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ON EQUATIONS INVOLVING THE NUMBER OF REDUCIBLE ELEMENTS IN Z_n

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Abstract: The purpose of this paper is to study equations involving two new number theoretic functions, the number of irreducible elements in Z_n and the number of reducible elements in Z_n. Complete characterizations of many of these equations are given using properties of these new functions while partial results for other equations are obtained.

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1. Introduction

Equations involving the well-known number theoretic functions, such as Euler's ϕ-function $\phi(n)$, the sum of divisors function $\sigma(n)$ and others, have been extensively studied. The problem of determining the perfect numbers, solutions to the equation $\sigma(n) = 2n$, dates back to antiquity. Since the turn of

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the twentieth century, numerous equations involving the function $\phi(n)$ can be found in the literature. The problem of determining all values of n such that $\phi(n)$ divides n, that is $n = k\phi(n)$, is a standard problem in an introductory course in number theory. In 1932, D.H. Lehmer introduced the equations $n = k\phi(n) + 1$ and $n = k\phi(n) - 1$, see [12]. Since then, many papers have been written regarding these equations, see [13] and [1]. The problems of determining all solutions to both equations are still open. In 1917 and 1918, Ratat [15] and Goormaghtigh [9] have recorded values of n satisfying the equation $\phi(n) = \phi(n+1)$. A list of all such values of $n < 3000$ was given by Klee [11]. Erdös, see [6], mentioned that the equation $\phi(n) = \phi(n+1) = \phi(n+2)$ has a solution, but did not give the solution, and conjectured that the equation $\phi(n) = \phi(n+1) = ... = \phi(n+k)$ is solvable for every positive integer k. In 1949, Moser [14] listed all solutions of $\phi(n) = \phi(n+1)$ with $n < 10000$. His list included five new solutions, the two largest being 5186 and 5187. The solution $n = 5186$ satisfies $\phi(n) = \phi(n+1) = \phi(n+2)$. In his paper, Moser also mentioned that the equation $\phi(n) = \phi(n+2)$, where n is a positive integer is satisfied by $n = 2(2p-1)$ whenever p and $2p-1$ are both primes, $n = 2^{2^a+1}$ whenever $2^{2^a}+1$ is a Fermat prime, and by some other solutions not of these forms. Recently, the second author obtained several classes of solutions to the equations $\phi(n) = \phi(n+2), k\phi(n) = \phi(n+1)$ and $k\phi(n+1) = \phi(n)$, see [3] and [5], using solutions of equations generalizing Lehmer's equations, see [4]. The purpose of this paper is to study equations involving Euler's $\phi-$function and two new number theoretic functions, $\psi(n)$ and $\rho(n)$, introduced by El-Kassar and Kojok in [2], where $\psi(n)$ is the number of irreducible elements in Z_n and $\rho(n)$ is the number of reducible elements in Z_n.

2. The Functions $\psi(n)$ and $\rho(n)$

In this section, we summarize the results obtained by Kassar and Kojok in [2]. In particular, formulas for the number of irreducible elements $\psi(n)$ and the number of reducible elements $\rho(n)$ are given. These new functions resulted from the study of divisibility in Z_n. The notion of divisibility in commutative rings with identity R is usually studied when R is an integral domain. Very few results can be found about divisibility in rings with zero divisors. In [8], Galovich formulated the concept of unique factorization in terms of irreducible elements in any commutative ring with identity and gave complete structure theorems for unique factorization rings with zero divisors. In that paper, Galovich posed the open problems of determining the irreducible elements of Z_n and of determining the number of such elements. In [2], Kassar and Kojok gave complete solutions to both problems using a slight modification of the definition of the irreducible elements. Next we state the main results in [2]. Let us first recall the definitions given in [2]. We say that two elements a and b of R are associates, denoted by $a \sim b$, when $a \mid b$ and $b \mid a$. Hence, $a \sim b$ if and only if we have the following equality of ideals $< a >=< b >$. We call a proper divisor of a any element b which is neither a unit nor an associate of a. Finally a nonunit element a of R is called irreducible when a has no proper divisors; that is, the only divisors of a are the units and the associates. Now we give the main results in [2].

Theorem 1. *An element $[r]$ in Z_n is irreducible if and only if $\gcd(n,r)$ is a prime integer.*

Theorem 2. *The number of irreducible elements of Z_n is $\sum\limits_{p|n}\phi(n/p)$, where the sum runs over the prime divisors of n.*

Theorem 3. *Let n be any integer. Write $n = \overline{n}m$, where $\overline{n}$ is squarefree, m is squarefull and $(\overline{n}, m) = 1$. Then, $\psi(n) = \phi(n)\left[\sum\limits_{p|\overline{n}}\frac{1}{p-1} + \sum\limits_{p|m}\frac{1}{p}\right]$, where the first sum runs over the prime divisors of $\overline{n}$ and the second one runs over the prime divisors of m.*

Since $\phi(n)$ is the number of units, invertible elements, in Z_n and since any element in Z_n is either a unit, irreducible or reducible, we have that for any positive integer n

$$n = \phi(n) + \psi(n) + \rho(n)\,. \tag{1}$$

3. Properties of $\psi(n)$ and $\rho(n)$

In this section we give some properties of the functions $\psi(n)$ and $\rho(n)$. In particular, we determine the values of n for which the numbers $\psi(n)$ and $\rho(n)$ are even and those for which the numbers $\psi(n)$ and $\rho(n)$ are odd. The characterization of such values will be used in the study of equations involving the functions $\psi(n)$, $\rho(n)$ and $\phi(n)$.

Theorem 4. *Let n be a squarefree positive integer. Then, $\psi(n)$ is even if and only if n is divisible by at least two distinct odd primes.*

Proof. Let n be a squarefree positive integer such that n is divisible by at least two distinct odd primes. Then, $n = 2p_1p_2...p_r$, or $n = p_1.p_2...p_r$, where

$r \geq 2$. In the first case,

$$\psi(n) = (2-1)(p_1-1)(p_2-1)...(p_r-1)\left[\frac{1}{2-1} + \sum_{i=1}^{r}\frac{1}{p_i-1}\right]$$
$$= (p_1-1)(p_2-1)...(p_r-1) + (p_2-1)...(p_r-1)$$
$$+ (p_1-1)(p_3-1)...(p_r-1) + ... + (p_1-1)(p_2-1)...(p_{r-1}-1)\,.$$

Since $r \geq 2$, each term in the last sum is even and hence $\psi(n)$ is even. Similarly, we show that $\psi(n)$ is even when $n = p_1p_2...p_r$ and $r \geq 2$. Now suppose that n is a squarefree integer not divisible by at least two distinct odd primes. Then, $n = 2, n = p$ or $n = 2p$, where p is an odd prime. In each of these three cases, $\psi(n)$ is odd since $\psi(2) = \psi(p) = 1$ and $\psi(2p) = (2-1)(p-1)\left[\frac{1}{2-1} + \frac{1}{p-1}\right] = (p-1)+1 = p$. □

Theorem 5. *Let n be a positive number. Then, $\psi(n)$ is odd if and only if $n = 2, 4, p$, or $2p$, where p is an odd prime.*

Proof. It is clear that when $n = 2, 4, p$, or $2p$, $\psi(n)$ is odd. Now, suppose that $\psi(n)$ is odd. The integer n is either prime, prime power or divisible by at least two distinct primes. In the case that $n = p^a$ with $a > 1$, we have $\psi(n) = \phi(p^{a-1})$ hence, $p^{a-1} = 2$ and $n = 4$.

If $n = p_1^{a_1}...p_r^{a_r}$, $r \geq 3$, then for every p_j dividing n, $\frac{n}{p_j} > 2$ so $\phi(\frac{n}{p_j})$ is even. Hence, $\psi(n)$ is even. Thus, the only remaining case is when $n = p_1^{a_1}p_2^{a_2}$. This implies that $\psi(n) = \phi(p_1^{a_1-1}p_2^{a_2}) + \phi(p_1^{a_1}p_2^{a_2-1})$; and since $\psi(n)$ is odd we have that either $\phi(p_1^{a_1-1}p_2^{a_2})$ or $\phi(p_1^{a_1}p_2^{a_2-1})$ is odd so $n = 2p$. □

From the above theorem we obtain the following characterization of positive integers n with $\psi(n)$ being even.

Corollary 6. *Let n be a positive integer. Then, $\psi(n)$ is even if and only if one of the following is true:*

1. *n is divisible by at least two odd primes;*
2. *$n = 2^a$, where $a > 2$;*
3. *$n = 2^bp$, where $b > 1$, and p is an odd prime.*

Another application of Theorem 5 is the following characterization of positive integers n with $\psi(n) = 1$.

Corollary 7. *Let n be a positive integer. Then, $\psi(n) = 1$ if and only if $n = 4$ or n is prime.*

In the following, we give a characterization of the integers n for which $\rho(n)$ is even.

Theorem 8. *Let n be a positive integer. Then, $\rho(n)$ is even if and only if n is prime or n is even not of the form $2p$, where p is an odd prime.*

Proof. It is clear that if n is prime or n is even not of the form $2p$, where p is an odd prime then $\rho(n)$ is even. Now suppose that $\rho(n)$ is even. If $\rho(n) = 0$, then n is prime. Otherwise, n is divisible by pq, where p and q are primes not necessarily distinct, and $[pq] = [p][q]$ is reducible in Z_n.

Let us consider now the case where $\rho(n) > 0$. If n is odd and not prime, then $n = p^a$ with p odd prime and $a > 1$ or n is divisible by at least two distinct odd primes. But, $n = p^a$ implies $\rho(n) = p^a - p^{a-1}(p-1) - p^{a-1}(p-1)\frac{1}{p}$, which is odd. Also, if n is divisible by at least two distinct odd primes, then Corollary 6 gives that $\psi(n)$ is even and so $\rho(n)$ is odd. Hence, $\rho(n) > 0$ and $\rho(n)$ even give n even so apart from the case where n is divisible by at least two distinct odd primes, the possible values of n are $n = 2^a$ and $n = 2^a p^b$, where p is prime and $a \geq 1$ and $b \geq 1$. We have $\rho(2^a) = 2^{a-2}$ hence a could not be 2. For $a \geq 2$ or $b \geq 2$, the numbers $\rho(2p^a)$, $\rho(2^a p)$ and $\rho(2^a p^b)$ are even and $\rho(2p) = 1$. □

From this we deduce a characterization of the integers n for which $\rho(n)$ is odd.

Corollary 9. *Let n be a positive integer. Then, $\rho(n)$ is odd if and only if n is an odd integer not prime or n is even of the form $2p$, where p is prime.*

4. Equations Involving $\phi(n), \psi(n)$ and $\rho(n)$

In this section we apply the properties of the functions $\psi(n)$ and $\rho(n)$ to solve number-theoretic equations involving these functions and the ϕ-function.

Theorem 10. *Let n be a positive integer greater than 1. Then, $\psi(n)$ divides $\phi(n) + 1$ if and only if $n = 2p$ or $n = p$, where p is a prime integer.*

Proof. It is clear that the integers $n = 2p$ and $n = p$, where p is a prime integer, satisfy the equation

$$k\psi(n) = \phi(n) + 1\,, \tag{2}$$

where k is a positive integer.

First we consider the case where $\psi(n) = \phi(n) + 1$. Then $n > 2$ so that $\phi(n)$ is even and $\psi(n)$ is odd. By Theorem 5, the possible values for n are $4, p$ and $2p$, where p is an odd prime integer. However, $n = 4$ and $n = p$ do not satisfy the equation $\psi(n) = \phi(n) + 1$. For $n = 2p$ we have $\psi(n) = \phi(n) + 1$. Now suppose that $k\psi(n) = \phi(n) + 1$ with $k > 1$. If $n = 2$ then $\phi(n) = 1$ and $\psi(n) = 1$ hence the equation (2) implies that $k = 2$. If $n > 2$ then $\phi(n)$ is even and so $\psi(n)$ is odd. By Theorem 5, $\psi(n)$ is odd if and only if $n = 4$ or $n = p$ or $n = 2p$, where p is an odd prime integer. For $n = 4$, $\phi(n) = 2$ and $\psi(n) = 1$ so the equation (2) implies that $k = 3$. For $n = p$, $\phi(n) = n - 1$ and $\psi(n) = 1$ so the equation (2) implies that $k = n$ and $k > 1$ since n is a prime integer. For $n = 2p$, $\phi(n) = \frac{n}{2} - 1$ and $\psi(n) = \frac{n}{2}$ so the equation (2) implies that $k = 1$ which is impossible since $k > 1$. Therefore the equation (2) with $k > 1$ implies that $n = 4$ or $n = p$, where p is a prime integer. □

Theorem 11. *Let n be a positive integer greater than 1. Then, $\psi(n)$ divides $\phi(n) - 1$ if and only if $n = 3$, $n = 4$ or $n = p$, where p is a prime integer greater than 3.*

Proof. It is clear that $\psi(3) = \phi(3) - 1$, $\psi(4) = \phi(4) - 1$ and $\psi(p)$ divides $\phi(p) - 1$ if p is a prime integer greater than 3 since, in this case $\psi(p) = 1$. First we consider the case where $\psi(n) = \phi(n) - 1$. Then $n > 2$ and $\psi(n)$ is odd. By Theorem 5, the possible values for n are $4, p$ and $2p$, where p is an odd prime integer. For $n = p$, $\phi(p) = p - 1$ and $\psi(p) = 1$ so that $\psi(p) = \phi(p) - 1$ implies $1 = p - 1 - 1$ and $n = 3$. For $n = 2p$, $\phi(2p) = p - 1$ and $\psi(2p) = p$. Hence, $\psi(n) = \phi(n) - 1$ implies $p = p - 2$ which is impossible. Therefore $\psi(n) = \phi(n) - 1$ implies $n = 3$ or $n = 4$. Now suppose that $k\psi(n) = \phi(n) - 1$ with $k > 1$. Since $n > 2$, $\phi(n)$ is even and so $\psi(n)$ and k are odd. By Theorem 5, $n = 4$ or $n = p$ or $n = 2p$, where p is an odd prime integer. For $n = 4$, $\phi(n) = 2$ and $\psi(n) = 1$ so $k\psi(n) = \phi(n) - 1$ implies $k = 1$. For $n = p$, $\phi(n) = n - 1$ and $\psi(n) = 1$ so $k\psi(n) = \phi(n) - 1$ implies $k = p - 2$. Since $k > 1$, p must be greater than 3. For $n = 2p$, $\phi(n) = \frac{n}{2} - 1$ and $\psi(n) = \frac{n}{2}$ so $k\psi(n) = \phi(n) - 1$ implies $k = 1 - \frac{2}{p}$ so $p = 1$ or $p = 2$ which is impossible since p is an odd prime integer. Therefore $k\psi(n) = \phi(n) - 1$, $k > 1$ with $n > 2$ implies $n = p$, where p is a prime integer greater than 3. □

Theorem 12. *Let n be an even positive integer greater than 1. Then, $\psi(n)$ divides $n - 1$ if and only if $n = 2$ or $n = 4$.*

Proof. It is clear that the integers $n = 2$ and $n = 4$ satisfy the equation

$$k\psi(n) = n - 1\,. \tag{3}$$

Now suppose that the equation (3) is satisfied. For the case $k = 1$, we have $\psi(n) = n - 1$. Since n is even, $n - 1$ is odd and so $\psi(n)$ is odd. By Theorem 5, $n = 2$ or $n = 4$ or $n = 2p$, where p is an odd prime integer. We have $\psi(2) = \psi(4) = 1$ and $\psi(2p) = p$. For $n = 2$ the equation (3) is satisfied while for $n = 4$ this equation is not satisfied. For $n = 2p$ the equation (3) gives $p = 2p - 1$ which is impossible since p is a prime integer. For the case $k > 1$, we have $k\psi(n) = n - 1$. Since n is even, $n - 1$ is odd so $\psi(n)$ is odd and by Theorem 5, $n = 4$ or $n = 2p$, where p is an odd prime integer. For $n = 4$, the equation (3) gives $k = 3$. For $n = 2p$, the equation (3) gives $k\frac{n}{2} = n - 1$ which implies that $k = 2 - \frac{1}{p}$ hence $p = 1$ which is impossible since p is an odd prime integer. □

Theorem 13. *Let n be an even positive integer greater than 1. Then, $\psi(n)$ divides $n + 1$ if and only if $n = 2$ or $n = 4$.*

Proof. It is obvious that the values 2 and 4 of n satisfy the equation

$$k\psi(n) = n + 1\,. \tag{4}$$

Now suppose that the equation (4) is satisfied. Since n is even, $n + 1$ is odd and so $\psi(n)$ and k are odd. By Theorem 5, $n = 4$ or $n = p$ or $n = 2p$, where p is a prime integer. For $n = 4$, $\psi(n) = 1$ so $k = 5$. For $n = p$, the only possible value of p is 2 so $k = 3$. For $n = 2p$, $\psi(n) = \frac{n}{2}$ so the equation (4) implies that $k = 2 + \frac{1}{p}$ hence $p = 1$ which is impossible since p is a prime integer. Therefore the equation (4) implies that $n = 2$ or $n = 4$ if n is an even positive integer greater than 1. □

Theorem 14. *Let n be a positive integer greater than 1. Then, $\phi(n)$ divides $\psi(n) - 1$ if and only if $n = 2p$, where p is an odd prime integer.*

Proof. It is obvious that the equation

$$k\phi(n) = \psi(n) - 1 \tag{5}$$

is satisfied by the integers of the form $n = 2p$, where p is an odd prime integer. Suppose now that the equation (5) is satisfied so we have $\psi(n) \neq 1$ hence $n \neq 4$ and n is not a prime integer. Since $n \neq 2$, $\phi(n)$ is even so $\psi(n)$ is odd. By Theorem 5, $n = 4$ or $n = p$ or $n = 2p$, where p is an odd prime integer. Since

$n \neq 4$ and n is not a prime integer, the only possible form of n is $2p$, where p is an odd prime integer. For $n = 2p$, $\phi(n) = \frac{n}{2} - 1$ and $\psi(n) = \frac{n}{2}$ so the equation (5) implies that $k = 1$. $\square$

Theorem 15. *Let n be a positive integer greater than 1. Then, $\phi(n)$ divides $\psi(n) + 1$ if and only if $n = 2$ or $n = 3$ or $n = 4$ or $n = 6$.*

Proof. The integers $n = 2$, $n = 3$, $n = 4$ and $n = 6$ satisfy the equation

$$k\phi(n) = \psi(n) + 1\,. \tag{6}$$

Suppose now that the equation (6) is satisfied. For $n = 2$, this equation implies that $k = 2$. If $n > 2$ then $\phi(n)$ is even and $k\phi(n) - 1 = \psi(n)$ is odd so, by Theorem 5, $n = 4$ or $n = p$ or $n = 2p$, where p is an odd prime integer. For $n = 4$, the equation (6) implies that $k = 1$. For $n = p$, the equation (6) implies that $k(p - 1) = 2$ so $k = 2$ and $p = 2$ or $k = 1$ and $p = 3$. Since p is odd, we just have the case $p = 3$. Finally, for $n = 2p$, the equation (6) implies that $p(k - 1) = k + 1$ hence $p = 2$ or $p = 3$ and $n = 4$ or $n = 6$. $\square$

Let us now consider the equations $k\phi(n) = \rho(n) + 1$ and $k\phi(n) = \rho(n) - 1$, where k is a positive integer. In the following theorem, we characterized some values of n satisfying these equations. In particular we obtain even integers verifying these equalities. We leave as open problems the complete solving of these equations.

Theorem 16. *i) Let n be a positive integer greater than 1. If $\phi(n)$ divides $\rho(n) \pm 1$ then n is cubefree.*

ii) The only positive integer n which is a multiple of 4 and such that $\phi(n)$ divides $\rho(n) \pm 1$ is 4 itself.

iii) The only even integers n such that $\phi(n)$ divides $\rho(n) + 1$ are $n = 2, 4, 6$.

iv) If n is even and $\phi(n)$ divides $\rho(n) - 1$ then $n = 2$ or $n = 2p$, where p is an odd prime.

Proof. i) Suppose $n = p_1^{\alpha_1}...p_t^{\alpha_t} q_1...q_j$, where $p_1, ..., p_t, q_1, ..., q_j$ are distinct primes and, for each integer i between 1 and t, $\alpha_i \succ 1$ and $\alpha_1 \succeq 3$, then $\phi(n) = p_1^{\alpha_1 - 1}...p_t^{\alpha_t - 1}(p_1 - 1)...(p_t - 1)(q_1 - 1)...(q_j - 1)$.

Since $\rho(n) = n - \psi(n) - \phi(n)$, $\phi(n) \mid \rho(n) \pm 1$ implies $\phi(n) \mid n \pm 1 - \psi(n)$ so $\frac{n \pm 1}{\phi(n)} - \sum_{i=1}^{t} \frac{1}{p_i} - \sum_{i=1}^{j} \frac{1}{q_i - 1}$ is an integer.

Let μ_{1i}, $i = 1, ..., t$, denote the power of p_1 in $p_i - 1$ and μ_{1i}, $i = t+1, ..., t+j$, denote the power of p_1 in $q_{i-t} - 1$.

Let $\frac{A}{B}$ (resp. $\frac{C}{D}$) be the reduced form of $\frac{n \pm 1}{\phi(n)}$ (resp. $\sum_{i=1}^{t} \frac{1}{p_i} + \sum_{i=1}^{j} \frac{1}{q_i - 1}$).

The power of p_1 in B is $a = \alpha_1 - 1 + \sum_{i=1}^{t+j} \mu_{1i}$.

Denoting by b the power of p_1 in D, we see that b is less or equal to $\max\{1, \mu_{1i}, i = t+1, ..., t+j\}$.

Since $\alpha_1 \succeq 3$, we get that $b < a$ therefore $\frac{A}{B} - \frac{C}{D}$ cannot be an integer.

ii) Suppose that n is a multiple of 4 and that n is different from 4. We know now by i) that n cannot be a multiple of 8, so there exists at least one odd prime p dividing n. Considering $p_1 = 2$ in the proof of i) and with the same notations, we have $a = \alpha_1 - 1 + \sum_{i=1}^{t+j} \mu_{1i} = 1 + \sum_{i=1}^{t+j} \mu_{1i}$ while $b \leq \max\{1, \mu_{1i}, i = t+1, ..., t+j\}$. The sum appearing in a is non empty so $a > b$.

iii) and iv) Suppose that n is a multiple of 2. Using i) and ii) we can assume that $n > 2$ and that n is of the form $n = 2p_1^{\alpha_1}...p_t^{\alpha_t} q_1...q_j$. With the notations of i), we have $a = 1 - 1 + \sum_{i=1}^{t+j} \mu_{1i}$. However $b \leq \max\{1, \mu_{1i}, i = t+1, ..., t+j\}$. Thus if $\frac{A}{B} - \frac{C}{D}$ is an integer then $a = b$ and $\sum_{i=1}^{t+j} \mu_{1i} \leq \max\{1, \mu_{1i}, i = t+1, ..., t+j\}$. Now we can prove that with this inequality, we can have at most one odd prime dividing n. Suppose there are at least two odd primes dividing n. If there is no prime in the q_j part then $\max\{1, \mu_{1i}, i = t+1, ..., t+j\} = 1$ and $a \geq 4$. Suppose there is at least one odd prime in the q_j part then the sum in a will contain the non-zero contribution of this prime and if there is another odd prime then the sum in a will contain the non-zero contribution of this other one too so we would have $a > b$. So finally the only remaining case is $n = 2p$, where p is an odd prime. The case $\phi(n)$ dividing $\rho(n) + 1$ implies $\frac{n+1}{\phi(n)} - 1 - \frac{1}{q-1}$ is an integer so $\frac{q+1}{q-1}$ is an integer which gives $q = 3$ and $n = 6$. The case $\phi(n)$ dividing $\rho(n) - 1$ implies $\frac{n-1}{\phi(n)} - 1 - \frac{1}{q-1}$ is an integer so $\frac{q-1}{q-1}$ is an integer which proves iv). □

5. Conclusion

Properties of the number of reducible and the number of irreducible elements in Z_n were examined in this paper. In particular, characterizations of the integers for which these functions are even and odd were obtained. Using the properties of these functions, solutions to many equations involving these functions and Euler's ϕ-function were characterized. Partial results for the equations $k\phi(n) = \rho(n) + 1$ and $k\phi(n) = \rho(n) - 1$ were obtained. We leave the complete characterization of these equations as open problems. In particular it still remains to characterize the odd integers satisfying these two equations.

References

[1] G.L. Cohen, P. Hagis, On the number of prime factors of n if $\phi(n)|(n-1)$", *Niew Archief voor Wiskunde*, **28**, No. 3 (1980), 177-185.

[2] A.N. El-Kassar, Badrié Kojok, The number of irreducible elements in Z_n, *Mathematical Sciences Research Journal*, **10**, No. 10 (October 2002), 515-519.

[3] A.N. El-Kassar, On values for which $\phi(n+2) = \phi(n)$, *International Journal Of Applied Mathematics*, **7**, No. 1 (1999), 829-838.

[4] A.N. El-Kassar, A generalization of Lehmer's equations, In: *Proceedings of the International Conference on Scientific Computations*, Lebanese American University, Beirut, Lebanon (1999), 141-151.

[5] A.N. El-Kassar, On the equations $k\phi(n) = \phi(n+1)$ and $k\phi(n+1) = \phi(n)$, In: *Proceedings of an International Conference on Number Theory and Related Topics*, Yonesi University, Seoul, South Korea (1998), 95-109.

[6] P. Erdös P., Some remarks on Euler's ϕ function and related problems, In: *Bull. Amer. Math. Soc.*, **51** (1945), 540-545.

[7] Joseph A. Gallian, *Contemporary Abstract Algebra*, 4-th Edition, Houghton Mifflin Co., Boston (1998).

[8] Galovich Steven, Unique factorization rings with zero divisors, *Math. Magazine*, **51** (1978), 277-283.

[9] R. Goormaghtigh, *L'intermédiaire des Math.*, **25** (1918), 42-44.

[10] S.W. Graham, Jeffrey J. Holt, Carl Pomerance, On the solutions to $\phi(n) = \phi(n+k)$, *Number Theory in Progress*, de Gruyter, Berlin, **2** (1999), 867-882.

[11] V.L. Klee, Some remarks on Euler's totient, *Amer. Math. Monthly*, **54** (1947), 332.

[12] D.H. Lehmer, On Euler's totient function, *Bull. Amer. Math. Soc.*, **38** (1932), 745-751.

[13] E. Lieuwens, Do there exist composite numbers m for which $k\phi(M) = M-1$ holds?, *Niew Archief Voor Wiskunde.*, **18**, No. 3 (1970), 165-169.

[14] L. Moser, Some equations involving Euler's totient function, *Amer. Math. Monthly*, **56** (1949), 22-23.

[15] R. Ratat, *L'intermédiaire des Math.*, **24** (1917), 101-102.