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UNIQUENESS RESULTS FOR ELLIPTIC EQUATIONS VMO-COEFFICIENTS

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Abstract: In this paper we prove some uniqueness results for the Dirichlet problem for linear elliptic equations with locally VMO coefficients in unbounded domains.

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1. Introduction

Let Ω be an unbounded open subset of $\mathbf{R}^n$, $n \geq 3$, with some suitable regularity property. Consider in Ω the uniformly elliptic operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{1.1}$$

and suppose that its coefficients satisfy the following basic conditions:

$$a_{ij} = a_{ji} \in L^\infty(\Omega) \,, \quad i,j = 1, \ldots, n \,, \tag{1.2}$$

$$a_i \in L^{t_1}_{\text{loc}}(\bar{\Omega}) \,, \quad i = 1, \ldots, n \,, \tag{1.3}$$

$$a \in L^{t_2}_{\text{loc}}(\bar{\Omega}) \,, \quad a \geq a_0 > 0 \ \text{ a.e. in } \Omega \,, \tag{1.4}$$

where the summability exponents t_1 and t_2 will be specified later.

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The aim of this paper is to obtain, under suitable restrictions on the coefficients of L, a uniqueness result for the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ Lu = 0\,, \end{cases} \tag{$\mathcal{D}$}$$

with $p \in]1, +\infty[$.

Observe that, when $p \geq n$ and $t_1, t_2 \geq n$, the uniqueness of the solution of ($\mathcal{D}$) follows easily from classical results of Alexandrov-Pucci concerning the case of bounded open sets (see for instance [8] and [7]). In fact, note first that in this case any solution u of ($\mathcal{D}$) satisfies the conditions

$$u \in C^0(\bar{\Omega})\,, \quad u_{|\partial\Omega} = 0\,, \quad \lim_{|x| \to +\infty} u(x) = 0\,.$$

Assume by contradiction that $u(x_0) = \max u > 0$; consider an open subset G of $\mathbf{R}^n$ such that

$$G \subset\subset \Omega\,, \quad u_{|\Omega \setminus G} \leq u(x_0)/2\,,$$

and put $\varphi = u_{|\partial G}$. Then u is also a solution of the problem

$$u_{|G} \in W^{2,n}(G)\,, \quad Lu = 0 \ \text{ in } \ G\,, \quad u_{|\partial G} = \varphi\,.$$

On the other hand, $v = u(x_0)/2$ is a solution of the problem

$$v \in C^0(\bar{G}) \cap W^{2,n}(G)\,, \quad -Lv \leq 0 \ \text{ in } \ G\,, \quad v \geq \varphi \ \text{ on } \ \partial G\,,$$

and hence Theorem III of [8] yields that

$$u(x) \leq u(x_0)/2\,, \quad \forall x \in G\,,$$

a contradiction because $x_0 \in G$. Since $-u$ is likewise a solution of ($\mathcal{D}$), the above argument proves that $u = 0$.

It is well known that, in order to obtain a uniqueness result for the problem ($\mathcal{D}$) in the case $p < n$, additional hypotheses on the leading coefficients are necessary, even if Ω is bounded. Many results on this latter situation can be found in literature; among others, the condition that the a_{ij}'s are of class VMO seems to be of special interest (see for instance [6], [11], [12]). In the case of unbounded domains, when $p \in \ ]n/2, n[$, $t_1 > n$, $t_2 \geq p$ and

$$a_{ij} \in VMO_{\text{loc}}(\Omega)\,, \tag{1.5}$$

it has been proved that the problem $(\mathcal{D})$ with the additional condition

$$\lim_{|x|\to+\infty} u(x) = 0$$

has only the zero solution (see [3], Corollary 4.3). Moreover, if $p \leq n/2$, $t_1 > n$, $t_2 > n/2$,

$$a_{ij} \in VMO(\Omega)\,, \tag{1.6}$$

and the coefficients a_i and a belong to suitable subspaces of $L^{t_1}_{\text{loc}}(\bar{\Omega})$ and $L^{t_2}_{\text{loc}}(\bar{\Omega})$, respectively, a uniqueness result holds for problem $(\mathcal{D})$ (see [5]).

In this paper we will suppose that the a_{ij}'s are in the class $VMO_{\text{loc}}(\bar{\Omega})$, and that at infinity they are "close" to certain functions α_{ij} of class $VMO(\Omega)$. In the last section an example will be given to show that such conditions on the a_{ij}'s are properly weaker than (1.6). Our main theorem is a uniqueness result for problem $(\mathcal{D})$ in this situation.

2. Notation and Function Spaces

Let Ω be an open subset of $\mathbf{R}^n$, and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω. For each $E \in \Sigma(\Omega)$, we denote by $|E|$ and χ_E the Lebesgue measure and the characteristic function of E, respectively. Moreover, put

$$E(x,r) = E \cap B(x,r)\,, \quad \forall x \in \mathbf{R}^n, \quad \forall r \in \mathbf{R}_+,$$

where $B(x,r)$ is the open ball of $\mathbf{R}^n$ of radius r centered at x, and

$$B_r = B(0,r)\,, \quad \forall r \in \mathbf{R}_+.$$

If $X(\Omega)$ is a space of functions defined on Ω, we denote by $X_{\text{loc}}(\bar{\Omega})$ the space of all functions $g : \Omega \longrightarrow \mathbf{R}$ such that $\zeta g \in X(\Omega)$ for every $\zeta \in \mathrm{D}(\bar{\Omega})$, where $\mathrm{D}(\bar{\Omega}) = \{\xi_{|\Omega} \mid \xi \in C^\infty_0(\mathbf{R}^n)\}$.

For $p \in [1,+\infty[$, denote by $M^p(\Omega)$ the set of all functions $g \in L^p_{\text{loc}}(\bar{\Omega})$ such that

$$||g||_{M^p(\Omega)} = \sup_{x\in\Omega} ||g||_{L^p(\Omega(x,1))} < +\infty\,, \tag{2.1}$$

endowed with the norm defined by (2.1). Moreover, $\tilde{M}^p(\Omega)$ denotes the closure of $L^\infty(\Omega)$ in $M^p(\Omega)$. It is known that a function g (in $M^p(\Omega)$) belongs to $\tilde{M}^p(\Omega)$ if and only if the function

$$\tau_g(t) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega}|E(x,1)|\leq t}} ||\chi_E g||_{M^p(\Omega)}\,, \quad t \in \mathbf{R}_+\,, \tag{2.2}$$

vanishes when t goes to zero. Then a *modulus of continuity* of g in $\tilde{M}^p(\Omega)$ is a map $\tilde{\sigma}_p[g] : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ such that

$$\tilde{\sigma}_p[g](t) \ \geq \ \tau_g(t) \ \ \forall t \in \mathbf{R}_+ \, , \qquad \lim_{t \to 0^+} \tilde{\sigma}_p[g](t) = 0 \, .$$

If Ω has the property

$$|\Omega(x,r)| \ \geq \ A\, r^n \, , \quad \forall x \in \Omega \, , \ \ \forall r \in]0,1] \, , \tag{2.3}$$

where A is some positive constant independent of x and r, one can consider the space $BMO(\Omega, t)$, $t \in \mathbf{R}_+$, consisting of all functions g in $L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x \in \Omega \\ r \in]0,t]}} \fint_{\Omega(x,r)} |g - \fint_{\Omega(x,r)} g| \ < +\infty \, , \tag{2.4}$$

where

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, with

$$t_A = \sup_{t \in \mathbf{R}_+} \Big(\sup_{\substack{x \in \Omega \\ r \in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \Big) \, ,$$

we say that g is in $VMO(\Omega)$ when

$$\lim_{t \to 0^+} [g]_{BMO(\Omega,t)} = 0 \, .$$

Moreover, a function $\eta[g] : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$\eta[g](t) \geq [g]_{BMO(\Omega,t)} \ \ \forall t \in \mathbf{R}_+ \, , \qquad \lim_{t \to 0^+} \eta[g](t) = 0 \, .$$

A more detailed account of properties of the above defined function spaces can be found in [9] and [10].

3. An a priori Estimate

Let Ω be an unbounded open subset of $\mathbf{R}^n$, $n \geq 3$, satisfying the following $C^{1,1}$-regularity property.

($\mathcal{P}_\Omega$) There exist a locally finite open covering $(U_i)_{i\in\mathbf{N}}$ of $\partial\Omega$ and $C^{1,1}$-diffeomorphisms $\Phi_i : U_i \longrightarrow B_1$, $i \in \mathbf{N}$, such that:

- $\{x \in \Omega \,:\, \mathrm{dist}\,(x, \partial\Omega) < \delta\} \subset \bigcup_{i\in\mathbf{N}} \Phi_i^{-1}(B_{1/2})$ for some $\delta > 0$;
- $\Phi_i(U_i \cap \Omega) = \{x \in B_1 \,:\, x_n > 0\}$ for each $i \in \mathbf{N}$;
- there is $m_0 \in \mathbf{N}$ such that the intersection of any $m_0 + 1$ distinct U_i's is empty;
- the components of Φ_i and Φ_i^{-1} have bounded $C^{1,1}$-norms independently of i.

Observe that any open set Ω with the property ($\mathcal{P}_\Omega$) also satisfies the condition (2.3) above.

Let $p \in]1, +\infty[$, and let L be the operator in Ω defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a\,. \tag{3.1}$$

Consider the following conditions on the coefficients of L:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\mathrm{loc}}(\bar{\Omega})\,, \quad i, j = 1, \dots, n\,, \\ \exists \nu \in \mathbf{R}_+ : \displaystyle\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \ \text{ a.e. in } \Omega, \ \forall \xi \in \mathbf{R}^n\,; \end{cases} \tag{h_1}$$

$$\begin{cases} a_i \in \tilde{M}^{t_1}(\Omega), \text{ where } t_1 \geq p,\, t_1 \geq n,\, t_1 > p \text{ if } p = n, \\ a \in \tilde{M}^{t_2}(\Omega)\,, \text{ where } t_2 \geq p,\, t_2 \geq n/2,\, t_2 > p \text{ if } p = n/2\,, \end{cases} \tag{h_2}$$

there exist functions α_{ij}, $i, j = 1, \dots, n$, g and $\mu \in \mathbf{R}_+$ such that

$$\begin{cases} \alpha_{ij} = \alpha_{ji} \in L^\infty(\Omega) \cap VMO(\Omega)\,, \quad i, j = 1, \dots, n\,, \\ \displaystyle\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \ \text{ a.e. in } \Omega, \ \forall \xi \in \mathbf{R}^n\,, \\ g \in L^\infty(\Omega), \quad \displaystyle\lim_{r\to+\infty} \sum_{i,j=1}^{n} ||\alpha_{ij} - g\, a_{ij}||_{L^\infty(\Omega\setminus B_r)} = 0. \end{cases} \tag{h_3}$$

Observe that, under the assumptions (h_1) and (h_2), the operator

$$L : W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is bounded (see for instance [4], Theorem 3.2).

Put

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad A = -\sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.2}$$

Moreover, if $r \in \mathbf{R}_+$ and ζ_r is a function of class $C_0^\infty(\mathbf{R}^n)$ such that

$$0 \leq \zeta_r \leq 1, \quad \zeta_{r|B_r} = 1, \quad \operatorname{supp} \zeta_r \subset B_{2r}\,, \tag{3.3}$$

we put also

$$\begin{cases} h_{ij}^r = \zeta_{2r} a_{ij} + (1-\zeta_{2r})\alpha_{ij}, \quad i,j = 1,\ldots,n\,, \\ H^r = -\displaystyle\sum_{i,j=1}^{n} h_{ij}^r \frac{\partial^2}{\partial x_i \partial x_j}\,. \end{cases} \tag{3.4}$$

For each $r \in \mathbf{R}_+$ the coefficients h_{ij}^r are symmetric, elliptic (with ellipticity constant $\min\{\nu,\mu\}$) and belong to $L^\infty(\Omega) \cap VMO(\Omega)$.

We can now prove the following result.

Theorem 3.1. *Assume that the conditions $(h_1), (h_2), (h_3)$ hold. Then there exist positive real numbers r_0, c such that*

$$||u||_{W^{2,p}(\Omega)} \leq c\left(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}\right), \ \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \tag{3.5}$$

where c depends on $n, p, t_1, t_2, \Omega, \nu, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||\alpha_{ij}||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \eta[\alpha_{ij}], \tilde{\sigma}_{t_1}[a_i], \tilde{\sigma}_{t_2}[a]$.

Proof. Fix $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. It follows from (3.3), (3.4) and from Theorem 5.1 of [4] that for each $r \in \mathbf{R}_+$ there exists $c_1(r) \in \mathbf{R}_+$, dependent on $n, p, \Omega, \nu, ||a_{ij}||_{L^\infty(\Omega)}, \eta[\zeta_{2r} a_{ij}]$, such that

$$\begin{aligned} ||\zeta_r u||_{W^{2,p}(\Omega)} &\leq c_1(r)\big(||H^r(\zeta_r u)||_{L^p(\Omega)} + ||\zeta_r u||_{L^p(\Omega)}\big) \\ &= c_1(r)\big(||L_0(\zeta_r u)||_{L^p(\Omega)} + ||\zeta_r u||_{L^p(\Omega)}\big). \end{aligned} \tag{3.6}$$

A second application of Theorem 5.1 of [4] yields that there exists another constant $c_2 \in \mathbf{R}_+$, depending on $n, p, \Omega, \mu, ||\alpha_{ij}||_{L^\infty(\Omega)}, \eta[\alpha_{ij}]$, such that

$$||(1-\zeta_r)u||_{W^{2,p}(\Omega)} \tag{3.7}$$

$$\leq c_2\big(||A((1-\zeta_r)u)||_{L^p(\Omega)} + ||(1-\zeta_r)u||_{L^p(\Omega)}\big)$$
$$\leq c_2\big(||(A-gL_0)((1-\zeta_r)u)||_{L^p(\Omega)}$$
$$+||g||_{L^\infty(\Omega)}||L_0((1-\zeta_r)u)||_{L^p(\Omega)} + ||(1-\zeta_r)u||_{L^p(\Omega)}\big).$$

Thus by (h_3) we have that there exist $r_0, c_3 \in \mathbf{R}_+$, depending on the same parameters as c_2, such that

$$||(1-\zeta_{r_0})u||_{W^{2,p}(\Omega)} \tag{3.8}$$
$$\leq c_3\big(||g||_{L^\infty(\Omega)}||L_0((1-\zeta_{r_0})u)||_{L^p(\Omega)} + ||(1-\zeta_{r_0})u||_{L^p(\Omega)}\big).$$

Therefore it follows from (3.6) and (3.8) that

$$||u||_{W^{2,p}(\Omega)} \leq ||\zeta_{r_0}u||_{W^{2,p}(\Omega)} + ||(1-\zeta_{r_0})u||_{W^{2,p}(\Omega)} \tag{3.9}$$
$$\leq c_1(r_0)\big(||L_0(\zeta_{r_0}u)||_{L^p(\Omega)} + ||\zeta_{r_0}u||_{L^p(\Omega)}\big)$$
$$+c_3\big(||g||_{L^\infty(\Omega)}||L_0((1-\zeta_{r_0})u)||_{L^p(\Omega)} + ||(1-\zeta_{r_0})u||_{L^p(\Omega)}\big)$$
$$\leq c_4\big(||L_0u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||u_x||_{L^p(\Omega)}\big),$$

where $c_4 \in \mathbf{R}_+$ depends on $n, p, \Omega, \nu, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||\alpha_{ij}||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}], \eta[\alpha_{ij}]$. Moreover, by (h_2) we obtain that for each $\varepsilon \in \mathbf{R}_+$ there exists $c(\varepsilon) \in \mathbf{R}_+$ such that

$$||\sum_{i=1}^n a_iu_{x_i} + au||_{L^p(\Omega)} \leq \varepsilon||u||_{W^{2,p}(\Omega)} + c(\varepsilon)\big(||u_x||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}\big), \tag{3.10}$$

with $c(\varepsilon)$ depending on $n, p, t_1, t_2, \Omega, \tilde{\sigma}_{t_1}[a_i], \tilde{\sigma}_{t_2}[a]$ (see [4], Corollary 3.3). Using now (3.9) and (3.10), we have

$$||u||_{W^{2,p}(\Omega)} \leq c_5\big(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} + ||u_x||_{L^p(\Omega)}\big), \tag{3.11}$$

for some $c_5 \in \mathbf{R}_+$ depending on $n, p, t_1, t_2, \Omega, \nu, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||\alpha_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}, \eta[\zeta_{2r_0}a_{ij}], \eta[\alpha_{ij}], \tilde{\sigma}_{t_1}[a_i], \tilde{\sigma}_{t_2}[a]$. On the other hand, there exists $K = K(p,\Omega) \in \mathbf{R}_+$ such that

$$||u_x||_{L^p(\Omega)} \leq K\varepsilon||u||_{W^{2,p}(\Omega)} + \frac{K}{\varepsilon}||u||_{L^p(\Omega)}\,, \tag{3.12}$$

for any $\varepsilon \in]0,1]$ (see for instance [1], Theorem 4.14). Therefore the statement follows from (3.11) and (3.12). □

4. Some Regularity Results

The aim of this section is to prove two regularity results, one of local type and the other of global type. For the local regularity case, we will assume only the condition (h_1) on the leading coefficients, while for the coefficients of lower order the condition (h_2) will be replaced by the assumption

$$\begin{cases} a_i \in L^{t_1}_{\mathrm{loc}}(\bar{\Omega}), \text{ where } t_1 \geq p,\ t_1 > n\,, \\ a \in L^{t_2}_{\mathrm{loc}}(\bar{\Omega})\,, \text{ where } t_2 \geq p,\ t_2 > n/2\,, \end{cases} \tag{h_2'}$$

in other words, we will assume in this case a higher summability but only of local type. The following result can be proved.

Lemma 4.1. *Suppose that the assumptions $(h_1), (h_2')$ hold, and let u be a solution of the problem*

$$\begin{cases} u \in W^{2,q}_{loc}(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,q}_{loc}(\bar{\Omega}), \\ Lu \in L^{p}_{loc}(\bar{\Omega}), \end{cases} \tag{$\mathcal{R}_1$}$$

where $q \in]1, p]$. Then u belongs to $W^{2,p}_{loc}(\bar{\Omega})$.

Proof. It can obviously be assumed that $q < p$, and it is enough to prove that $\zeta_r u \in W^{2,p}(\Omega)$ for each $r \in \mathbf{R}_+$. Let k be the positive integer such that

$$k - 1 < n\Big(\frac{1}{q} - \frac{1}{p}\Big) \leq k\,, \tag{4.1}$$

so that

$$\frac{1}{q} - \frac{h}{n} > \frac{1}{p}\,,\quad h = 0, \dots, k-1\,,\qquad \frac{1}{q} - \frac{k}{n} \leq \frac{1}{p}\,. \tag{4.2}$$

If we put

$$\frac{1}{q_h} = \frac{1}{q} - \frac{h}{n}\,,\quad h = 0, \dots, k-1\,,\qquad \frac{1}{q_k} = \frac{1}{p}\,, \tag{4.3}$$

it is enough to show that

$$\zeta_r u \in W^{2,q_h}(\Omega) \Longrightarrow \zeta_r u \in W^{2,q_{h+1}}(\Omega) \tag{4.4}$$

for all $r \in \mathbf{R}_+$ and $h = 0, \dots, k-1$. Note first that

$$\zeta_r u \in W^{2,q_h}(\Omega) \cap \overset{\circ}{W}{}^{1,q_h}(\Omega)\,. \tag{4.5}$$

Moreover, if

$$B^r = \zeta_{2r} L + (1 - \zeta_{2r})\Delta\,, \tag{4.6}$$

it follows from (3.3) that

$$B^r(\zeta_r u) = L(\zeta_r u) = \zeta_r Lu - 2\sum_{i,j=1}^{n} a_{ij}(\zeta_r)_{x_j} u_{x_i} \tag{4.7}$$

$$+\Big(-\sum_{i,j=1}^{n} a_{ij}(\zeta_r)_{x_i x_j} + \sum_{i=1}^{n} a_i(\zeta_r)_{x_i}\Big)u = \zeta_r Lu + \sum_{i=1}^{n}\alpha_i(\zeta_{2r}u)_{x_i} + \alpha\zeta_{2r}u\,,$$

where

$$\alpha_i = -2\sum_{j=1}^{n} a_{ij}(\zeta_r)_{x_j}\,, \quad i = 1,\dots,n\,, \tag{4.8}$$

and

$$\alpha = -\sum_{i,j=1}^{n} a_{ij}(\zeta_r)_{x_i x_j} + \sum_{i=1}^{n} a_i(\zeta_r)_{x_i}. \tag{4.9}$$

Since $\zeta_r Lu$ belongs to $L^p(\Omega)$, we have also

$$\zeta_r Lu \,\in L^{q_{h+1}}(\Omega); \tag{4.10}$$

on the other hand, $(\zeta_{2r}u)_{x_i}$ lies in $W^{1,q_h}(\Omega)$, and hence it follows from (4.2), (4.3) and from the Sobolev Embedding Theorem that

$$\alpha_i(\zeta_{2r}u)_{x_i} \,\in\, L^{q_{h+1}}(\Omega)\,, \quad i = 1,\dots,n\,. \tag{4.11}$$

Finally, as $\alpha \in L^{t_1}(\Omega)$ and $\zeta_{2r}u \in W^{2,q_h}(\Omega)$, we may deduce from (4.2), (4.3) and from Theorem 3.2 of [4] that

$$\alpha\zeta_{2r}u \,\in\, L^{q_{h+1}}(\Omega)\,. \tag{4.12}$$

Therefore application of (4.7), (4.10), (4.11) and (4.12) yields that

$$B^r(\zeta_r u) \,\in\, L^{q_{h+1}}(\Omega)\,. \tag{4.13}$$

It follows from (4.5), (4.13), from the assumptions $(h_1), (h_2')$ and from Lemma 4.2 of [5] that $\zeta_r u$ belongs to $W^{2,q_{h+1}}(\Omega)$. The proof is complete. □

We prove now the following regularity result of global type.

Lemma 4.2. *Suppose that the conditions $(h_1), (h_2)$ (with $t_1 > n$ and $t_2 > n/2$) and (h_3) hold, and let u be a solution of the problem*

$$\begin{cases} u \in W^{2,q}_{loc}(\bar\Omega) \cap \overset{\circ}{W}{}^{1,q}_{loc}(\bar\Omega) \cap L^{q_0}(\Omega), \\ Lu \,\in\, L^p(\Omega), \end{cases} \tag{$\mathcal{R}_2$}$$

where $q \in]1,p]$ and $q_0 \in [1,p]$. Then u belongs to $W^{2,p}(\Omega)$.

Proof. By Lemma 4.1 we have

$$u \in W^{2,p}_{\text{loc}}(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,p}_{\text{loc}}(\bar{\Omega}) . \tag{4.14}$$

It follows from $(\mathcal{P}_\Omega)$ that there exists $d \in]0,1[$ such that for any $x \in \Omega$ either $B(x,d) \subset \Omega$ or $B(x,d) \subset U_i$ for some positive integer i. Consider $\rho, R \in \mathbf{R}_+$ with $\rho < R < d$ and a function $\Phi \in C^\infty_0(\mathbf{R}^n)$ such that

$$\begin{cases} 0 \leq \Phi \leq 1, \ \ \Phi_{|B_\rho} = 1 , \ \ \operatorname{supp} \Phi \subset B_R , \\ \sup_{\mathbf{R}^n} |D^\alpha \Phi| \leq c_\alpha (R-\rho)^{-|\alpha|} \quad \forall \alpha \in \mathbf{N}^n_0 , \end{cases}$$

where $c_\alpha \in \mathbf{R}_+$ depends only on α. For each $x \in \Omega$, consider the function

$$\psi : y \in \mathbf{R}^n \longmapsto \Phi(y-x) \in [0,1] ;$$

then

$$\begin{cases} \psi \in C^\infty_0(\mathbf{R}^n), \ \ \psi_{|B(x,\rho)} = 1 , \ \ \operatorname{supp} \psi \subset B(x,R) , \\ \sup_{\mathbf{R}^n} |D^\alpha \psi| \leq c_\alpha (R-\rho)^{-|\alpha|} \quad \forall \alpha \in \mathbf{N}^n_0 . \end{cases} \tag{4.15}$$

It follows from (4.14) that ψu belongs to $W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$, so that by Theorem 3.1 there exists $c \in \mathbf{R}_+$ such that

$$||\psi u||_{W^{2,p}(\Omega)} \leq c\big(||L(\psi u)||_{L^p(\Omega)} + ||\psi u||_{L^p(\Omega)}\big) , \tag{4.16}$$

with c independent of ψ and u. The statement follows now from (4.16) using the same argument of the proof of Theorem 5.1 of [4]. □

5. Main Results

In this section we will prove our uniqueness results.

Theorem 5.1. *Suppose that the assumptions* $(h_1), (h'_2)$ *are satisfied, and that also the condition*

$$\exists \, a_0 \in \mathbf{R}_+ \ \textit{such that} \ a \geq a_0 \ \textit{a.e. in} \ \Omega \tag{5.1}$$

holds. Then the solution of the problem

$$\begin{cases} u \in W^{2,p}_{loc}(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,p}_{loc}(\bar{\Omega}) , \\ \lim\limits_{|x| \to +\infty} u(x) = 0 , \\ Lu = 0 , \end{cases} \tag{$\mathcal{D}_1$}$$

is zero in Ω*.*

Proof. Assume first that $p > n/2$. Then by the Sobolev embedding theorem we have that u belongs to $C^0(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,p}_{\text{loc}}(\bar{\Omega})$, and hence $u_{|\partial\Omega} = 0$. In this case the statement follows from Corollary 4.3 of [3].

Suppose now that $p \in]1, n/2]$, so that by Lemma 4.1 u belongs to $W^{2,p'}_{\text{loc}}(\bar{\Omega})$ with $p' = \min\{t_1, t_2\}$. It follows that u is a solution of the problem $(\mathcal{D}_1)$ with p' instead of p. Since $p' > n/2$, the previous case can be applied to complete the proof. □

Theorem 5.2. *Suppose that the assumptions* $(h_1), (h_2')$ *and (5.1) hold, and if* $p \le n$ *suppose that also the conditions* (h_2) *(with* $t_1 > n$, $t_2 > n/2$*) and* (h_3) *are satisfied. Then the solution of the problem*

$$\begin{cases} u \in W^{2,p}_{loc}(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ Lu = 0\,, \end{cases} \qquad (\mathcal{D}_2)$$

is zero in Ω.

Proof. Suppose first that $p > n$, so that $\lim_{|x| \to +\infty} u(x) = 0$. Then u is also a solution of $(\mathcal{D}_1)$, and hence it is zero in Ω by Theorem 5.1.

Assume now that $p \in]n/2, n]$, so that by the Sobolev Embedding Theorem u belongs to $C^0(\bar{\Omega}) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$ and hence $u_{|\partial\Omega} = 0$. On the other hand, it follows from Lemma 4.2 that $u \in W^{2,p}(\Omega)$, so that again the Sobolev Embedding Theorem yields that u lies in $W^{1,\tilde{p}}(\Omega)$, where $\frac{1}{\tilde{p}} \in \left[\frac{1}{p} - \frac{1}{n}, \frac{1}{n}\right[$. Thus $\lim_{|x| \to +\infty} u(x) = 0$, and the statement follows in this case by Corollary 4.3 of [3].

Suppose finally that $p \in]1, n/2]$, so that by Lemma 4.2 the function u belongs to $W^{2,p'}(\Omega)$ with $p' = \min\{t_1, t_2\}$, and hence $u \in W^{2,p'}(\Omega) \cap \overset{\circ}{W}{}^{1,p'}(\Omega)$. As $p' > n/2$, the previous cases can be used to complete the proof. □

6. An Example

The aim of this short section is to construct an example of functions a_{ij} which satisfy the conditions (h_1) and (h_3), but do not belong to $VMO(\Omega)$.

Consider a function $\gamma \in C^0(]-1, 1[)$ for which the following conditions hold:

$$\gamma(-x) = -\gamma(x) \qquad \forall x \in]-1, 1[, \tag{6.1}$$

$$\exists\, \alpha \in \mathbf{R}_+ \quad \text{such that} \quad \lim_{x \to 0^+} \frac{|\gamma(x)|}{x^\alpha} = \ell \in \mathbf{R}_+. \tag{6.2}$$

For each $r \in \mathbf{R}_+$, define γ_r by the position

$$\gamma_r(x) = r^\alpha \gamma(x-r) \quad \text{for} \quad |x-r| < 1\,.$$

It follows from (6.1) and (6.2) that there exist c_0, $t_0 \in]0,1]$ such that

$$I_{\gamma_r}(r,t) = \fint_{r-t}^{r+t} |\gamma_r(x) - \fint_{r-t}^{r+t} \gamma_r(y)\,dy|\,dx = \frac{r^\alpha}{t}\int_0^t |\gamma(z)|\,dz \geq\ c_0 r^\alpha t^\alpha\,,$$

for any $t \leq t_0$. Thus, if we choose $r \geq 1/t_0$, we obtain that

$$I_{\gamma_r}(r,1/r)\ \geq\ c_0\,. \tag{6.3}$$

Consider now a function $f \in C^0(\bar{\mathbf{R}}_+) \cap L^\infty(\mathbf{R}_+)$ such that $f(x) = \gamma_k(x)$ for $x \in [k-1/k, k+1/k]$ and for any positive integer $k \geq 2$. Then, for k large enough, by (6.3) we have that $I_f(k,1/k) \geq c_0 > 0$. Therefore f is a function in $VMO_{\text{loc}}(\bar{\mathbf{R}}_+) \cap L^\infty(\mathbf{R}_+)$ which does not belong to $VMO(\mathbf{R}_+)$.

It easily follows that the function

$$\psi : x \in \bar{\mathbf{R}}^n_+ \mapsto f(x_n) \in \mathbf{R}\,,$$

lies in $VMO_{\text{loc}}(\bar{\mathbf{R}}^n_+) \cap L^\infty(\mathbf{R}^n_+)$ but it does not belong to $VMO(\mathbf{R}^n_+)$. The function ψ allows now to construct the a_{ij}'s with the required properties. In fact, choose $\Omega = \mathbf{R}^n_+$ and consider the functions a_{ij} defined by the position

$$a_{ij}(x) = \Big((5 - 2\sin|x|)(\psi_0 - \psi(x)) + \frac{\psi^2(x)}{1+|x|^2}\Big)\delta_{ij}\,, \quad i,j = 1,\dots,n\,,$$

where $\psi_0 > \sup\limits_{\mathbf{R}^n_+} |\psi(x)|$.

Put finally

$$g(x) = \frac{1}{\psi_0 - \psi(x)} \quad \text{and} \quad \alpha_{ij}(x) = (5 - 2\sin|x|)\delta_{ij}, \quad i,j = 1,\dots,n\,.$$

Easy computations prove that the functions a_{ij}'s satisfy (h_1) and (h_3), but they are not in $VMO(\mathbf{R}^n_+)$.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York (1975).

[2] A.D. Aleksandrov, Majorization of solutions of second-order linear equations, *Vestnik Leningrad. Univ.*, **21**, No.1 (1966), 5-25; English Translation in: *Amer. Math. Soc. Transl.*, **68** (1968), 120-143.

[3] L. Caso, P. Cavaliere, M. Transirico, On the maximum principle for elliptic operators, To Appear.

[4] P. Cavaliere, M. Longobardi, A. Vitolo, Imbedding estimates and elliptic equations with discontinuous coefficients in unbounded domains, *Matematiche (Catania)*, **51** (1996), 87-104.

[5] P. Cavaliere, M. Transirico, M. Troisi, Uniqueness result for elliptic equations in unbounded domains, *Matematiche (Catania)*, **54** (1999), 139–146.

[6] F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.*, **336** (1993), 841–853.

[7] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer, Berlin-Heidelberg (1983).

[8] C. Pucci, Limitazioni per soluzioni di equazioni ellittiche, *Ann. Mat. Pura Appl.*, **4**, No. 74 (1966), 15-30.

[9] M. Transirico, M. Troisi, A. Vitolo, Spaces of Morrey type and elliptic equations in divergence form on unbounded domains, *Boll. Un. Mat. Ital.*, **7**, No. 9B (1995), 153-174.

[10] M. Transirico, M. Troisi, A. Vitolo, BMO spaces on domains of $\mathbf{R}^n$, *Ricerche Mat.*, **45** (1996), 355-378.

[11] C. Vitanza, $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients, *Matematiche (Catania)*, **47** (1992), 177–186.

[12] C. Vitanza, A new contribution to the $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients, *Matematiche (Catania)*, **48** (1993), 287–296.