International Journal of Pure and Applied Mathematics

Volume 14 No. 2 2004, 189-198

EXACT SOLUTIONS TO THE (3+1)-DIMENSIONAL POTENTIAL KdV-ZAKHAROV-KUZNETSOV (P-KdV-Z-K) EQUATION

De-Sheng Li[1 §], Cheng-Xin Luo[2], Zhen-Sheng Yu[3], Feng Gao[4]

[1]Science School
Shenyang University of Technology
Shenyang, 110023, P.R. CHINA
e-mail: lds_6638@163.com

[2]Department of Mathematics
Shenyang Normal University
Shenyang, 110034, P.R. CHINA

[3,4]Department of Applied Mathematics
Dalian University of Technology
Dalian, 116024, P.R. CHINA

Abstract: In this paper, by combining Fan's extended tanh-function method with Gao's generalized tanh-function method, and with the aid of symbolic computation system *Maple*, we obtain a large number of new exact solutions of (3+1)-dimensional potential KdV-Z-K equation.

AMS Subject Classification: 35C05, 35L05, 35Q51
Key Words: (3+1)-dimensional potential KdV-Zakharov-Kuznetsov equation, extended tanh-function method, generalized hyperbolic-function method, symbolic computation system *Maple*, exact solutions

1. Introduction

It is well known that the closed-form solutions of nonlinear PDEs and ODEs facilitate greatly the testing of numerical solvers, and provide much help in the stability analysis of solutions.

Received: April 10, 2004

[§]Correspondence author

There is a variety of special methods to find solitary wave solutions of nonlinear PDEs and ODEs. Among the more algorithmic techniques are the Hirota method [16], the real exponential method [13], [15], and the singular manifold method [5], [6], [7], [11], [14]. Some of these direct methods are straightforward to implement in any computer algebra system.

More comprehensive methods to find exact solutions of PDEs and ODEs are chiefly based on similarity reductions via Lie point symmetry methods, see [17], [12] and the references cited therein, for example. It should be pointed out that it is the studies on the similar solutions generated in the course of similarity reductions via Lie point symmetry methods that the above mentioned direct methods to find exact solutions of PDEs and ODEs are obtained and developed.

Travelling wave solutions of many nonlinear PDEs, ODEs from soliton theory (and beyond) can be expressed as polynomials of the hyperbolic tangent and secant functions. One explanation is given in Hereman et al. [16] and Hereman and Takaoka [15]. The tanh method provides a straightforward algorithm to compute such particular solutions for a large class of nonlinear PDEs. See Malfliet [18], Malfliet and Hereman [19], and Das and Sarma [3] for a multitude of references to tanh-based techniques and various applications. Recently, the above mentioned tanh method has been improved and extended further, such as Fan's extended tanh-function method and Gao's generalized hyperbolic-function method, see [8], [9], [10] for more details.

In this paper, we combine Fan's and Gao's methods, and apply it to the (3+1)-dimensional Potential KdV-Zakharov-Kuznetsov (P-KdV-Z-K) equations. Some new exact solutions are derived.

Maple, V. 7 offers a package *PDEtools*, which contains the functions *pdesolve* to find exact solutions of classes of linear and nonlinear PDEs. For more information consult Cheb-Terrab [2]-[1]. The methods presented in this paper are dependent on these efforts.

2. New Solutions

The KdV-Zakharov-Kuznetsov (KdV-Z-K) equation takes the following form [4]:

$$u_t + \alpha u u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0. \tag{1}$$

We set

$$u = w_x, \tag{2}$$

Inserting equation (2) in equation (1), we have

$$w_{xt} + \alpha w_x w_{xx} + w_{xxxx} + w_{xxyy} + w_{xxzz} = 0. \tag{3}$$

Equation (3) is called the (3+1)-dimensional Potential KdV-Zakharov-Kuznetsov (P-KdV-Z-K) equation. It is equivalent to

$$w_t + \frac{\alpha}{2} w_x^2 + w_{xxx} + w_{xyy} + w_{xzz} = h(y,z,t), \tag{4}$$

where $h(y,z,t)$ is an arbitrary function of the indicated variables. To seek for the solutions of equation (4), according to Fan's and Gao's methods, we assume that

$$w(x,y,z,t) = f(y,z,t) + g(y,z,t)\phi(\xi), \tag{5}$$

$$\xi = p(y,z,t)x + q(y,z,t), \tag{6}$$

and

$$\phi' = \delta + \phi^2, \tag{7}$$

where f, g, p, q are functions to be determined, $\phi = \phi(\xi), \phi' = \frac{d\phi}{d\xi}$.

The Riccati equation (7) has the general solutions

$$\begin{cases} \phi &= -\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \quad \delta < 0, \\ \phi &= -\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \quad \delta > 0, \\ \phi &= -\frac{1}{\xi}, \quad \delta = 0, \\ \phi &= \sqrt{\delta}\tan\sqrt{\delta}\xi, \quad \delta > 0, \\ \phi &= -\sqrt{\delta}\cot\sqrt{\delta}\xi, \quad \delta > 0. \end{cases} \tag{8}$$

Substituting (5)-(6) into (4), collecting the coefficients of ϕ^i and setting them to zero, we derive the following system:

$$\begin{cases} 2\delta^2 gp[p_z^2 + p_y^2] = 0, \quad 8\delta gp[p_z^2 + p_y^2] = 0, \quad 6gp[p_z^2 + p_y^2] = 0, \\ \delta g[4\delta pp_z q_z + 4\delta pp_y q_y + p_t] = 0, \\ g[16\delta pp_y q_y + p_t + 16\delta pp_z q_z] = 0, \\ \frac{1}{2} gp[\alpha pg + 12p^2 + 12q_z^2 + 12q_y^2] = 0, \quad 12gp[p_y q_y + p_z q_z] = 0, \\ 2g_y p_y \delta - h(y,z,t) + f_t + gp_{zz}\delta + \frac{1}{2}\alpha g^2 p^2 \delta^2 \\ +2g_z p_z \delta + 2gp\delta^2 q_y^2 + p\delta g_{yy} + g\delta q_t + 2gp\delta^2 q_z^2 \\ \qquad + gp_{yy}\delta + p\delta g_{zz} + 2gp^3\delta^2 = 0, \\ 4g_z p\delta q_z + g_t + 2gp\delta q_{yy} + 2gp\delta q_{zz} + 4g\delta p_z q_z \\ \qquad + 4p\delta g_y q_y + 4g\delta p_y q_y = 0, \\ 2\delta[gpp_{yy} + 2gp_z^2 + 2pg_y p_y + 2gp_y^2 + 2pg_z p_z + gpp_{zz}] = 0, \\ gp_{yy} + 8gp\delta q_y^2 + gq_t + gp_{zz} + 2g_z p_z + 8gp^3\delta + \alpha g^2 p^2 \delta + pg_{zz} \\ \qquad + pg_{yy} + 2g_y p_y + 8g\delta pq_z^2 = 0, \\ 4gp_y q_y + 2gpq_{zz} + 4gp_z q_z + 4pg_y q_y + 2gpq_{yy} + 4pg g_z q_z = 0, \\ 2gpp_{yy} + 4g_y pp_y + 4gp_y^2 + 4gp_z^2 + 4pg_z p_z + 2gpp_{zz} = 0\,. \end{cases} \tag{9}$$

From (9) we conclude that

$$p(y,z,t) = p(t). \tag{10}$$

Substituting (10) into (9), we have

$$\begin{cases} \frac{1}{2}gp[\alpha pg + 12p^2 + 12q_y^2 + 12q_z^2] = 0, \\ 2p[gq_{yy} + gq_{zz} + 2g_zq_z + 2g_yq_y] = 0, \\ 2gp\delta q_{zz} + g_t + 4p\delta g_zq_z + 4p\delta g_yq_y + 2gp\delta q_{yy} = 0, \quad gp_t = 0, \\ -h(y,z,t) + f_t + pg_{zz}\delta + 2gp\delta^2q_y^2 + p\delta g_{yy} + \frac{1}{2}\alpha g^2p^2\delta^2 \\ \qquad\qquad + 2gp\delta^2q_z^2 + g\delta gq_t + 2gp^3\delta^2 = 0, \\ gq_t + \alpha g^2p^2\delta + 8gp\delta q_y^2 + pg_{zz} + pg_{yy} + 8gp\delta q_z^2 + 8gp^3\delta = 0\,. \end{cases} \tag{11}$$

It follows from (11) that $p(y,z,t)$ equals to constant. At this stage, we consider

$$f(y,z,t) = f(y,t), \quad g(y,z,t) = g(y,t). \tag{12}$$

Inserting equation (12) in (11) leads to

$$\begin{cases} 2p[gq_{yy} + gq_{zz} + 2g_yq_y] = 0, \\ \frac{1}{2}gp[\alpha pg + 12p^2 + 12q_y^2 + 12q_z^2] = 0, \\ 2gp\delta q_{zz} + g_t + 4p\delta g_yq_y + 2gp\delta q_{yy} = 0, \\ 2gp^3(\delta)^2 - h(y,t) + f_t + \frac{1}{2}\alpha g^2p^2(\delta)^2 + p\delta g_{yy} \\ \qquad\qquad + 2gp(\delta)^2q_y^2 + 2gp(\delta)^2q_z^2 + g\delta q_t = 0, \\ 8gp\delta q_y^2 + 8gp^3\delta + gq_t + \alpha g^2p^2\delta + pg_{yy} + 8gp\delta q_z^2 = 0. \end{cases} \tag{13}$$

With the aid of symbolic computation, we obtain the solutions of (13):

$$\begin{cases} f = \int h(y,t)dt + F_1(y), \quad g = -12\dfrac{C_2^2 + p^2}{p\alpha}, \\ q = [4pC_2^2t + 4p^3t]\delta + C_2z + C_1; \end{cases} \tag{14}$$

$$\begin{cases} f = \int h(y,t)dt + F_1(y), \quad g = -12\dfrac{p}{\alpha}, \\ q = (-iy + z)C_3 + 4p^3\delta t + C_1 + C_2; \end{cases} \tag{15}$$

$$\begin{cases} f = \int h(y,t)dt + F_1(y), \quad g = -3\dfrac{C_1}{p^2\delta\alpha}, \\ q = \dfrac{1}{2}\dfrac{\sqrt{p\delta(-4p^3\delta - 4C_2^2p\delta + C_1)}z + 2p\delta(C_1t + C_2y + C_3)}{p\delta}. \end{cases} \tag{16}$$

Similarly, taking $p =$ constant and

$$f(y,z,t) = f(z,t), \quad g(y,z,t) = g(z,t). \tag{17}$$

Inserting equation (17) in (11), we derive

$$\begin{cases} 2p[gq_{yy}+gq_{zz}+2g_zq_z]=0, \quad \dfrac{1}{2}gp[\alpha pg+12p^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad +12q_y^2+12q_z^2]=0, \\ 2gp\delta q_{zz}+g_t+4p\delta g_zq_z+2gp\delta q_{yy}=0, \\ 2gp^3\delta^2-h(z,t)+f_t+\dfrac{1}{2}\alpha g^2p^2\delta^2q_t+g\delta q_t+p\delta g_{zz} \\ +2gp\delta^2q_z^2+2gp\delta^2q_y^2+g\delta q_t+p\delta g_{zz}=0, \\ 8gp\delta q_y^2+8gp\delta q_z^2+\alpha g^2p^2\delta+8gp^3\delta+gq_t+pg_{zz}=0. \end{cases} \tag{18}$$

With the aid of symbolic computation system *Maple*, we obtain the following solutions of (18):

$$\begin{cases} f=\int h(z,t)dt+F_1(z), \quad g=-12\dfrac{C_2^2+p^2}{p\alpha}, \\ q=(4pC_2^2t+4p^3t)\delta+C_2y+C_1; \end{cases} \tag{19}$$

$$\begin{cases} f=\int h(z,t)dt+F_1(z), \quad g=-12\dfrac{p}{\alpha}, \\ q=(-iy+z)C_3+4p^3\delta t+C_1+C_2; \end{cases} \tag{20}$$

$$\begin{cases} f=\int h(z,t)dt+F_1(z), \\ g=-\dfrac{4}{5}\dfrac{\sqrt{-11p^4+28C_1^2p^2+64C_1^4}+13p^2+8pC_1^2}{p\alpha}, \\ q=C_1y-\dfrac{1}{15}\sqrt{-30p^2-105C_1^2+15\sqrt{-11p^4+28C_1^2p^2+64C_1^4}}z, \\ +\dfrac{4}{15}pt\delta\sqrt{-11p^4+28C_1^2p^2+64C_1^4}+\dfrac{1}{15}(52p^3+32pC_1^2)t\delta+C_2, \end{cases} \tag{21}$$

$$\begin{cases} f=\int h(z,t)dt+F_1(z), g=-3\dfrac{C_1}{p^2\alpha\delta}, \\ q=\dfrac{1}{2}\dfrac{\sqrt{p\delta(-4p^3\delta+C_1)}y+2p\delta(C_1t+C_2)}{p\delta}. \end{cases} \tag{22}$$

Substituting (14)-(16) and (19)-(22) into (5) and using (8), we derive the following solutions of equation (3):

$$\begin{cases} w_{11} &= \int h(y,t)dt + F_1(y) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0,\\ w_{12} &= \int h(y,t)dt + F_1(y) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0,\\ w_{13} &= \int h(y,t)dt + F_1(y) + 12\dfrac{C_2^2+p^2}{p\alpha\xi}, \delta=0,\\ w_{14} &= \int h(y,t)dt + F_1(y) - 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0,\\ w_{15} &= \int h(y,t)dt + F_1(y) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta>0.\\ \xi &= px + [4pC_2^2t + 4p^3t]\delta + C_2z + C_1; \end{cases}$$

$$\begin{cases} w_{21} &= \int h(y,t)dt + F_1(y) + 12\dfrac{p}{\alpha}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0,\\ w_{22} &= \int h(y,t)dt + F_1(y) + 12\dfrac{p}{\alpha}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0,\\ w_{23} &= \int h(y,t)dt + F_1(y) + 12\dfrac{p}{\xi\alpha}, \delta=0,\\ w_{24} &= \int h(y,t)dt + F_1(y) - 12\dfrac{p}{\alpha}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0,\\ w_{25} &= \int h(y,t)dt + F_1(y) + 12\dfrac{p}{\alpha}\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta>0,\\ \xi &= px + (-iy+z)c_2 + 4p^3\delta t + C_1 + C_2; \end{cases}$$

$$\begin{cases} w_{31} = \int h(y,t)dt + F_1(y) + 3\dfrac{C_1}{p^2\delta\alpha}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0,\\ w_{32} = \int h(y,t)dt + F_1(y) + 3\dfrac{C_1}{p^2\delta\alpha}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0,\\ w_{33} = \int h(y,t)dt + F_1(y) - 3\dfrac{C_1}{p^2\delta\alpha}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0,\\ w_{34} = \int h(y,t)dt + F_1(y) + 3\dfrac{C_1}{p^2\delta\alpha}\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta>0,\\ \xi = px + \dfrac{1}{2}\dfrac{\sqrt{p\delta(-4p^3\delta - 4C_2^2p\delta + C_1)}z + 2p\delta(C_1t + C_2y + C_3)}{p\delta}. \end{cases}$$

$$\begin{cases} w_{41} &= \int h(z,t)dt + F_1(z) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0, \\ w_{42} &= \int h(z,t)dt + F_1(z) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0, \\ w_{43} &= \int h(z,t)dt + F_1(z) + 12\dfrac{C_2^2+p^2}{p\alpha\xi}, \delta=0, \\ w_{44} &= \int h(z,t)dt + F_1(z) - 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0, \\ w_{45} &= \int h(z,t)dt + F_1(z) + 12\dfrac{C_2^2+p^2}{p\alpha}\sqrt{-\delta}\cot\sqrt{-\delta}\xi, \delta>0, \\ \xi &= px + [4pC_2^2t + 4p^3t]\delta + C_2y + C_1; \end{cases}$$

$$\begin{cases} w_{51} &= \int h(z,t)dt + F_1(z) + 12\dfrac{p}{\alpha}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0, \\ w_{52} &= \int h(z,t)dt + F_1(z) + 12\dfrac{p}{\alpha}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0, \\ w_{53} &= \int h(z,t)dt + F_1(z) + 12\dfrac{p}{\alpha\xi}, \delta=0, \\ w_{54} &= \int h(z,t)dt + F_1(z) - 12\dfrac{p}{\alpha}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0, \\ w_{55} &= \int h(z,t)dt + F_1(z) + 12\dfrac{p}{\alpha}\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta>0, \\ \xi &= px + (-iy+z)C_3 + 4p^3\delta t + C_1 + C_2; \end{cases}$$

$$\begin{cases} w_{61} &= \int h(z,t)dt + F_1(z) + g\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta<0, \\ w_{62} &= \int h(z,t)dt + F_1(z) + g\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta<0, \\ w_{63} &= \int h(z,t)dt + F_1(z) + g\dfrac{1}{\xi}, \delta=0, \\ w_{64} &= \int h(z,t)dt + F_1(z) - g\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta>0, \\ w_{65} &= \int h(z,t)dt + F_1(z) + g\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta>0, \end{cases}$$

$$\begin{cases} g = -\dfrac{4}{5}\dfrac{\sqrt{-11p^4+28C_1^2p^2+64C_1^4}+13p^2+8pC_1^2}{p\alpha}, \\ \xi = px + C_1y \\ \qquad -\dfrac{1}{15}\sqrt{-30p^2-105C_1^2+15\sqrt{-11p^4+28C_1^2p^2+64C_1^4}}z \\ +\dfrac{4}{15}pt\delta\sqrt{-11p^4+28C_1^2p^2+64C_1^4} + \dfrac{1}{15}(52p^3+32pC_1^2)t\delta + C_2; \end{cases}$$

$$\begin{cases} w_{71} &= \int h(z,t)dt + F_1(z) + 3\dfrac{C_1}{p^2\alpha\delta}\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \delta < 0, \\ w_{72} &= \int h(z,t)dt + F_1(z) + 3\dfrac{C_1}{p^2\alpha\delta}\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \delta < 0, \\ w_{73} &= \int h(z,t)dt + F_1(z) - 3\dfrac{C_1}{p^2\alpha\delta}\sqrt{\delta}\tan\sqrt{\delta}\xi, \delta > 0, \\ w_{74} &= \int h(z,t)dt + F_1(z) + 3\dfrac{C_1}{p^2\alpha\delta}\sqrt{\delta}\cot\sqrt{\delta}\xi, \delta > 0, \\ \xi &= px + \dfrac{1}{2}\dfrac{\sqrt{p\delta(-4p^3\delta + C_1)}y + 2p\delta(C_1 t + C_2)}{p\delta}. \end{cases}$$

3. Conclusion and Discussion

In summary, in the present paper, we use a combined method of Fan's extended tanh-function method and Gao's generalized hyperbolic-function method to solve the (3+1)-dimensional Potential KdV-Zakharov-Kuznetsov (P-KdV-Z-K) equation. With the aid of symbolic computation system *Maple*, a large number of new exact solutions of the P-KdV-Z-K equation are obtained. It is known, the solutions obtained by Fan's method are only travelling wave solution, while ours can be non-travelling wave solution. In particular, when take the coefficient functions in the solutions to be constant, we derive Fan's solutions. On the other hand, while Gao's method can obtain non-travelling wave solution, his solution is only a special case of ours.

The method used in our paper can be also applied to other (2+1)-dimensional or (3+1)-dimensional nonlinear evolution equation(s).

Acknowledgements

This work is supported by the National Key Basic Research Development of China (Grant No. 1998030600) and the National Nature Science Foundation of China (Grant No. 10072013).

References

[1] E.S. Cheb-Terrab, *Private Communication. PDETools* Project, Center for Experimental and constructive Mathematics, Simon Fraser University, Vancouver, British Columbia, Canada (2001).

[2] E.S. Cheb-Terrab, K. Von Bulow, A computational approach for the analytical solving of systems of partial differential equations, *Comp. Phys. Comm.*, **90** (1995), 102-116.

[3] G.C. Das, J. Sarma, Response to "Comment on 'A new mathematical approach for finding the solitary waves in dusty plasmas'", *Phys. Plasmas*, **6** (1999), 4393-4397.

[4] K.P. Das, F. Verheest, Ion-acoustic solitons in magnetized multi-component plasmas including negative ions, *J. Plasma Phys.*, **41** (1989), 139-155.

[5] P.G. Estevez, P.G. Gordor, Non-classical symmetries and the singular manifold method: Theory and six Examples, *Stud. Appl. Math.*, **95** (1995), 73-113.

[6] P.G. Estevez, P.G. Gordor, Non-classical symmetries and the singular manifold method: a further two examples, *J. Phys. A: Math. Gen.*, **31** (1998), 7511-7519.

[7] P.G. Estevez, G.A. Hernaez, Non-isospectral problem in 2+1 dimensions, *J. Phys. A: Math. Gen.*, **33** (2000), 2131-2143.

[8] E.G. Fan, Extended tanh-function and its applications to nonlinear equations, *Phys. Lett. A.*, **277** (2000), 212-218.

[9] E.G. Fan, Travelling wave solutions for nonlinear equations using symbolic computation, *Comp. Math. Appl.,* **43** (2002), 671-680.

[10] Y.T. Gao, B. Tian, Generalized hyperbolic-function method with computerized symbolic computation to construct the solitonic to nonlinear equations of mathematical physics, *Comp. Phys. Comm.*, **133** (2001), 158-164.

[11] W. Hereman, Application of a Macsyma program for the Painleve test to the Fizhugh-Nagumo equation, In: *Partialy Integrable Evolution Equations in Physics, Proc. Winter School Theoretical Physics*, (Ed-s: R. Conte, N. Boccara) Dordrecht, Kluwer Academic Publishers (1990), 585-586.

[12] W. Hereman, Symbolic software for lie symmetry analysis, In: *CRC Handbook of Lie Group Analysis of Differential Equations*, Volume **3:** *New Trends in Theoretical Developments and Computations Methods* (Ed. N.H. Ibragimov), Chapter 13, Boca Raton, Florida, CRC Press (1966), 367-413.

[13] W. Hereman, P.P. Banerjee, A. Korpel, G. Assanto, A. Van Immerzeele, A. Meerpoel, Exact solitary wave solutions of non-linear evolution and wave equations using a direct algebraic method, *J. Phys. A: Math. Gen.*, **19** (1986), 607-628.

[14] W. Hereman, A. Nuseir, Symbolic methods to construct exact solutions of nonlinear partial differential equations, *Math. Comp. in Simul.*, **43** (1997), 13-27.

[15] W. Hereman, M. Takaoka, Solitary wave solutions of nonlinear evolution and wave equations using Macsyma, *J. Phys. A: Math. Gen.*, **23** (1990), 4805-4822.

[16] W. Hereman, W. Zhuang, Symbolic software for soliton theory, *Acta Appl. Math.*, **39** (1995), 361-378.

[17] D.K. Ludlow, P.A. Clarkson, A.P. Basssom, Similarity reductions and exact solutions of the two-dimensional incompressible Navier-Stokes equations, *Stud. appl. Math.*, **103** (1999), 183-240.

[18] W. Malfliet, Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.*, **60** (1992), 650-654.

[19] W. Malfliet, W. Hereman, The tanh method. I exact solutions of nonlinear evolution and wave equations, *Phys. Scripta*, **54** (1996), 563-568.