DISCRETE REALIZATIONS OF CONTACT
AND INTERSECTION GRAPHS

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Abstract: Known realizations of geometric representations of graphs, like contact, intersection, etc., are "continuous", in the sense that the geometric objects are drawn in Euclidean space with real numbers as coordinates. In this paper, we initiate the study of dicrete versions of contact and intersection graphs and examine their relation to their continuous counterparts. The classes of graphs arising appear to have interesting properties and are thus interesting in their own right. We also study realizability, characterizations as well as intractability questions for the resulting new classes of graphs.

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1. Introduction

Classes of graphs having realizations either as intersection or contact graphs of given geometric objects have attracted the attention of several researchers in the literature, e.g. see the extensive literature cited in [9, 12]. Problems raised include realization, characterization, as well as intractability. Graphs thus realized have interesting applications ranging from scheduling theory [9] to motion planning [1].

Known studies of such graph-classes have only been concerned with "continuous" realizations of the concepts of "intersection" and "contact", in which the contact or intersection points have real numbers as coordinates. However, practical realizations of these concepts would seem to require the more natural concepts of discrete "intersection" and "contact" points, in the sense that the contact or intersection points have rationals or even "integers with a restricted range" as coordinates. By this we mean that the geometric objects have a predetermined set of points and only these points can serve as contact or intersection points among the objects.

A coin is a closed circular disc in the plane. Coin graphs are constructed from discs with disjoint interiors. Vertices of the graph are the coins. Two coins are adjacent if they touch each other. Such graphs are planar and have been considered previously in the literature [12]. A remarkable theorem attributed to P. Koebe by H. Sachs [12] states that every finite simple planar graph is a coin graph. The most recent proof of this theorem is given in [4]. It is intersesting to note that the discs required in the above representation of the planar graph are of arbitrary size. In addition, a related result in [7] shows that planar graphs are precisely the contact graphs of isosceles triangles.

Another class of graphs we will study are discrete intersection graphs. Among these are interval and unit square graphs which have been studied extensively in the literature [9]. Linear recognition algorithms for these graphs have appeared in [9, 5, 6], etc.

1.1. Preliminaries and Notation

The class of coin graphs considered here are constructed from a single size coin.

Definition 1.1. The class of coin graphs, denoted by $\mathcal{C}$, is defined as follows. Vertices are represented by unit discs such that the interiors of any pair of such discs have empty intersection. Two discs are adjacent if they touch.

Clearly the graphs in $\mathcal{C}$ are planar. The problem of recognizing whether a given graph is in the class $\mathcal{C}$ has been shown recently to be NP-complete [3].

Another class of graphs we will consider in this paper is the class of unit interval graphs (all intervals considered in the sequel are closed).

Definition 1.2. The class of unit interval graphs, denoted by $\mathcal{I}$, is defined as follows. Vertices are represented by unit intervals. Two intervals are adjacent if they have a nonempty intersection.

It is known that unit interval graphs are precisely the interval graphs containing no induced copy of $K_{1,3}$ [9]. Thus the discrete unit interval graphs are a refinement of the unit interval graphs.

Notation. From now on and for the rest of the paper if $\mathcal{G}$ denotes a class of graphs then $\mathcal{G}^n$ denotes the set of graphs in $\mathcal{G}$ with exactly n vertices. For example, we have the classes $\mathcal{I}^n, \mathcal{C}^n$, etc.

1.2. Results of the Paper

In this paper we initiate the study of dicrete versions of the above mentioned classes of graphs. The classes of graphs arising appear to have interesting properties and are thus interesting in their own right. We also study realizability, characterizations as well as intractability questions for the resulting new graphs.

2. Discrete Coin Graphs

In the sequel we define new classes of planar coin graphs. For each integer $k \geq 3$ consider a regular k-gon P_k.

Definition 2.1. The class of polygonal coin graphs, denoted by $\mathcal{P}_k$ is defined as follows. Vertices are represented by isothets of the regular k-gon P_k such that the interiors of any pair of such k-gons have empty intersection. Two k-gons are adjacent if they touch at a polygon vertex.

Clearly the graphs in the class $\mathcal{P}_k$ are planar. We can prove the following theorem.

Theorem 2.1. *All polygonal coin graphs are coin graphs, i.e. $\mathcal{P}_k \subseteq \mathcal{C}$, for all integers k.*

Proof. The theorem is easy for k even. Consider the layout of regular k-gons representing a given graph. Inscribe each polygon in a disc. Since k is even the resulting discs have mutually disjoint interiors. Moreover, the contact graph
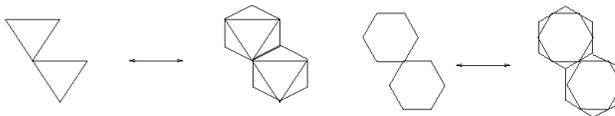
Figure 1: The polygonal transformations required to prove that all polygonal coin graphs are coin graphs, for the case of equilateral triangles and hexagons

realized by these discs is identical to the contact graph realized by the k-gons. (See also Figure 1.) Therefore, we need only prove the assertion $\mathcal{P}_k \subseteq \mathcal{C}$ in the case k is odd.

In the sequel it will be convenient to use the notation $\mathcal{P}_k^{ve}$ as identical to the notation $\mathcal{P}_k$ in order to indicate that adjacency is determined by polygons touching on a polygon vertex. It will also be convenient to consider "adjacency" for polygonal layouts in which two k-gons are adjacent if they touch at a whole edge.   □

More precisely we have.

Definition 2.2. The class $\mathcal{P}_k^{ed}$ is defined as follows. Vertices are represented by isothets of the regular k-gon P_k such that the interiors of any pair of such k-gons have empty intersection. Two k-polygons are adjacent if two polygon edges match.

Thus, by definition we have $\mathcal{P}_k^{ed} = \emptyset$, if k is odd. We can prove the following lemma which easily implies Theorem 2.1.

Lemma 2.1. *For each integer $k \geq 3$ we have that:*

1. *if k is even then $\mathcal{P}_k^{ve} \subseteq \mathcal{P}_{kl}^{ve}$, for all integers l,*

2. *if k is odd then $\mathcal{P}_k^{ve} = \mathcal{P}_{2k}^{ed}$,*

3. *if k is even then $\mathcal{P}_k^{ve} = \mathcal{P}_k^{ed}$.*

Proof. Consider polygon P_k. The two consecutive sides of P_k forms an angle $\pi - 2\pi/k$ or an external angle $2\pi/k$ (see Figure 2). Let a, b be the exterior angles formed by two k-gons intersecting at a vertex. It is clear that $a + b = 4\pi/k$. If follows easily from angle considerations of the corresponding k-gons that if k is even then $a = b = 2\pi/k$, while if k is odd then one of the angles is π/k while the other is $3\pi/k$.

Now we can give the proof of the lemma. We transform the given polygon P_k into a new polygon Q_k. For given l, we inscribe P_k into a regular kl-gon
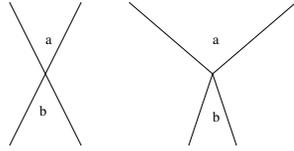
Figure 2: The angles a, b between edges of two touching polygons. In the picture to the left k is even, while in the picture to the right k is odd

Q_{kl} such that the vertices of P_k are also vertices of Q_{kl}. Now suppose that P_k and P'_k are two touching k-gons. We consider separately the case where k is even and k is odd. It is easy to show that if k is even then the kl-gons Q_{kl} and Q'_{kl} touch at a vertex if and only if the k-gons P_k and P'_k touch at a vertex. This implies that if k is even then $\mathcal{P}^{ve}_k \subseteq \mathcal{P}^{ve}_{kl}$, for all integers l. If k is odd and the polygons P_k and P'_k touch at a vertex then the previous observations imply that one of the exterior angles at the contact point of the polygons is π/k while the other is $3\pi/k$. It follows that the corresponding polygons Q_{kl} and Q'_{kl} have nontrivial intersection. Moreover, this intersection is a polygon edge exactly when $l = 2$. It follows that if k is odd then $\mathcal{P}^{ve}_k \subseteq \mathcal{P}^{ed}_{2k}$.

To prove that $\mathcal{P}^{ed}_{2k} \subseteq \mathcal{P}^{ve}_k$ we essentially reverse the above argument. For a given $2k$-gon P_{2k} inscribe a k-gon as follows: form the k-gon Q_k with vertices the mid-points of the edges of P_{2k}. It follows that two $2k$-gons P_{2k} and P'_{2k} touch at an edge if and only if the k-gons Q_k and Q'_k touch at a vertex. This completes the proof of the assertion.

The assertion $\mathcal{P}^{ve}_k = \mathcal{P}^{ed}_k$ for k even follows essentially the same ideas. One inscribes the given k-gon P_k into a k-gon Q_k such that the vertices of P_k are midpoints of edges of Q_k, and vice versa. Details of the proof are left to the reader.

This completes the proof of Lemma 2.1 and hence also the proof of Theorem 2.1. $\square$

The statement of Theorem 2.1 raises the following interesting problem.

Problem 2.1. Is there a constant c such that $\mathcal{P}_k = \mathcal{C}$, for all $k \geq c$?

We show this is impossible by proving the following theorem.

Theorem 2.2. *For each k, there exist an n-vertex unit-disc contact graph with $n \in O(k)$ which is not representable by regular k-gons.*

Proof. We consider the contact graph depicted in Figure 3. It consists of the following components. An internal circle C, straight line double chains L
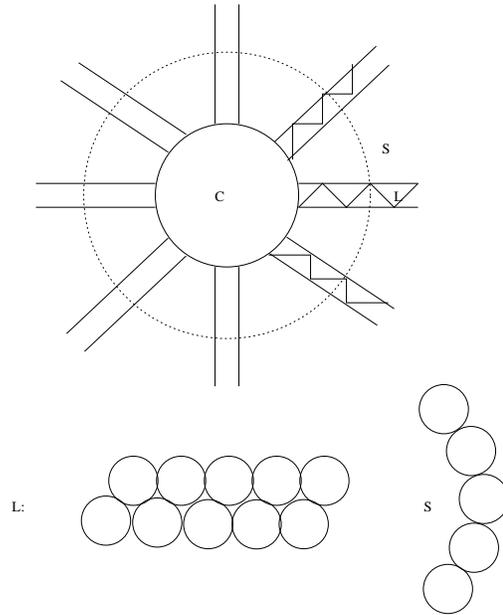
Figure 3: Proving the lower bound in discretizing disc graphs. The graph is constructed from the components L, S and the circle C which is formed of a circular chain of unit discs

around the internal circle and at equal angles, and an external circle. Both internal and external circles consist of unit circles. The external circle consists of circular chain segments S. These parts are connected in such a way that an immobile structure is formed. The corresponding parts can be constructed with unit discs thus forming a contact disc graph.

Let successive segments of type L be such that they are forming an angle of size $2\pi/n$. It is clear that we can achieve this by using a total of $O(n)$ unit discs in the graph. If $k \notin \Omega(n)$ then we note by using only regular k-gons it will be impossible to form the graph depicted in Figure 3. $\qquad \square$

2.1. Complexity of Recognition Problem

Theorem 2.3.   *For each k, the problem of recognizing whether a given graph is in the class $\mathcal{P}_{4k}$ is NP-complete.*

Proof. We give first the argument for the class $\mathcal{P}_4$.

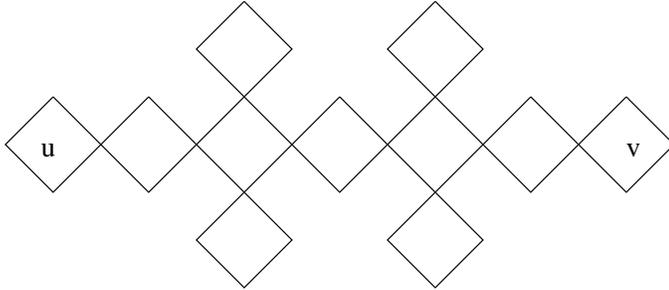We show how to reduce the problem of laying out a graph on a grid with

Figure 4: The caterpillar $C(u,v)$ replacing the edge $\{u,v\}$

dilation one to the above graph recognition problem. The former problem is known to be NP-complete [2, 11] even for the case when the graph is restricted to be a tree. For a given graph $G = (V, E)$ we define a new graph $G' = (V', E')$ such that

$$G' \in \mathcal{P}_4 \Leftrightarrow G \text{ is embeddable on a grid with dilation 1.} \qquad (1)$$

It follows that the recognition problem for the class $\mathcal{P}_4$ is NP-complete. Since G is a tree if and only if G' is a tree it will also follow that the problem of recognizing whether a given tree is in the class $\mathcal{P}_4$ is NP-complete.

The graph G' is defined from G as follows. We replace each edge $\{u,v\}$ with the following caterpillar $C(u,v)$

- vertices: u, v, u_1, u_2, u_3, u_4, u_5 and x_1, x_2, y_1, y_2.

- edges:

$$\{u, u_1\}, \{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_5\}, \{u_5, v\}, \text{ and}$$

$$\{x_1, u_2\}, \{x_2, u_4\}, \{y_1, u_2\}, \{y_2, u_4\}.$$

The caterpillar itself is depicted in Figure 4. To prove the equivalence (1) observe that if the caterpillar $C(u,v)$ is represented in $\mathcal{P}_4$ then the centers of the squares representing the verices u, v must both lie either on a vertical or horizontal line. This completes the proof of the theorem. $\qquad\square$

An interesting problem is the following.

Problem 2.2. For each k, is the problem of recognizing whether a given graph is in the class $\mathcal{P}_k$, NP-complete?

3. Discrete Intersection Graphs

Interval and unit square graphs have been extensively studied in the literature [9]. Linear recognition algorithms for these graphs have appeared in [9, 5, 6], etc. In the sequel we define new classes of unit interval graphs.

3.1. Discretizing Intersection Graphs

Intersection graphs of convex objects can be discretized by replacing intersection points with rational numbers which are sufficiently close to the original intersection points. To illustrate this point the following argument which is stated for the classes of intersection graphs of unit squares can be used for intersection graph representation by any type of convex objects.

Theorem 3.1. *Intersection graphs of unit squares can be realized as intersection graphs of unit squares all of whose corners have rational endpoints.*

Proof. We consider the Manhattan norm for points in the plane, $|(x, y)| = x + y$. Let $\{S_u : u \in V\}$ be a collection of closed unit squares representing the graph $G = (V, E)$, i.e. for all vertices u, v,

$$E(u, v) \Leftrightarrow S_u \cap S_v \neq \emptyset.$$

Let c_u be the center of the square I_u. Clearly, there exists a real $r > 0$ such that for all u, v

$$|c_u - c_v| < 1 \Rightarrow |c_u - c_v| < 1 - r, \text{ and } |c_u - c_v| > 1 \Rightarrow |c_u - c_v| > 1 + r.$$

Choose a sequence $\{r_u : u \in V\}$ of pairs of reals such that for all u, v:

- $|r_u| < r/2$,

- $c_u + r_u$ has both components rational, and

- if $c_u - c_v$ has both components integers then $r_u = r_v$.

Now consider the unit squares T_u with center $c_u + r_u$. It is easy to verify that this sequence of squares realizes the same graph G. Since all the corners of the squares are rationals it is straightforward that we can find an integer k such that $\mathcal{S}^n = \mathcal{S}_k^n$. $\qquad \square$

One of the problems to be studied in this paper is the following.

Problem 3.1. What is the complexity of a discrete realization of a geometric graph when corresponding points are only allowed to have integer coordinates with "limited range"?

3.2. Discrete Unit Interval Graphs

Definition 3.1. Let k be a given integer. To define the class of k-discrete unit interval graphs we consider intervals of identical size over the integers. A graph belongs to the class $\mathcal{I}_k$ if it can be represented by a collection of *pairwise distinct* intervals over the integers such that each interval has length exactly k and such that interval intersections can only occur at integer points.

Since all intervals are of equal length it is clear that $\mathcal{I}_k \subseteq \mathcal{I}$, for all k.

Theorem 3.2. *For all n there exists an integer $k \leq 2^{n-1}$ such that $\mathcal{I}^n = \mathcal{I}_k^n$.*

Proof. Let $\{I_u : u \in V\}$ be a collection of closed unit intervals representing the graph $G = (V, E)$, i.e. for all vertices u, v, $E(u, v) \Leftrightarrow I_u \cap I_v \neq \emptyset$.

We prove the existence of a unit interval realization of the graph such that no two endpoints of intervals of the new realization are identical.

Let $l(I), r(I)$ denote the left and right endpoints of the interval I. We prove the existence of a unit interval representation $\{J_u : u \in V\}$ of the graph in such a way that for all u, v:

- $l(I_u) < l(I_v) \Leftrightarrow l(J_u) < l(J_v)$,

- $|\{l(J_u), r(J_u) : u \in V\}| = 2n$,

- for any $x \neq y$ in the set $\{l(J_u), r(J_u) : u \in V\}$ we have that $|x - y| \geq 1$.

The proof is by induction on n. The inductive step $n = 1$ is trivial. Assume the induction hypothesis is true for interval graphs with $n - 1$ vertices. Remove the interval with the smallest left endpoint, say I_a. Consider all intervals $\{I_u : u \neq a\}$ intersecting I_a. Since all intervals have equal lengths, it is easy to show that the set $\{I_u : u \neq a, I_u \cap I_a \neq \emptyset\}$ forms a clique. By the induction hypothesis the set $\{J_u : u \neq a, I_u \cap I_a \neq \emptyset\}$ also forms a clique. In particular their left endpoints must lie within an interval of length 2^{n-2}. Since by the induction hypothesis no two endpoints can coincide, in follows that these left endpoints must lie within an interval of length $2^{n-2} - 1$. Let r be the rightmost left endpoint among the intervals $\{J_u : u \neq a, I_u \cap I_a \neq \emptyset\}$. Let J_a be the closed interval

$$J_a = \left[r - 2^{n-2} + \frac{1}{2}, r + \frac{1}{2} \right].$$

For each u define the intervals $\bar{J}_u = [2l(J_u), 2r(J_u)]$. It is clear that each interval has length 2^{n-1}. Moreover, it is easy to see that the resulting sequence

$\{\bar{J}_u : u \in V\}$ of intervals represents the same graph G and satisfies all the conditions of the inductive steps. This proves the theorem. $\qquad\square$

It is interesting to note that Theorem 3.2 improves on Theorem 3.2 of [5] which implies that for all n there exists an integer $k \leq n!$ such that $\mathcal{I}^n = \mathcal{I}_k^n$.

Theorem 3.3.   *There exist graphs $G \in \mathcal{I}_{n/2-1}^n \setminus \mathcal{I}_{n/2-2}^n$, for $n \geq 2$.*

Proof. Let $n = 2m$. Consider the unit interval graph with vertex set $V = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$. Edges are defined as follows. Vertex u_i is adjacent exactly to the vertices

$$u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_{i-1},$$

and vertex v_i is adjacent exactly to the vertices

$$v_1, v_2, \ldots, v_m, u_{i+1}, v_{i+2}, \ldots, u_m.$$

Since the degree sequence of the graph attains m different values it is easy to see that $G \notin \mathcal{I}_{n/2-2}^n$. The rest of the statement of the theorem is immediate. $\qquad\square$

Definition 3.2.   An interesting parameter for a (unit) interval graph G is $\mathcal{I}(G)$ which is the smallest integer m such that a discrete unit interval representation of the graph can be drawn with all integers of all interval endpoints in the range $[0, m]$.

Theorem 3.2 implies that for any unit interval graph G, $\mathcal{I}(G) \leq n2^{n-1}$. We can also prove the following lower bound.

Theorem   3.4.   *There   exist   graphs G $\in$ $\mathcal{I}^n$ such   that $\mathcal{I}(G) \in \Omega(n \log n)$.*

Proof. Let $i(n)$ be the number of unit interval graphs on n vertices. Assume we could draw all unit interval graphs within the integer range $[1, m]$ with all (pairwise distinct) intervals of length exactly k. Since unit intervals are uniquely determined by a unique point (say, their cenrter) it is clear that there exist at most $\binom{m}{n}$ such graphs. Hence, $i(n) \leq \binom{m}{n}$. This implies that $n! \cdot i(n) \leq m^n$. Trivial calculations show that this implies that $m \in \Omega(n \log n)$. $\qquad\square$

An interesting problem is the following.

Problem 3.2.   Give an algorithm for recognizing membership in the class $\mathcal{I}_k$. Is this an NP-complete problem? A related question is the following. For each graph $G \in \mathcal{I}^n$ compute $\text{disc}_\mathcal{I}(G) = \min_k\{G \in \mathcal{I}_k^n\}$. A similar question is the following. For each graph $G \in \mathcal{I}^n$ compute $\mathcal{I}(G)$. Can we tighten the bounds $cn \log n \leq \mathcal{I}(G) \leq n2^{n-1}$?

3.3. Discrete Unit Square Intersection Graphs

Definition 3.3. The class of unit square graphs, denoted by $\mathcal{S}$, is defined as follows. Vertices are represented by unit squares. Two vertices are adjacent if they have a nonempty intersection.

Definition 3.4. Let k be a given integer. To define the class of k-discrete unit square graphs we consider squares of identical side over the integer square lattice of integers. A graph belongs to the class $\mathcal{S}_k$ if it can be represented by a collection of squares over the square lattice of integers such that the side of each square is an interval of length exactly k.

Since all squares are of equal length it is clear that $\mathcal{S}_k \subseteq \mathcal{S}$, for all k. We can ask similar questions as with unit interval graphs.

Theorem 3.5. *For all n there exists an integer $k \leq 2^{n-1}$ such that $\mathcal{S}^n = \mathcal{S}_k^n$.*

Proof. Consider the two unit interval graphs formed by projecting the squares representing a given graph on the x and y axis. Then use the result of Theorem 3.2. □

Problem 3.3. Give an algorithm for recognizing membership in the class $\mathcal{S}_k^n$. Is this an NP-complete problem? A related question is the following. For each graph $G \in \mathcal{S}^n$ compute $\mathrm{disc}_\mathcal{S}(G) = \min_k \{G \in \mathcal{S}_k^n\}$.

We conclude by stating two theorems which are not difficult to prove but are useful in understanding the structure of discrete unit square graphs.

Theorem 3.6. *For each graph $G \in \mathcal{S}$ let G_x, G_y be the unit interval graphs obtained by projecting the squares representing G on the x- and y-axis. Then $\max\{\mathrm{disc}_\mathcal{I}(G_x), \mathrm{disc}_\mathcal{I}(G_y)\} \leq \mathrm{disc}_\mathcal{S}(G) \leq \mathrm{disc}_\mathcal{I}(G_x) \cdot \mathrm{disc}_\mathcal{I}(G_y)$.*

Theorem 3.7. *If $G_1, G_2 \in \mathcal{I}^n$ such that $\mathrm{disc}_\mathcal{I}(G_1) = \mathrm{disc}_\mathcal{I}(G_2) = k$ then there exists a graph $G \in \mathcal{S}^n$ such that $G_x = G_1, G_y = G_2$ and $\mathrm{disc}_\mathcal{S}(G) = k$.*

4. Conclusion

In this paper we considered discrete realizations of contact and intersection graphs. Contact graphs of regular polygons form a hierarchy of planar graphs and the recognition problem for such graphs is NP-hard. In contrast to this, intersection graphs are always descretizable and bounds on the complexity of the resulting discetizations are given.

Acknowledgements

Remark

An extended abstract of this paper has appeared in the proceedings of the *5-th Symposium on Graph Drawing, GD'97* (Ed. DeBatista), held in Rome, Italy, September 18-20, 1997, *Springer Lecture Notes in Computer Science*, **1352** (1997), 359-370.

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