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RECIPROCAL SOLUTION OF A QUARTIC EQUATION

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Abstract: Reciprocal type of a solution of a quartic equation is given through a cubic equation, with zero-measure exception leading to a compound quadratic equation or a product of quadratic equations, the condition of the exception being expressed explicitly in terms of the coefficients of the equation.

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1. Introduction

Historically quartic equations were solved algebraically by many persons such as Ferrari in [1], Euler, Lagrange, Christianson [2], Neumark [3], and Yacoub and Fraidenraich in [4], using a decomposed cubic equation. Here is reciprocal solution except for cases leading to a substantially quadratic equation is presented (measure of zero for a set of coefficients).

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2. Analysis

2.1. General

Let a quartic equation be expressed as

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0\,. \tag{1}$$

The quantity $A^3 - 4AB + 8C$ corresponding to the polynomial $x^4 + Ax^3 + Bx^2 + Cx + D$ is an invariant with respect to the shift (parallel translation) $x \to x + \text{constant}$. In case of $A^3 - 4AB + 8C \neq 0$, let β be one of the roots of

$$\begin{aligned}&\left(4\beta^3 + 3A\beta^2 + 2B\beta + C\right)^2 \\ &= (A + 4\beta)^2 \left(\beta^4 + A\beta^3 + B\beta^2 + C\beta + D\right)\ ,\end{aligned} \tag{2}$$

which is identical to

$$\begin{aligned}&\left(A^3 - 4AB + 8C\right)\beta^3 + \left(A^2B + 2AC - 4B^2 + 16D\right)\beta^2 \\ &+ \left(A^2C + 8AD - 4BC\right)\beta + A^2D - C^2 = 0\ .\end{aligned} \tag{3}$$

Equation (3) itself is the same as in [4] by Yacoub and Fraidenraich. Let w be Sylvester's resultant of $4y^3+3Ay^2+2By+C = 0$ and $y^4+Ay^3+By^2+Cy+D = 0$, that is,

$$w = \begin{vmatrix} 4 & 3A & 2B & C & 0 & 0 & 0 \\ 0 & 4 & 3A & 2B & C & 0 & 0 \\ 0 & 0 & 4 & 3A & 2B & C & 0 \\ 0 & 0 & 0 & 4 & 3A & 2B & C \\ 1 & A & B & C & D & 0 & 0 \\ 0 & 1 & A & B & C & D & 0 \\ 0 & 0 & 1 & A & B & C & D \end{vmatrix}\ . \tag{4}$$

2.2. In Case of $A^3 - 4AB + 8C \neq 0$ and $w \neq 0$

If $x = \alpha z+\beta$ ($\beta :$ an arbitrarily chosen root of the cubic equation (3)) and $\alpha^2 \equiv \left(4\beta^3 + 3A\beta^2 + 2B\beta + C\right)/\left(A + 4\beta\right)$, then equation (1) becomes a reciprocal equation with respect to z, since $\alpha \neq 0$, which gives

$$x = \beta + \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\alpha^2}\right)\ , \tag{5}$$

where T is any one of the root of the following quadratic equation:

$$T^2 + (A + 4\beta)\,T + \frac{16\beta^3 + 12A\beta^2 + 3A^2\beta + AB - 2C}{A + 4\beta} = 0\,. \tag{6}$$

Finally equation (5) gives four roots. In case that A, B, C, and D are all real, there exists at least one real root β in equation (3) such that

$$\left(A^3 - 4AB + 8C\right)(A + 4\beta) > 0\,. \tag{7}$$

For this value of β, equation (6) produces real roots.

2.3. In Case of $A^3 - 4AB + 8C \neq 0$ and $w = 0$

Among the roots β's of equation (3), it is possible to choose β such that $4\beta^3 + 3A\beta^2 + 2B\beta + C = 0$. If $x \equiv z + \beta$, then

$$z^4 + (A + 4\beta)\,z^3 + \left(B + 3A\beta + 6\beta^2\right) z^2 = 0\ , \tag{8}$$

which is not a reciprocal equation and becomes

$$z = 0 \text{ or } z^2 + (A + 4\beta)\,z + B + 3A\beta + 6\beta^2 = 0\ . \tag{9}$$

However, since equation (5) is analytic, equation (5) and equation (6) hold in the current case provided $\alpha \equiv 0$.

2.4. In Case of $A^3 - 4AB + 8C = 0$

Translation $x = z - A/4$ leads to a compound quadratic equation in z, which gives

$$x = -\frac{A}{4} + \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\alpha^2}\right)\ , \tag{10}$$

$$\alpha^2 \equiv \sqrt{\frac{5}{256}A^4 - \frac{1}{16}A^2B + D}\ , \tag{11}$$

$$T^2 \equiv 2\alpha^2 + \frac{3}{8}A^2 - B\ . \tag{12}$$

3. Conclusion

Reciprocal type of a solution of a quartic equation is given, with exception (zero measure) reducing to a compound qudratic equation or a product of quadratic equations.

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