International Journal of Pure and Applied Mathematics

Volume 13 No. 4 2004, 457-461

OUTER PRODUCT AND ORIENTED AREA OF A POLYGON IN EUCLIDEAN PLANE

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Abstract: The oriented area of an n–gon can be expressed by the outer product of n–gons. Some simple formulas in terms of oriented areas of some n–gons associated to a given n–gon can be proved.

AMS Subject Classification: 51M25
Key Words: outer product, oriented area, polygon

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Let

$$\mathbf{a_1} = (x_1, y_1),\ \mathbf{a_2} = (x_2, y_2)$$

be vectors in real Euclidean plane E_2 represented by their coordinates in some Cartesian coordinate system. The real number

$$[\mathbf{a_1}, \mathbf{a_2}] = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \tag{1}$$

is said to be the *outer product* of these two vectors. It is obvious that the outer multiplication of vectors is anticommutative and bilinear.

Received: February 7, 2004

We identify any point of E_2 with its radius–vector in the considered coordinate system.

Let $n \in \{3, 4, \dots\}$. An *n–gon* is an ordered n–tuple

$$\mathcal{A} = (\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n}) \tag{2}$$

of points. The addition of n–gons and the multiplication of an n–gon by a real number are defined by

$$\mathcal{A} + \mathcal{B} = (\mathbf{a_1} + \mathbf{b_1}, \mathbf{a_2} + \mathbf{b_2}, \dots, \mathbf{a_n} + \mathbf{b_n}),$$

$$a\mathcal{A} = (a\mathbf{a_1}, a\mathbf{a_2}, \dots, a\mathbf{a_n}),$$

where

$$\mathcal{B} = (\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_n}). \tag{3}$$

In the set $\mathbf{P_n}$ of all n–gons we have according to [1] an important mapping ζ defined by

$$\zeta(\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_n}) = (\mathbf{a_2}, \mathbf{a_3}, \dots, \mathbf{a_n}, \mathbf{a_1}).$$

This mapping is linear, i.e. we have always

$$\zeta(a\mathcal{A} + b\mathcal{B}) = a\,\zeta(\mathcal{A}) + b\,\zeta(\mathcal{B}). \tag{4}$$

Moreover,

$$\zeta^n = 1, \tag{5}$$

where 1 is the identical mapping on $\mathbf{P_n}$.

The *outer product* of two n–gons $\mathcal{A}$ and $\mathcal{B}$ from (2) and (3) is defined by

$$[\mathcal{A}, \mathcal{B}] = \sum_{i=1}^{n} [\mathbf{a_i}, \mathbf{b_i}].$$

According to the properties of outer multiplication (1) it follows that the outer multiplication of n–gons is anticommutative and bilinear, i.e. we have identically

$$[\mathcal{B}, \mathcal{A}] = -[\mathcal{A}, \mathcal{B}], \tag{6}$$

$$[a\mathcal{A} + b\mathcal{B}, \mathcal{C}] = a[\mathcal{A}, \mathcal{C}] + b[\mathcal{B}, \mathcal{C}], \tag{7}$$

$$[\mathcal{C}, a\mathcal{A} + b\mathcal{B}] = a[\mathcal{C}, \mathcal{A}] + b[\mathcal{C}, \mathcal{B}]. \tag{8}$$

Moreover,

$$[\zeta\mathcal{A}, \zeta\mathcal{B}] = [\mathcal{A}, \mathcal{B}]. \tag{9}$$

From (6) it follows immediately

$$[\mathcal{A}, \mathcal{A}] = 0. \tag{10}$$

The notion of outer product of n–gons is very closely related to the concept of oriented area of polygons. Indeed, we can prove the following theorem.

Theorem 1. *If φ is the oriented area on the set $\mathbf{P}$ of all polygons, then*

$$\varphi(\mathcal{A}) = \frac{1}{2}[\mathcal{A}, \zeta\mathcal{A}]. \tag{11}$$

Proof. The proof is by induction on n. For $n = 3$ let $\mathcal{A} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ be any triangle, where $\mathbf{a_i} = (x_i, y_i)$ $(i = 1, 2, 3)$. Then we have successively

$$\begin{aligned}\varphi(\mathcal{A}) &= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}\right) \\ &= \frac{1}{2}([\mathbf{a_1}, \mathbf{a_2}] + [\mathbf{a_2}, \mathbf{a_3}] + [\mathbf{a_3}, \mathbf{a_1}]) \\ &= \frac{1}{2}[(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}), (\mathbf{a_2}, \mathbf{a_3}, \mathbf{a_1})] = \frac{1}{2}[\mathcal{A}, \zeta\mathcal{A}].\end{aligned}$$

Now, suppose that (11) is valid for all $(n-1)$–gons $(n \geq 4)$ and let $\mathcal{A}$ be any n–gon given by (2). Then

$$\varphi(\mathcal{A}) = \varphi(\mathcal{B}) + \varphi(\mathcal{C}), \tag{12}$$

where $\mathcal{B} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_{n-1}})$ is an $(n-1)$–gon and $\mathcal{C} = (\mathbf{a_1}, \mathbf{a_{n-1}}, \mathbf{a_n})$ is a triangle. By the previous proof we have

$$\varphi(\mathcal{C}) = \frac{1}{2}[\mathcal{C}, \zeta\mathcal{C}] = \frac{1}{2}([\mathbf{a_1}, \mathbf{a_{n-1}}] + [\mathbf{a_{n-1}}, \mathbf{a_n}] + [\mathbf{a_n}, \mathbf{a_1}]), \tag{13}$$

and from the hypothesis of induction we have

$$\varphi(\mathcal{B}) = \frac{1}{2}[\mathcal{B}, \zeta\mathcal{B}] = \frac{1}{2}(\sum_{i=1}^{n-2}[\mathbf{a_1}, \mathbf{a_{i+1}}] + [\mathbf{a_{n-1}}, \mathbf{a_1}]). \tag{14}$$

But, $[\mathbf{a_1}, \mathbf{a_{n-1}}] + [\mathbf{a_{n-1}}, \mathbf{a_1}] = 0$ because of anticommutativity of the outer multiplication (1) and by (12), (13) and (14) it follows

$$\varphi(\mathcal{A}) = \frac{1}{2}(\sum_{i=1}^{n-1}[\mathbf{a_i}, \mathbf{a_{i+1}}] + [\mathbf{a_n}, \mathbf{a_1}]) = \frac{1}{2}[\mathcal{A}, \zeta\mathcal{A}],$$

i.e. the equality (11) for n–gons. $\square$

As an application of Theorem 1 let us prove some generalizations of a result from [2].

For any $k \in \mathbb{N}$ and any n–gon $\mathcal{A}$ we can define a new n–gon $\mathcal{A}_k$ by

$$\mathcal{A}_k = \frac{1}{2}(\mathcal{A} + \zeta^k \mathcal{A}). \tag{15}$$

Therefore, the vertices of $\mathcal{A}_1$ are the midpoints of the sides of $\mathcal{A}$, the vertices of $\mathcal{A}_2$ are the midpoints of the first diagonals of $\mathcal{A}$, etc. Let $f = \varphi(\mathcal{A})$ and $f_k = \varphi(\mathcal{A}_k)$ for every $k \in \mathbb{N}$. From (11) and (15) according to (4), (7) and (8) we obtain successively

$$\begin{aligned} f_k &= \varphi(\mathcal{A}_k) = \frac{1}{2}[\mathcal{A}_k, \zeta\mathcal{A}_k] = \frac{1}{2}[\frac{1}{2}(\mathcal{A} + \zeta^k\mathcal{A}), \frac{1}{2}(\zeta\mathcal{A} + \zeta^{k+1}\mathcal{A})] \\ &= \frac{1}{8}([\mathcal{A}, \zeta\mathcal{A}] + [\mathcal{A}, \zeta^{k+1}\mathcal{A}] + [\zeta^k\mathcal{A}, \zeta\mathcal{A}] + [\zeta^k\mathcal{A}, \zeta^{k+1}\mathcal{A}]). \end{aligned}$$

However, by (9) and (6) we have

$$\begin{aligned} [\zeta^k\mathcal{A}, \zeta^{k+1}\mathcal{A}] &= [\mathcal{A}, \zeta\mathcal{A}] = 2\varphi(\mathcal{A}) = 2f, \\ [\zeta^k\mathcal{A}, \zeta\mathcal{A}] \quad &= [\zeta^{k-1}\mathcal{A}, \mathcal{A}] = -[\mathcal{A}, \zeta^{k-1}\mathcal{A}], \end{aligned}$$

and therefore

$$f_k = \frac{1}{8}(4f + [\mathcal{A}, \zeta^{k+1}\mathcal{A}] - [\mathcal{A}, \zeta^{k-1}\mathcal{A}]) \quad (k = 1, 2, \ldots). \tag{16}$$

Theorem 2. *(i) If $n = 2m + 1$ ($m \in \mathbb{N}$), then for any n–gon*

$$f_1 + f_2 + \ldots + f_m = \frac{2m-1}{4} f = \frac{n-2}{4} f. \tag{17}$$

(ii) If $n = 4m + 2$ ($m \in \mathbb{N}$), then

$$f_1 + f_3 + \ldots + f_{2m-1} + \frac{1}{2} f_{2m+1} = \frac{2m+1}{4} f = \frac{n}{8} f, \tag{18}$$

$$f_2 + f_4 + \ldots + f_{2m} = \frac{2m-1}{4} f = \frac{n-4}{8} f. \tag{19}$$

(iii) If $n = 4m$ ($m \in \mathbb{N}$), then

$$f_1 + f_3 + \ldots + f_{2m-1} = \frac{m}{2} f = \frac{n}{8} f, \tag{20}$$

$$f_2 + f_4 + \ldots + f_{2m-2} + \frac{1}{2} f_{2m} = \frac{m-1}{2} f = \frac{n-4}{8} f. \tag{21}$$

Proof. (i) It follows by addition of equalities (16) for $k = 1, 2, \dots, m$, where we use (10), the equality $[\mathcal{A}, \zeta\mathcal{A}] = 2f$ and the result

$$[\mathcal{A}, \zeta^{m+1}\mathcal{A}] = [\zeta^m\mathcal{A}, \zeta^{2m+1}\mathcal{A}] = [\zeta^m\mathcal{A}, \mathcal{A}] = -[\mathcal{A}, \zeta^m\mathcal{A}],$$

which is implied by (9), (5) and (6).

(ii) (18) follows by addition of equalities (16) for $k = 1, 3, \dots, 2m-1$ and the equality (16) for $k = 2m+1$ multiplied by $\frac{1}{2}$, where we use the equalities

$$[\mathcal{A}, \zeta^{2m+2}\mathcal{A}] = -[\mathcal{A}, \zeta^{2m}\mathcal{A}].$$

Equality (19) can be proved analogously , using the equalities

$$[\mathcal{A}, \zeta^{2m+1}\mathcal{A}] = [\zeta^{2m+1}\mathcal{A}, \mathcal{A}] = -[\mathcal{A}, \zeta^{2m+1}\mathcal{A}]\, i.e.\, [\mathcal{A}, \zeta^{2m+1}\mathcal{A}] = 0.$$

(iii) The proof is analogous, but we must use the equalities

$$[\mathcal{A}, \zeta^{2m}\mathcal{A}] = 0, \quad [\mathcal{A}, \zeta^{2m+1}\mathcal{A}] = -[\mathcal{A}, \zeta^{2m-1}\mathcal{A}]. \qquad \square$$

It must be noted that f_{2m+1} in (18) is the oriented area of a $(2m+1)$–gon counted twice and therefore $\frac{1}{2}f_{2m+1}$ is the oriented area of the same $(2m+1)$–gon with the vertices in midpoints of principal diagonals of $(4m+2)$–gon $\mathcal{A}$. The same is true for f_{2m} in (21).

Adding (18) and (19), resp. (20) and (21), we obtain in both cases the same equality

$$f_1 + f_2 + \dots + f_{\frac{n}{2}-1} + \frac{1}{2}f_{\frac{n}{2}} = \frac{n-2}{4},$$

the analogue of (17) in the case of an even n.

In [2] the equality (18) is proved for $m = 1$, i.e. for $n = 6$, i.e. the equality

$$f_1 + \frac{1}{2}f_3 = \frac{3}{4}f\,,$$

for hexagons.

References

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