0^m are plotted in Figure 1.

Definition 4. The space of periodic splines of order m with period T and knot interval T/N is a subspace

$$S^m = [\varphi_l^m]_{l=0}^{N-1} \tag{6}$$

of $L^2[0, T]$ spanned by the periodic B-splines

$$\varphi_l^m(t) = \varphi_0^m(t - lT/N), \quad l = 0, 1, \cdots, N-1, \tag{7}$$

where the inner product of $f, g \in S^m$ is

$$(f, g) = \frac{1}{T}\int_0^T f(t)\overline{g(t)}dt \tag{8}$$

and the norm of $f \in S^m$ is

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2 dt}. \tag{9}$$

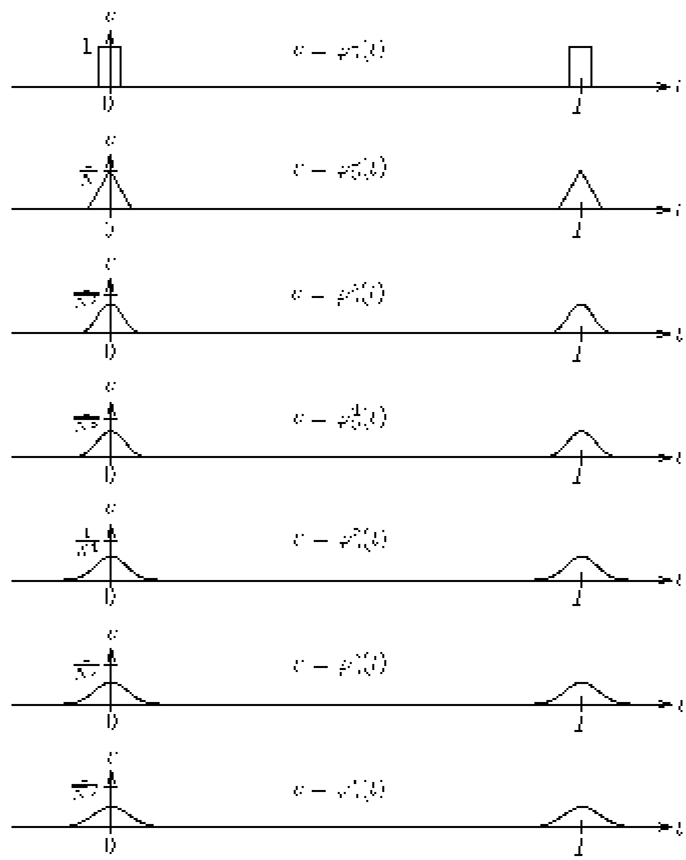

Figure 1: Examples of the periodic B-spline φ_0^m in the case $N = 21$

As the analogy of equation (1.19) on [2, p. 61], the inner products of periodic B-splines were obtained in [3] as follows.

Proposition 5.

$$(\varphi_{l_1}^m, \varphi_{l_2}^m) = \varphi_0^{2m}((l_2 - l_1)T/N), \qquad l_1, l_2 = 0, 1, \cdots, N-1. \tag{10}$$

An orthonormal basis for S^m has been given by [3] as follows.

Proposition 6. *A linear transform $\{\psi_k^m\}_{k=0}^{N-1}$ of the periodic B-spline basis $\{\varphi_l^m\}_{l=0}^{N-1}$ defined by*

$$\psi_k^m = \frac{1}{N\sqrt{\theta_k^m}} \sum_{l=0}^{N-1} e^{i2\pi lk/N} \varphi_l^m, \qquad k = 0, 1, \cdots, N-1, \tag{11}$$

where

$$\theta_k^m = \sum_{q=-\infty}^{\infty} \left\{ \frac{\sin(\pi k/N)}{\pi(k+qN)} \right\}^{2m}, \qquad k = 0, 1, \cdots, N-1, \tag{12}$$

constitutes an orthonormal basis for S^m*, i.e.*

$$(\psi_{k_1}^m, \psi_{k_2}^m) = \begin{cases} 1, & k_1 = k_2, \\ 0, & k_1 \neq k_2. \end{cases} \tag{13}$$

This orthonormal basis for the case $m = 4$ and $N = 21$ is plotted in Figure 2. Its shape is similar to the Fourier exponential functions. In fact, it approaches the exponential functions as asserted in the following proposition [3].

Proposition 7. *In the case that* N *is odd, when* m *approaches infinity, it holds good that*

$$\psi_k^m(t) \to \begin{cases} e^{i2\pi kt/T}, & k = 0, 1, \cdots, (N-1)/2, \\ e^{i2\pi(k-N)t/T}, & k = (N+1)/2, \cdots, N-1. \end{cases} \tag{14}$$

This orthonormal basis may be useful in constructing a least-square approximation of a given function by a periodic spline. But it has a major drawback in comparison with the periodic B-splines. The orthonormal functions in Figure 2 are ever oscillating throughout the domain while the periodic B-splines in Figure 1 are locally supported.

3. Orthonormal Basis Closest to Periodic B-splines

Another orthonormal basis to be presented in this section is constructed in an attempt to combine orthonormality and locality. First, Theorem 8 presents a new basis and shows its orthonormality. Then Theorem 10 shows that the orthonormal basis is the only one closest to the periodic B-spline basis.

Theorem 8. *A linear transform* $\{\chi_j^m\}_{j=0}^{N-1}$ *of the periodic B-splines* $\{\varphi_l^m\}_{l=0}^{N-1}$ *defined by*

$$\chi_j^m = N^{-3/2} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \frac{1}{\sqrt{\theta_k^m}} e^{i2\pi k(l-j)/N} \varphi_l^m, \tag{15}$$

$j = 0, 1, \cdots, N-1$*, constitute an orthonormal basis for* S^m*.*

Proof. Since functions $\{\chi_j^m\}_{j=0}^{N-1}$ are linear combinations of the B-splines $\{\varphi_l^m\}_{l=0}^{N-1}$ that span S^m, they constitute an orthonormal basis for S^m if they are orthonormal to each other.

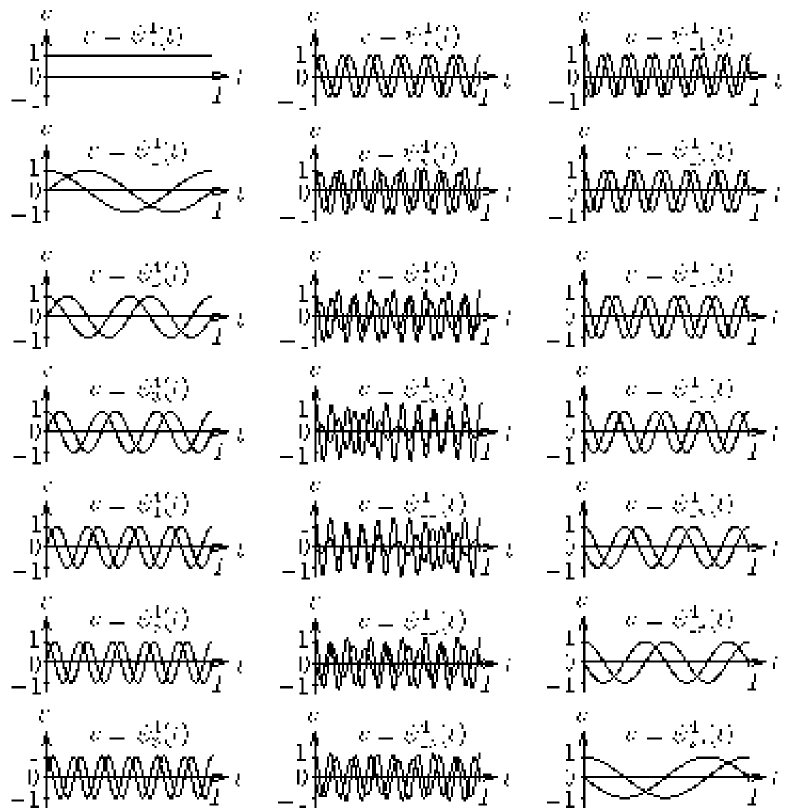

Figure 2: An example of orthonormal basis $\{\psi_k^m\}_{k=0}^{N-1}$ for the case $m = 4$ and $N = 21$ (solid and dotted curves represent the real and imaginary parts, respectively)

Equations (11) and (15) imply

$$\chi_j^m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left((\theta_k^m)^{-1/2} \frac{1}{N} \sum_{l=0}^{N-1} e^{-i2\pi kl/N} \varphi_l^m \right) e^{i2\pi kj/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \psi_k^m e^{-i2\pi kj/N}.$$

Then we have

$$\begin{aligned}(\chi_{j_1}^m, \chi_{j_2}^m) &= \left(\frac{1}{\sqrt{N}} \sum_{k_1=0}^{N-1} e^{i2\pi k_1 j_1/N} \psi_{k_1}^m, \frac{1}{\sqrt{N}} \sum_{k_2=0}^{N-1} e^{i2\pi k_2 j_2/N} \psi_{k_2}^m \right) \\ &= \frac{1}{N} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} e^{i2\pi(k_1 j_1 - k_2 j_2)/N} (\psi_{k_1}^m, \psi_{k_2}^m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{i2\pi k(j_1-j_2)/N} \\ &= \begin{cases} 1, & j_1 = j_2, \\ 0, & j_1 \neq j_2. \end{cases} \qquad \square\end{aligned}$$

Corollary 9. *Functions $\{\chi_j^m\}_{j=0}^{N-1}$ are real valued. In the case N is even,*

$$\chi_j^m = \sum_{l=0}^{N-1} \left(\frac{1}{\sqrt{\theta_0^m}} + \frac{1}{\sqrt{\theta_{N/2}^m}} (-1)^{N(l-j)/2} + 2 \sum_{k=1}^{N/2-1} \frac{\cos 2\pi k(l-j)/N}{\sqrt{\theta_k^m}} \right) \varphi_l^m, \quad j = 0, 1, 2, \cdots, N-1. \tag{16}$$

In the case N is odd,

$$\chi_j^m = \sum_{l=0}^{N-1} \left(\frac{1}{\sqrt{\theta_0^m}} + 2 \sum_{k=1}^{(N-1)/2} \frac{\cos 2\pi k(l-j)/N}{\sqrt{\theta_k^m}} \right) \varphi_l^m, \quad j = 0, 1, 2, \cdots, N-1. \tag{17}$$

Examples of χ_0^m in the case $N = 21$ are plotted in Figure 3. They are not locally supported. But their amplitude decays so rapidly that their truncation in a short interval can do almost the same in practice.

The following theorem shows that the orthonormal basis is closest to the periodic B-spline basis, where the distance between those bases is measured as the total sum of L^2-metric between each basis function and its counterpart.

Theorem 10. *For any orthonormal basis $\{\alpha_j^m\}_{j=0}^{N-1}$ for S^m, it holds good that*

$$\sum_{j=0}^{N-1} \|\chi_j^m - \varphi_j^m\|^2 \leqq \sum_{j=0}^{N-1} \|\alpha_j^m - \varphi_j^m\|^2, \tag{18}$$

where the equality holds if and only if $\alpha_j^m = \chi_j^m$, $j = 0, 1, 2, \cdots, N-1$.

Proof. Define vectors $\boldsymbol{\chi} = [\chi_0^m, \chi_1^m, \cdots, \chi_{N-1}^m]$, $\boldsymbol{\varphi} = [\varphi_0^m, \varphi_1^m, \cdots, \varphi_{N-1}^m]$, $\boldsymbol{\alpha} = [\alpha_0^m, \alpha_1^m, \cdots, \alpha_{N-1}^m]$, a unitary matrix

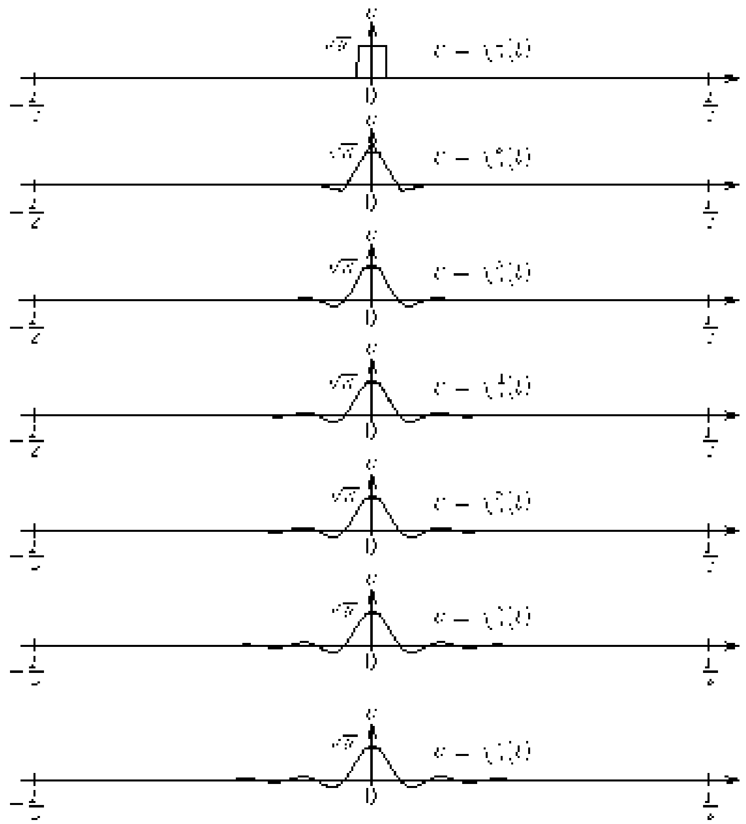

Figure 3: Examples of χ_0^m in the case $N = 21$

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i2\pi/N} & e^{i4\pi/N} & \cdots & e^{i2\pi(N-1)/N} \\ 1 & e^{i4\pi/N} & e^{i8\pi/N} & \cdots & e^{i4\pi(N-1)/N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{i2\pi(N-1)/N} & e^{i4\pi(N-1)/N} & \cdots & e^{i2\pi(N-1)(N-1)/N} \end{bmatrix},$$

and a diagonal matrix

$$\Theta = N \begin{bmatrix} \theta_0^m & 0 & \cdots & 0 \\ 0 & \theta_1^m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \theta_{N-1}^m \end{bmatrix}.$$

Then it follows from (15) that

$$\boldsymbol{\chi} = \boldsymbol{\varphi} U^* \Theta^{-1/2} U. \tag{19}$$

There exists a unitary matrix V such that relates the two orthonormal bases as

$$\boldsymbol{\alpha} = \boldsymbol{\chi} V\,, \tag{20}$$

since any orthonormal bases are unitary transforms of each other. Equations (19) and (20) yield

$$\boldsymbol{\alpha} = \boldsymbol{\varphi} U^* \Theta^{-1/2} U V. \tag{21}$$

Then we can rearrange the right hand side of (18) as follows:

$$\begin{aligned}
\sum_{j=0}^{N-1} \|\alpha_j^m - \varphi_j^m\|^2 &= \sum_{j=0}^{N-1} \frac{1}{T}\int_0^T |\alpha_j^m(t) - \varphi_j^m(t)|^2\,dt \\
&= \operatorname{tr}\left\{\frac{1}{T}\int_0^T (\boldsymbol{\alpha}(t) - \boldsymbol{\varphi}(t))^*(\boldsymbol{\alpha}(t) - \boldsymbol{\varphi}(t))\,dt\right\} \\
&= \operatorname{tr}\left\{\frac{1}{T}\int_0^T (\boldsymbol{\varphi}(t)U^*\Theta^{-1/2}UV - \boldsymbol{\varphi}(t))^*(\boldsymbol{\varphi}(t)U^*\Theta^{-1/2}UV - \boldsymbol{\varphi}(t))\,dt\right\} \\
&= \operatorname{tr}\left\{\frac{1}{T}\int_0^T (\boldsymbol{\varphi}(t)(U^*\Theta^{-1/2}UV - I))^*\boldsymbol{\varphi}(t)(U^*\Theta^{-1/2}UV - I)\,dt\right\} \\
&= \operatorname{tr}\left\{(U^*\Theta^{-1/2}UV - I)^*\frac{1}{T}\int_0^T \boldsymbol{\varphi}^*(t)\boldsymbol{\varphi}(t)\,dt(U^*\Theta^{-1/2}UV - I)\right\}.
\end{aligned} \tag{22}$$

In the meantime, Proposition 5 and Definition 1 imply

$$\begin{aligned}
(\varphi_{l_1}^m, \varphi_{l_2}^m) &= \frac{1}{T}\int_0^T \varphi_{l_1}^m(t)\overline{\varphi_{l_2}^m(t)}\,dt \\
&= \varphi_0^{2m}((l_2 - l_1)T/N) \\
&= \sum_{p=-\infty}^{\infty} \{\sin(\pi p/N)/\pi p\}^{2m} e^{i2\pi p(l_2-l_1)/N} \\
&= \sum_{k=0}^{N-1}\sum_{q=-\infty}^{\infty} \left\{\frac{\sin(\pi(k+qN)/N)}{(\pi(k+qN))}\right\}^{2m} e^{i2\pi(k+qN)(l_2-l_1)/N} \\
&= \sum_{k=0}^{N-1}\sum_{q=-\infty}^{\infty} \left\{\frac{\sin(\pi k/N)}{\pi(k+qN)}\right\}^{2m} e^{i2\pi k(l_2-l_1)/N}
\end{aligned}$$

$$= \sum_{k=0}^{N-1} \theta_k^m e^{i2\pi k(l_2-l_1)/N}, \quad l_1, l_2 = 0, 1, 2, \cdots, N-1\,,$$

so that we have

$$\frac{1}{T}\int_0^T \boldsymbol{\varphi}^*(t)\boldsymbol{\varphi}(t)\,dt = U^*\Theta U. \tag{23}$$

Then we can further rearrange (22) by using (23) as follows:

$$\begin{aligned}
&\sum_{j=0}^{N-1} \|\alpha_j^m - \varphi_j^m\|^2 \\
= \;& \mathrm{tr}\left\{\{(U^*\Theta^{-1/2}UV - I)^*U^*\Theta U(U^*\Theta^{-1/2}UV - I)\}\right\} \\
= \;& \mathrm{tr}\left\{\{(V^*U^*\Theta^{-1/2} - I)U^*\Theta U(U^*\Theta^{-1/2}UV - I)\}\right\} \\
= \;& \mathrm{tr}\left\{(V^*U^*\Theta^{-1/2}UU^*\Theta UU^*\Theta^{-1/2}UV - IU^*\Theta UU^*\Theta^{-1/2}UV \right. \\
& \left. \quad - V^*U^*\Theta^{-1/2}UU^*\Theta U + IU^*\Theta UI)\right\} \\
= \;& \mathrm{tr}(I - U^*\Theta^{1/2}UV - V^*U^*\Theta^{1/2}U + U^*\Theta U) \\
= \;& \mathrm{tr}(I + U^*\Theta U) - \mathrm{tr}(U^*\Theta^{1/2}UV + V^*U^*\Theta^{1/2}U) \\
= \;& \mathrm{tr}(I + U^*\Theta U) - \mathrm{tr}(U^*\Theta^{1/2}UV + (U^*\Theta^{1/2}UV)^*) \\
= \;& \mathrm{tr}(I + U^*\Theta U) - 2\mathrm{Re}[\mathrm{tr}(U^*\Theta^{1/2}UV)] \\
= \;& \mathrm{tr}(I + U^*\Theta U) - 2\mathrm{Re}[\mathrm{tr}((\Theta^{1/4}U)^*(\Theta^{1/4}UV))].
\end{aligned} \tag{24}$$

Equation (24) implies that $\sum_{j=0}^{N-1} \|\alpha_j^m - \varphi_j^m\|^2$ is minimized when $\mathrm{Re}[\mathrm{tr}((\Theta^{1/4}\,U)^*(\Theta^{1/4}UV))]$ is maximized. Note that $\mathrm{tr}((\Theta^{1/4}U)^*(\Theta^{1/4}U))$ sums up the norm of all the column vectors constituting $\Theta^{1/4}U$. For any unitary matrix V, the column vectors of $\Theta^{1/4}UV$ are rotation of those of $\Theta^{1/4}U$. This rotation by V lowers the value of $\mathrm{Re}[\mathrm{tr}((\Theta^{1/4}U)^*(\Theta^{1/4}UV))]$ unless V is the identical matrix I. So $\mathrm{Re}[\mathrm{tr}((\Theta^{1/4}U)^*(\Theta^{1/4}UV))]$ is maximized and thus $\sum_{j=0}^{N-1} \|\alpha_j^m - \varphi_j^m\|^2$ is minimized if and only if $V = I$. This case is equivalent to $\alpha_j^m = \chi_j^m$, $j = 0, 1, 2, \cdots, N-1$ because of (20). □

4. Property at the Limit $m \to \infty$

By Proposition 7, the periodic spline basis $\{\psi_k^m\}_{k=0}^{N-1}$, in the case N is odd, approaches Fourier exponential functions at the limit when m tends to infinity.

The present basis $\{\chi_j^m\}_{j=0}^{N-1}$ should also approach some finite Fourier series. The following theorem shows that the extremal function is a periodic version of the cardinal function.

Theorem 11. *Denote by*

$$\sigma_j(t) = \frac{1}{\sqrt{N}} \frac{\sin \pi N(t/T - j/N)}{\sin \pi(t/T - j/N)}, \quad j = 0, 1, 2, \cdots, N-1, \tag{25}$$

a periodic version of the cardinal function. Then it holds good that

$$\left\| \chi_j^m - \sigma_j \right\| \to 0 \quad (m \to \infty) \tag{26}$$

in the case N is an odd natural number.

Proof. Equations (1) and (15) yield

$$\begin{aligned} \chi_j^m(t) &= N^{-3/2} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} (\theta_k^m)^{-1/2} e^{i2\pi k(l-j)/N} \\ &\quad \times \sum_{p=-\infty}^{\infty} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m e^{i2\pi p(t/T - l/N)} \\ &= N^{-3/2} \sum_{k=0}^{N-1} (\theta_k^m)^{-1/2} e^{-i2\pi kj/N} \\ &\quad \times \sum_{p=-\infty}^{\infty} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m e^{i2\pi p(t/T)} \sum_{l=0}^{N-1} e^{i2\pi(k-p)l/N}, \end{aligned}$$

of which last factor is

$$\sum_{l=0}^{N-1} e^{i2\pi(k-p)l/N} = \begin{cases} N & \text{if } k = p \bmod N, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have χ_j^m represented as a Fourier series

$$\chi_j^m(t) = N^{-1/2} \sum_{p=-\infty}^{\infty} (\theta_{p \bmod N}^m)^{-1/2} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m e^{i2\pi p(t/T - j/N)}, \quad j = 0, 1, 2, \cdots, N-1. \tag{27}$$

In the meantime, (25) is equivalent to

$$\sigma_j(t) = N^{-1/2} \sum_{p=-(N-1)/2}^{(N-1)/2} e^{i2\pi p(t/T - j/N)}, \quad j = 0, 1, 2, \cdots, N-1 \tag{28}$$

in the case N is odd. By (27), (28) and Parseval equality, we have

$$\left\|\chi_j^m - \sigma_j\right\|^2 = \frac{1}{N}\left\{\sum_{|p|\geq(N+1)/2}(\theta^m_{p \bmod N})^{-1}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{2m} + \sum_{|p|\leq(N-1)/2}\left|(\theta^m_{p \bmod N})^{-1/2}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{m} - 1\right|^2\right\}. \tag{29}$$

In the case $|p| \geq (N+1)/2$, an upper bound for the Fourier coefficient can be found by substituting (12) for $\theta^m_{p \bmod N}$ as follows:

$$\begin{aligned}
&(\theta^m_{p \bmod N})^{-1}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{2m}\\
&\quad= \left\{\sum_{q=-\infty}^{\infty}\left\{\frac{\sin(\pi(p \bmod N)/N)}{\pi(p \bmod N + qN)}\right\}^{2m}\right\}^{-1}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{2m}\\
&\quad= \left\{\sum_{q=-\infty}^{\infty}\left(\frac{p}{p+qN}\right)^{2m}\right\}^{-1} = \left(\frac{N}{2p}\right)^{2m}\left\{\sum_{q=-\infty}^{\infty}\left(\frac{2p}{N}+2q\right)^{-2m}\right\}^{-1}\\
&\quad\leq \left(\frac{N}{2p}\right)^{2m}.
\end{aligned} \tag{30}$$

Then the first part of (29) satisfies

$$\begin{aligned}
&\sum_{|p|\geq(N+1)/2}(\theta^m_{p \bmod N})^{-1}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{2m} = \sum_{|p|\geq(N+1)/2}\left(\frac{N}{2p}\right)^{2m}\\
&\quad= 2\sum_{p=(N+1)/2}^{\infty}\left(\frac{N}{2p}\right)^{2m} \leq 2\int_{N/2}^{\infty}\left(\frac{N}{2x}\right)^{2m}dx = \frac{N}{2m-1} \to 0\\
&\qquad (m \to \infty).
\end{aligned} \tag{31}$$

In the case $|p| \leq (N-1)/2$, we have

$$\begin{aligned}
&(\theta^m_{p \bmod N})^{-1/2}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{m}\\
&= \left\{\sum_{q=-\infty}^{\infty}\left\{\frac{\sin(\pi(p \bmod N)/N)}{\pi(p \bmod N + qN)}\right\}^{2m}\right\}^{-1/2}\left\{\frac{\sin(\pi p/N)}{\pi p}\right\}^{m}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \sum_{q=-\infty}^{\infty} \left(\frac{p}{p+qN} \right)^{2m} \right\}^{-1/2} \\
&= \left\{ 1 + \left(\frac{2p}{N} \right)^{2m} \sum_{q=1}^{\infty} \left\{ \left(\frac{2p}{N} + 2q \right)^{-2m} + \left(\frac{2p}{N} - 2q \right)^{-2m} \right\} \right\}^{-1/2} \\
&\leq \ 1. \qquad (32)
\end{aligned}
$$

For any $q = 1, 2, 3, \cdots$, the function $\left(\frac{2p}{N} + 2q\right)^{-2m} + \left(\frac{2p}{N} - 2q\right)^{-2m}$ of p in the domain $-N/2 \leq p \leq N/2$ comes to its maximum at $p = \pm N/2$. Then we have

$$
\begin{aligned}
&\sum_{q=1}^{\infty} \left\{ \left(\frac{2p}{N} + 2q \right)^{-2m} + \left(\frac{2p}{N} - 2q \right)^{-2m} \right\} \\
&\qquad < \sum_{q=1}^{\infty} \left\{ (1+2q)^{-2m} + (1-2q)^{-2m} \right\} = -1 + 2 \sum_{q=1}^{\infty} (2q-1)^{-2m} \\
&\qquad\qquad \leq -1 + 2 \sum_{q=1}^{\infty} (2q-1)^{-2} = \pi^2/4 - 1 \,, \qquad (33)
\end{aligned}
$$

for $|p| < (N-1)/2$. It follows from inequalities (32) and (33) that

$$
1 \geq (\theta^m_{p \bmod N})^{-1/2} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m \geq \left\{ 1 + \left(\frac{2p}{N} \right)^{2m} \left(\frac{\pi^2}{4} - 1 \right) \right\}^{-1/2} . \qquad (34)
$$

This implies that the second part of (29) satisfies

$$
\begin{aligned}
&\sum_{|p| \leq (N-1)/2} \left| (\theta^m_{p \bmod N})^{-1/2} \left\{ \frac{\sin(\pi p/N)}{\pi p} \right\}^m - 1 \right|^2 \\
&\quad \leq \sum_{|p| \leq (N-1)/2} \left| 1 - \left\{ 1 + \left(\frac{2p}{N} \right)^{2m} \left(\frac{\pi^2}{4} - 1 \right) \right\}^{-1/2} \right|^2 \to 0 \ \ (m \to \infty). \qquad (35)
\end{aligned}
$$

Eventually, it follows from (29), (31) and (35) that

$$
\left\| \chi^m_j - \sigma_j \right\| \to 0 \ \ (m \to \infty) \,,
$$

for odd natural number N. □

Although we have made much use of the precondition that N is odd in the proof, it is hard to see any mathematical reasons why Theorem 11 needs this condition. A better proof that allows N to be even is yet to be investigated.

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