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PROJECTIVE VARIETIES AND SETS OF OSCULATING SPACES IN UNIFORM POSITION

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Abstract: Let $X \subset \mathbf{P}^r$ be an integral projective variety. For every $P \in X_{reg}$ and $m > 0$ let $O(X, mP)$ be the osculating space of X. Here we give the set-up to show that in several cases there must be k-ples of distinct points $(P_1, \dots, P_k), (Q_1, \dots, Q_k) \in X_{reg}^k$ such that $\dim(\langle O(X, mP_1) \cup \cdots \cup O(X, mP_k)\rangle) \neq \dim(\langle O(X, mQ_1) \cup \cdots \cup O(X, mQ_k)\rangle)$.

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1. Sets in Multiple Uniform Position

Here we try to give a general set-up and a "measure" for the following type of questions. Let $C \subset \mathbf{P}^2$ be a smooth plane curve of degree $d \geq 3$. Why must C have a flex, i.e. why not all $P \in C$ are such that the tangent line to C at P has order of contact two with C at P? Take an abstract set $S \subset \mathbf{P}^r$; when all finite subsets of S with the same cardinality have the same properties, even from a differential point of view? For the first question (in the set-up of osculating spaces to projective varieties) see Section 2. In te first section we discuss the second question. We work over an algebraically closed field $\mathbb{K}$.

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Definition 1. Let $S \subset \mathbf{P}^n$ be a finite set. We will say that S is in multiple uniform position if for all $A \subseteq S$, $B \subseteq S$ such that $\mathrm{card}(A) = \mathrm{card}(B)$, say $A = \{P_1, \dots, P_c\}$ and $B = \{Q_1, \dots, Q_c\}$ and all integers t and $m_i > 0$, $1 \le i \le c$, we have $h^0(\mathbf{P}^n, \mathcal{I}_{m_1P_1 \cup \cdots m_cP_c}(t)) = h^0(\mathbf{P}^n, \mathcal{I}_{m_1Q_1 \cup \cdots m_cQ_c}(t))$.

Since we allow the case $m_i \ne m_j$ for some i, j, this is even a non-trivial condition taking $A = B$ and even taking $A = B = S$. However, since there are infinitely many degrees $\sum_{1 \le i \le c} \binom{m_i+n-1}{n-1}$ involved in the definition and the intersection of a countable (but infinite) family of Zariski open subset is not Zariski open, in general, me prefer the following definition.

Definition 2. Fix an integer $m > 0$. Let $S \subset \mathbf{P}^n$ be a finite set. For any $P \in \mathbf{P}^r$ and any integer $m > 0$ let mP denote the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_P)^m$ as ideal sheaf. We will say that S is in multiple uniform position up to order m if for all $A \subseteq S$, $B \subseteq S$ such that $\mathrm{card}(A) = \mathrm{card}(B)$, say $A = \{P_1, \dots, P_c\}$ and $B = \{Q_1, \dots, Q_c\}$ and all integers t and m_i with $0 \le m_i \le m$, $1 \le i \le c$, we have $h^0(\mathbf{P}^n, \mathcal{I}_{m_1P_1 \cup \cdots m_cP_c}(t)) = h^0(\mathbf{P}^n, \mathcal{I}_{m_1Q_1 \cup \cdots m_cQ_c}(t))$.

If the ambient space is not a projective space or if we want to work with vector bundles instead of line bundles, the following definition is very natural.

Definition 3. Let X be an integral projective variety, Δ a family of vector bundles on X and $S \subset X$ a finite subset. We will say that S is in multiple uniform position with respect to Δ if for all $A \subseteq S$, $B \subseteq S$ such that $\mathrm{card}(A) = \mathrm{card}(B)$, say $A = \{P_1, \dots, P_c\}$ and $B = \{Q_1, \dots, Q_c\}$ and all integers t and $m_i > 0$, $1 \le i \le c$, we have $h^0(X, \mathcal{I}_{m_1P_1 \cup \cdots m_cP_c} \otimes E) = h^0(X, \mathcal{I}_{m_1Q_1 \cup \cdots m_cQ_c} \otimes E)$ for every $E \in \Delta$. Similarly, for any fixed integer $m > 0$ we will say that S is in multiple uniform position up to order m if the previous condition is satisfied for all $m_i \le m$.

In Definition 3 we could take S infinite (e.g. take $S = X$). However, in this way we do not get a useful definition, as shown by the next very easy result, unless we give an upper bound for the number of elements of the finite sets A, B; we will call this multiple uniform position (or multiple uniform position with multiplicity $\le m$) for all subsets A, B of S with $\sharp(A) = \sharp(B) \le x$.

Here is a related definition, which is often more convenient to obtain sharp bounds.

Definition 4. Let $X \subseteq \mathbf{P}^r$ be an integral n-dimensional variety. For every integer $x > 0$ and every $P \in X_{reg}$, let xP denote the infinitesimal neighborhood of order $x - 1$ P in X, not in $\mathbf{P}^r$, i.e. the closed subscheme of X with $(\mathcal{I}_{P,X})^m$ as ideal sheaf. Set $0P := \emptyset$. Fix a locally closed subset Y of X_{reg} and integers $k > 0$, $m > 0$. We will say that this embedding of X is k-point uniform up to order m along Y if for all k-ples of distinct points $(P_1, \dots, P_k) \in X^k$ and $(Q_1, \dots, Q_k) \in X^k$ and all integers m_i, $1 \le i \le m_i$ with $0 \le m_i \le m$ we have $\dim(\langle m_1P_1 \cup \cdots \cup m_kP_k\rangle) = \dim(\langle m_1Q_1 \cup \cdots \cup m_kQ_k\rangle)$. If this is true for every m, then we will say that this embedding of X is k-point uniform along Y. If $Y = X_{reg}$, then we will omit the words "along Y". Abusing notation, when $m = 1$ we will allow the case in which Y contains some singular point of Y. For any $m > 0$ and $s > 0$ et $\tau(X, s, m) := \dim(\langle mP_1 \cup \cdots \cup mP_s\rangle)$ for general $(P_1, \dots, P_s) \in X^s$. We will say that X is k-uniform for the multiplicity m along $Y \subset X_{reg}$ if for every integer $s \le k$ and every $(Q_1, \dots, Q_s) \in Y^s$ with $Q_i \le Q_j$ for $i \ne j$ we have $\dim(\langle mQ_1 \cup \cdots \cup mQ_s\rangle) = \tau(X, s, m)$.

In the vector bundle case we have the following modification of Definition 4

Definition 5. For all integers $m > r > 0$ let $G(r, m)$ be the Grassmannian of all $(m-r)$-dimensional linear subspaces of $\mathbb{K}^m$. Hence $\dim(G(r, m)) = r(m-r)$. $G(r, m)$ is equipped with a rank r spanned vector bundle $Q_G(r, m)$ (called the universal quotient line bundle) such that $\det(Q_{G(r,m)}) \cong \mathcal{O}_{G(r,m)}(1)$ is the positive generator of $\mathrm{Pic}(G(r, m))$ and $h^0(G(r, m), Q_{G(r,m)}) = m+1$. Let $X \subseteq G(r, m)$ be an integral n-dimensional variety. For every integer $x > 0$ and every $P \in X_{reg}$, let xP denote the infinitesimal neighborhood of P in X, not in $G(r, m)$. Set $0P := \emptyset$. Fix a locally closed subset Y of X_{reg} and integers $k > 0$, $m > 0$. We will say that this embedding of X is k-point uniform up to order m along Y if for all k-ples of distinct points $(P_1, \dots, P_k) \in X^k$ and $(Q_1, \dots, Q_k) \in X^k$ and all integers m_i, $1 \le i \le m_i$ with $0 \le m_i \le m$ we have $h^0(G(r, m), \mathcal{I}_{m_1P_1 \cup \cdots \cup m_kP_k} \otimes Q_{G(r,m)}) = h^0(G(r, m), \mathcal{I}_{m_1Q_1 \cup \cdots \cup m_kQ_k} \otimes Q_{G(r,m)})$. If this is true for every m, then we will say that this embedding of X is k-point uniform along Y. If $Y = X_{reg}$, then we will omit the words "along Y". Abusing notation, when $m = 1$ we will allow the case in which Y contains some singular point of Y.

Example 1. Let $X = G/P$ be a rational homogeneous projective manifold ane E a rank $r \ge 1$ homogeneous vector bundle on X. Call $j : X \to G(r, h^0(X, E))$ the associated morphism. Since $j(X)$ is homogeously embedded, it is 1-point uniform. The automorphism group of $\mathbf{P}^1$ is 3-transitive and hence if $X = \mathbf{P}^1$, then $j(X)$ is 3-point uniform. The automorphism group of $\mathbf{P}^n$, $n \ge 2$, is 2-transitive and hence if $X = \mathbf{P}^1$, then $j(X)$ is 2-point uniform.

The following well-known result motivated our definitions.

Proposition 1. *Fix an integer* $m > 0$. *Let* $C \subset \mathbf{P}^n$ *be an integral nondegenere curve. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $n \ge 4$ *and* $\deg(C) \ge 25$. *Then* *there is a non-empty open subset U_m of $\mathbf{P}^{n*}$ such that for every $H \in U_m$ the set $H \cap C$ is in multiple uniform position up to order m. If we add the condition that $\mathbb{K}$ is uncountable, then there is a dense subset $U \in \mathbf{P}^{n*}$ such that for every $H \in U$ the set $H \cap C$ is in multiple uniform position.*

Proof. To prove the existence of U_m it is sufficient to note that by [1], Corollary 2.2 (case char$(\mathbb{K}) = 0$) or Theorem 2.5 (case char$(\mathbb{K}) > 0$), the monodromy group of the generic hyperplane section is at least $(\deg(C) - 2)$-transitive. For the last part set $U := \cap_{m \geq 1} U_m$ and use that over an uncountable field $\mathbb{K}$ the countable intersection of Zariski open non-empty subsets of $\mathbb{A}_{\mathbb{K}}$ is dense in $\mathbb{A}_{\mathbb{K}}$. □

Example 2. Let $X \subset \mathbf{P}^3$ be an integral surface of degree $d \geq 2$.

Here we will check the existence of $\{P_1, P_2, P_3, P_4\} \in X_{reg}$ with is not in multiple uniform position up to order one. It is sufficient to check the existence of collinear $\{P_1, P_2, P_3\} \in X_{reg}$ and then take P_4 general in Y. If $d = 2$, then X contains infinitely many lines and hence we are done. If $d \geq 3$ a general plane section of X contains many triples of smooth collinear points.

Proposition 2. *Fix integers $k > 0$, $m > 0$ and an integral subvariety $X \subseteq \mathbf{P}^n$. For every integer $y > 0$ let $v_{n,y} : \mathbf{P}^n \to \mathbf{P}^{N(n,y)}$, $N(n, y) := \binom{n+y}{y} - 1$, denote the Veronese embedding given by the linear system of all degree y hypersurfaces. Then for every integer $t \geq km$ every subset S of $v_{n,t}(X_{reg})$ with $\sharp(S) \leq k$ is in multiple uniform position up to order m.*

Proof. Set $x := \dim(X)$. Since $v_{n,t}$ is an embedding, for every integer $a > 0$ and any $P \in X_{reg}$ we have $a(v_{n,t}(P)) = v_{n,t}(aP)$ (as subschemes of $v_{n,t}(X)$). Just use that for all integers $s > 0$ and $m_i > 0$ such that $\sum_{i=1}^{s} m_i < t$ and all $Q_i \in \mathbf{P}^n$, $1 \leq i \leq s$, we have $\dim(\langle m_1Q_1 \cup \cdots \cup m_sQ_s\rangle) = -1 + \sum_{i=1}^{n} \binom{n+m_i}{n}$ and (when $Q_i \in X_{reg}$ for all i) restrict this relation to the linear span of the osculating spaces of $v_{n,y}(X)$ at $Q_1, \ldots, Q_s$. □

2. Osculating Spaces

Notation 1. Let $X \subset \mathbf{P}^r$ be an integral n-dimensional variety. For any $P \in X_{reg}$ and anu integer $x > 0$ let xP denote the infinitesimal neighborhood of order $x - 1$ of P in X, not in $\mathbf{P}^r$. Set $o(X, P, x) = \dim(\langle xP\rangle)$. The sequence $o(X, P) := \{o(X, P, x)\}_{x \geq 1}$ is called the order sequence of X at P. Set $o(X, x) := o(X, P, x)$ for P general in X. The sequence $\{o(X, x)\}_{x \geq 1}$ is called the order sequence of X.

Proposition 3. *Let X be a projective variety and L a line bundle on X such that $a := h^0(X, L) > 0$. Assume $h^0(X, \mathcal{I}_A \otimes L) = h^0(X, \mathcal{I}_B \otimes L)$ for all $A, B \subset X$ such that $\sharp(A) = \sharp(B) = a$. Then either $L \cong \mathcal{O}_X$ or $X \cong \mathbf{P}^1$.*

Proof. For a general $A \in X^a$ we have $h^0(X, \mathcal{I}_A \otimes L) = 0$. If L has at least one base point, choosing it instead of one of the points of A we obtain a contradiction. Hence L has no base points. If $a = 1$, this implies $L \cong \mathcal{O}_X$. Assume $a \geq 2$ and call $\psi : X \to \mathbf{P}(H^0(X, L)^*) \cong \mathbf{P}^{a-1}$ the morphism induced by the complete linear system $|L|$. As in the proof of the base point freeness of L we obtain that ψ is injective. The rational normal curve of $\mathbf{P}^{a-1}$ is the only subvariety of $\mathbf{P}^{a-1}$ such that any a points of it span $\mathbf{P}^{a-1}$. Hence X is a curve and $\phi(X) \cong \mathbf{P}^1$. Since ψ is injective and $\mathbf{P}^1$ is a normal variety, ψ is an isomorphism by Zariski Main Theorem. $\square$

Definition 6. Let X be an integral n-dimensional projective variety and $Y \subset X$ an integral locally closed x-dimensional variety, $0 < x < n$. We will say that Y is very positive if $Y \cap T = \emptyset$ for every closed irreducible $(n-x)$-dimensional integral subvariety T of X.

The following observation explains the strength of the previous definition.

Lemma 1. *Let X be aqn integral n-dimensional projective variety and $Y \subset X$ an integral very positive locally closed x-dimensional variety, $0 < x \leq n-2$. Then for every integer y with $1 \leq y \leq n-1-x$ and every y-dimensional closed integral subvariety D of X the set $Y \cap D$ contains an $(x+y-n)$-dimensional subvariety.*

Proof. Fix a y-dimensional closed integral subvariety D of X, a very ample linear system $|H|$ on X and $x+y-n$ general members $H_1, \dots, H_{x+y-n}$ of $|H|$. Set $T := D \cap D_1 \cap \cdots \cap D_{x+y-n}$. Thus T is a non-empty $(n-y)$-dimensional closed subvarety of X. Since Y is very positive, $Y \cap T \neq \emptyset$. By the generality of $H_1, \dots, H_{x+y-n}$ this implies $\dim(Y \cap D) \geq x+y-n$. □

Notation 2. For all integers $b > a > 0$ let $\mathrm{Grass}(a,b)$ denotes the set of all a-dimensional linear subspaces of $\mathbf{P}^b$. Fix integers $c > 0$ and $x \geq 0$ and a dimension c linear subspace $M \subset \mathbf{P}^b$. Set $\mathrm{Grass}(a,b;M,x) := \{A \in \mathrm{Grass}(a,b) : \dim(M \cap A) \geq x\}$ and

$$\mathrm{Grass}(a,b;M) := \mathrm{Grass}(a,b;M,0) = \{A \in \mathrm{Grass}(a,b) : A \cap M \neq \emptyset\}.$$

Up to a projective transformation, the Schubert cells $\mathrm{Grass}(a,b;M,x)$ and $\mathrm{Grass}(a,b;M)$ do not depend from the choice of M and hence we are allowed to define $\beta(a,b;c,x)$ (resp. $\beta(a,b;c)$) as the minimal integer t such that every t-dimensional closed subvariety of $\mathrm{Grass}(a,b)$ intersects $\mathrm{Grass}(a,b;M,x)$ (resp. $\mathrm{Grass}(a,b;M)$).

All the integers $\beta(a,b;c,x)$ are in principle computable.

Proposition 4. *Fix integers $m \geq 20$, $k > 0$ and an integral non-degenerate n-dimensional $X \subset \mathbf{P}^r$. Assume $\tau(X,k,2) = r - o(X,m) - 1$ and $\tau(X,k+1,m) = r$. Fix a general $(P_1, \dots, P_k) \in X^k$ and set $\Omega := \{P \in X_{reg} \backslash \{P_1, \dots, P_k\} : \langle mP_1 \cup \cdots \cup mP_k \cup mP \rangle \neq \mathbf{P}^r\}$. Then Ω is a non-empty open subset of X_{reg} which contains no projective curve.*

Proof. Set $M := \langle mP_1 \cup \cdots \cup mP_k \rangle$. Let $E \subset \mathrm{Grass}(o(X,m),r)$ be the set of all $o(X,m)$-dimensional linear subspaces of $\mathbf{P}^r$ intersecting M. Since $\dim(M) = r - o(X,m) - 1$, E is a non-empty hyperplane of the Grassmannian $G(o(X,m),r)$. We have $\Omega = \{P \in X_{reg} \backslash \{P_1, \dots, P_k\} : O(X,mP) \neq E$. Since $\mathcal{O}_{\mathrm{Grass}(o(X,m)+1,r+1)}(1)$ is very ample, we are done. □

Theorem 1. *Fix integers $m > 0$, $k > 0$, $x \geq 0$ and an integral non-degenerate n-dimensional $X \subset \mathbf{P}^r$. Assume $\tau(X,k,2)$ $ger - o(X,m) - 1 + x$ and $\tau(X,k+1,m) = r$. Fix a general $(P_1, \dots, P_k) \in X^k$ and set $\Omega := \{P \in$*

$X_{reg} \backslash \{P_1, \dots, P_k\} : \langle mP_1 \cup \cdots \cup mP_k \cup mP \rangle \neq \mathbf{P}^r$. *Then* Ω *contains no integral projective variety of dimension at least* $\beta(o(X,m), r; \tau(X,k,2), x)$ *on which the osculating rational map of order* m *from* X *into* $\mathrm{Grass}(o(X,m), r)$ *is generically finite.*

Proof. Set $M := \langle mP_1 \cup \cdots \cup mP_k \rangle$. Repeat the proof of Proposition 4, just using the definition of the integer $\beta(o(X,m), r; \tau(X,k,2), x)$. □

A similar proof (we leave the details to the interested reader) gives the following result.

Theorem 2. *Fix integers* $m_1 \geq 2$, $m_2 \geq 2$ *and* $x \geq 0$. *Let* $X \subset \mathbf{P}^r$ *be an integral non-degenerate variety. Set* $U(m_i) : P \in X_{reg} : \dim(\langle m_i P \rangle) = o(X, m_i)$. *Fix a general* $P_1 \in X_{reg}$ *and set* $\Omega := \{P \in U(m_2) : \dim(O(X, m_2 P_2) \cap O(X, m_1 P_1)) < x$. *Assume* $o(X, m_1) + o(X, m_2) + x \leq r - 1$. *Then* Ω *is an open subset of* $U(m_2)$ *not containing any irreducible variety of dimension* $\beta(o(X,m), r; o(X, m_1), x)$ *on which the osculating rational map of order* m_2 *from* X *into* $\mathrm{Grass}(o(X,m), r)$ *is generically finite; we allow the case* $\Omega = \emptyset$.

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References

[1] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276** (1987), 565-579.