International Journal of Pure and Applied Mathematics

Volume 14 No. 4 2004, 421-436

ON COMPARISON OF OPERATOR THEORETIC AND REGULARIZATION METHODS FOR NUMERICAL INVERSION OF LAPLACE TRANSFORM

M. Iqbal[1] §, Abdulaziz Al-Shuaibi[1]

[1,2]Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, SAUDI ARABIA
[1]e-mail: miqbal@kfupm.edu.sa
[2]e-mail: shuaaziz@kfupm.edu.sa

Abstract: In this paper we are using two types of methods for approximating the inverse Laplace transform.

AMS Subject Classification: 65R20, 65R30
Key Words: ill-posed problems, inversion of laplace transform, convolution equation, maximum likelihood method, filter function, regularization parameter, operator theoretic methods

1. Introduction

The Laplace transform inversion is a severely ill-posed problem in the terminology of improperly posed problems. Unfortunately, many problems of physical interest lead to Laplae transforms whose inverses are not readily expressed in terms of tabulated functions, although there exist extensive tables of transforms and their inverses. It is highly desirable, therefore, to have methods for approximate numerical inversion. However, no single method gives optimum results for all purposes and all occasions. For a detailed bibliography, the reader should consult Piessens [34], Piessens and Branders [35], and a review and comparison is given in Davies [13] and Talbot [40].

Received: December 20, 2003

§Correspondence author

The problem of the recovery of a real function $f(t)$ for $t \geq 0$, given its Laplace transform

$$\int_0^\infty e^{-st} f(t)dt = g(s)\,, \tag{1.1}$$

for real values of s, is an ill-posed problem in the sense of Hadamard [15], and is therefore affected by numerical instability. The ill-posedness of Laplace transform inversion in the case, where $f \in L^2(R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Mellin transform [31], and by means of maximum entropy method [3], [32], [33].

In practice, however $g(s)$ is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by Papoulis [36], Regularization methods have been discussed by Varah [42], Essah and Delves [14], and Brianzi [9], [10], Several other methods have been developed [12], [16], [20], [30], which, in general, work rather well even if they require a large computational cost and high precision arithmetic.

In (1.1), given $g(s)$ for $s \geq 0$ we wish to find $f(t)$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$, so that (1.1) holds. Frequently $g(s)$ measured at certain points. For testing purposes we assume $g(s)$ is given analytically with known $f(t)$ so that we can measure the error in the numerical solution.

2. The Regularization Method

2.1. Fredholm Equation of Convolution Type

The convolution integral equations are widely used in physics, especially in spectroscopy and the reader is referred to two special monographs by Blass et al [7] and Jansen [27].

We shall convert the Laplace transform into the first kind integral equation of convolution type, with the following substitution in equation (1.1)

$$s = a^x \text{ and } t = a^{-y}\,, \quad \text{where } a > 1. \tag{2.1.1}$$

Then

$$g\left(a^x\right) = \int_{-\infty}^{\infty} \log\, a e^{-a^{x-y}} f(\left(a^{-y}\right) a^{-y} dy. \tag{2.1.2}$$

Multipoying both sides of (2.2) by a^x we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x),\ -\infty \leq x \leq \infty\,, \tag{2.1.3}$$

where

$$\left.\begin{array}{rcl} G(x) & = & a^x g\left(a^x\right) = sg(s) \\ K(x) & = & \log aa^x e^{-a^x} = \log ase^{-s} \\ F(y) & = & f\left(a^{-y}\right) = f(t) \end{array}\right\}. \tag{2.1.4}$$

Equation (2.1.3) occurs widely in the applied sciences. K and G are known kernel and data functions respectively, and F is to be found. We shall assume that F, G and K lie in suitable function spaces, such as $L_2(R)$, so that their Fourier transform (FTs) exist. $^\wedge$ denotes FTs and $^\vee$ denotes inverse FTs).

2.2. Description of the Method

We assume that the support of each function F, F and K is essentially finite ad contained within the interval $[0, T]$, where T is the period and equal to Nh, where N is the number of data points and h is the spacing.

Let T_N be the space of trigonometric polynomials of degree at Most N and period T. We shall look for filtered solution of (2.1.3) within the space T_N for the folowing reasons:

a) Thediscretization error in the convolution may be made precisely zero at the grid points.

(b) Fast Fourier Transform (FFT) routines are easily employed in the solution procedure.

(c) The adoption of T_N as the approximating function space is itself a regularizing feature.

Let G and K be given at N equally spaced points $x_n = nh, n = 0, 1, 2, \ldots, N-1$, with spacing $h = T/N$. Then G ad K are interpolated by $G_N \in T_N$, where

$$G_N = \frac{1}{N} \sum_{q=0}^{N-1} \hat{G}_{N,q} \exp(i\omega_q x), \tag{2.2.1}$$

$$\hat{G}_{N,q} = \sum_{n=0}^{N-1} G_n \exp(-i\omega_q x_n), \tag{2.2.2}$$

where

$$G(x_n) = G_n = G_N(x_n), \quad \omega_q = \frac{2\pi q}{T}. \tag{2.2.3}$$

Similar expressions as (2.2.1) and (2.2.2) can be obtained for K_N.

Consider (2.1.3) the Fredholm integral equation of the first kind of convolution type

$$(KF)(x) \equiv \int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \quad -\infty \leq x \leq \infty, \tag{2.2.4}$$

where G and K are known functions in $L_2(R)$, and $F \in H^p(R)$ is to be found. Then from the convolution theorem we have

$$\hat{K}(\omega)\hat{F}(\omega) = \hat{G}(\omega)\,, \tag{2.2.5}$$

whence

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp(i\omega y)d\omega. \tag{2.2.6}$$

The ill-posedness of (2.2.4) is reflected by the fact that any small perturbation ϵ in G, whose transform $\hat{\epsilon}(\omega)$ does not decay faster than $\hat{K}(\omega)$ as $|\omega| \to \infty$, will result in a perturbation in $\frac{\hat{G}(\omega)}{\hat{K}(\omega)}$ which will grow without bound.

When G is inexact, therefore, we may seek a stable or filtered approximation to F given by

$$F_\lambda(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega;\lambda)\frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp(i\omega y)d\omega\,, \tag{2.2.7}$$

where $Z(\omega;\lambda)$ is a stabilizing or filter function dependent on a parameter λ.

In this paper we restrict attention to filters generated from regularization theory. The smoothing functional

$$C(F;\lambda) = ||KF - G||_2^2 + \lambda\Omega[F] \tag{2.2.8}$$

is minimized in an appropaiate subspace of L_2, where $\Omega[F]$ is a stabilizing functional in the form of a smoothing

$$\Omega[F] = ||LF||^2 \tag{2.2.9}$$

and L is a linear operator. The regularization parameter λ controls the tradeoff between smoothness, as imposed by Ω and the extent to which (2.2.4) is satisfied.

We restrict attention to regularization of order $p(p = 2$ in our case), where L in (2.2.9) is the p-th order differential operator, $LF = F^{(p)}$ and the norm in (2.2.9) is L_2. The minimizer of (2.2.8) in H^p is the given by (2.2.7) where

$$z(\omega;\lambda) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda\omega^{2p}}. \tag{2.2.10}$$

If (2.2.4) is now replaced by

$$(K_N F_N)(x) \equiv \int_0^T K_N(x-y)F_N(y)dy = G_N)x)\,, \tag{2.2.11}$$

where K_N is periodically continued outside $(0,T)$. Then we may prove (a) above.

Lemma. *Let $F \in T_N$ and $\underline{F} = (F(x_0),\ldots,F(x_{N-1}))^T \in R^N$. Then the $N \times N$ matrix*

$$K = \psi \mathrm{diag}\,(\hat{K}_{N,q})\psi^H\,, \tag{2.2.12}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi}{N} irs\right)\,, \quad r,s = 0,\ldots,N-1 \tag{2.2.13}$$

has the property

$$(K\underline{F})_n = (K_N F)(x_n). \tag{2.2.14}$$

Thus from the infinite support hyothesis and (2.2.8) it follows that at $\{x_n\}$, (2.1.3) is exactly equivalent to the discrete system

$$(K\underline{F}) = G_n\,, \tag{2.2.15}$$

where K is given i (2.2.12) and $\underline{F} = (F_N(x_0),\ldots,F_N(x_N))^T$.

In T_N it is easily shown that F_λ in (2.2.7) is approximated by

$$F_{N,\lambda}(x) = \sum_{q=0}^{N-1} Z_{q;\lambda} \frac{\hat{G}_{n,q}}{\hat{K}_{N,q}} \exp(i\omega_q x)\,, \tag{2.2.16}$$

where the discrete p-th order filter is

$$Z_{q;\lambda} = \frac{|\hat{K}_{N,q}|^2}{|\hat{K}_{N,q}|2 + N^2\lambda\tilde{\omega}_q^{2p}}, \tag{2.2.17}$$

and

$$\tilde{\omega}_q = \begin{cases} \omega_q, & 0 \le q < \dfrac{1}{2}N\,, \\ \omega_{N-q}, & \dfrac{1}{2}N \le q < N-1\,. \end{cases} \tag{2.2.18}$$

To show (b) above we note that $\sqrt{N}\psi^H$ is the discrete FT matrix representing (2.2.13), and so (2.2.15) is equivalent to the diagonal system

$$\hat{K}_{N,q}\hat{F}_{N,q} = \hat{G}_{N,q}. \tag{2.2.19}$$

After regularization, (2.2.19) is replaced by

$$\hat{K}_{N,q}\hat{F}_{N,q;\lambda} = Z_{q;\lambda}\hat{G}_{N,q}\,, \tag{2.2.20}$$

so that $F_{N,q;\lambda}(x)$ may be found by multiplying the FFT of $\{G_N\}$ by the filter, dividing by the FFT of $\{K_N\}$, and then taking the inverse FFT (fast Fourier transform).

2.3. The Filter in a Stochastic Setting

In this section we relate the pth order convolution filter (2.2.17) to certain spectral densities which play a role in the maximum likelihood (ML) optimization of λ in the next section.

Assume tht the data $\{G_n\}$ are noisy and that there is an underlying function $U_N \in T_N$ such that

$$G_n = U_n\,(x_n) + \epsilon_n \equiv U_n + \epsilon_n. \tag{2.3.1}$$

In the limit $N \to \infty$, $h \to 0$, for any discrete process X_n we may write (see, for example, ref. (39)).

$$X_n = \int_0^T \exp(2\pi i\omega n)d\,S_x(\omega)\,, \tag{2.3.2}$$

where $S_x(\omega)$ s a stochastic process defined on $[0, T]$.

Lemma. *The variance of any integral $\int \theta(\omega)dS_x(\omega)$ is given by $\int |^{\theta}(\omega)|^2 dG_x(\omega)$, where $dG_x(\omega) = E(|dS_x(\omega)|^2)$. $G_x(\omega)$ may be interpolated as a spectral distribution function, and accordingly we shall write $dG_x(\omega) = P_x(\omega)d\omega$, where $P_x(\omega)$ is a spectral density. Now consider $F_N \in T_N$, with $\underline{F} = (F_n) \equiv F_N(x_n)$ defined by $(K\underline{F}) = U_n, n = 0, \ldots, N-1$, with K given by (2.2.12). From (2.3.2), we have*

$$\begin{aligned} F_n = \int_{m=0}^{N-1} \left\{(K^{-1})_{mn} \int_0^T \exp(2\pi i\omega m)dS_u(\omega)\right\} \\ = \int_0^T \left[\hat{K}_N(\omega)\right]^{-1} \exp(2\pi i\omega n)dS_u(\omega)\,, \end{aligned} \tag{2.3.3}$$

where

$$\hat{K}_N(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} K_n \exp(-2\pi i\omega n). \tag{2.3.4}$$

Assume that F_n is estimated by $\sum_{m=0}^{N-1} l_m G_{n-m}$, where $\{l_m\}$ is a filter which we shall relate to $Z_{q;\lambda}$ and $\{G_n\}$ is periodicallycontinued for $n \notin [0, N)$. Then the error

$$F_n - \sum_{m=0}^{N-1} l_m G_{n-m} \tag{2.3.5}$$

is given by

$$\int_0^T \exp(2\pi i\omega n)(\hat{K}_N(\omega))^{-1}\hat{l}_N(\omega)dS_u(\omega) - \int_0^T \exp(2\pi i\omega n)\hat{l}_N(\omega)ds_\epsilon(\omega)\,, \tag{2.3.6}$$

where $\hat{l}_N(\omega)$ is defined as in (2.3.4).

From Lemma (2.3.1) the variance of this error is clearly

$$\int_0^T |[\hat{K}_N(\omega)]^{-1} - \hat{l}_N(\omega)|^2 P_U(\omega)d(\omega) + \int_0^T |\hat{l}_N(\omega)|^2 P_\epsilon(\omega)d\omega\,, \tag{2.3.7}$$

which is minimized when

$$\hat{l}_{N(\omega)\hat{K}_N(\omega)} = \frac{P_U(\omega) + P_\in(\omega)}{P} \tag{2.3.8}$$

and since the Fourier coefficients of the filtered solution must satisfy

$$\hat{F}_{N,q;\lambda} = \hat{l}_{n,q}\hat{G}_{N,q} = Z_{q;\lambda}\hat{G}_{N,q}[\hat{K}_{N,q}]^{-1}, \tag{2.3.9}$$

we find from (2.3.8)

$$Z_{q;\lambda} = \hat{l}_{N,q}\hat{K}_{N,q} = \frac{P_U(qh)}{P_U(qh) + P_\epsilon(qh)}\,, \tag{2.3.10}$$

where $\hat{l}_{N,q} = \hat{l}_N(qh)$ and $\hat{K}_{N,q} = \hat{K}_N(qh)$.

2.4. Optimization of Regularization Parameter by ML Method

We now simply relate the filter (2.3.9) to the p-th order filter (2.2.17). Assuming that the errors are uncorrelated, $P_\epsilon(\omega)$ has the form

$$P_\epsilon(\omega) = \sigma^2 - \text{constant}, \tag{2.4.1}$$

where σ is the unknown variance of the noise in the data.

Choosing

$$P_U(\omega) = \frac{\sigma^2 |\hat{K}_N(\omega)|^2}{\lambda \omega^{2p}}, \tag{2.4.2}$$

where

$$\tilde{\omega} = \begin{cases} 2\pi N\omega, & 0 \le \omega < \frac{1}{2}T, \\ 2\pi N(T-\omega), & \frac{1}{2}T \le \omega < T. \end{cases}$$

Then yields (2.2.17) from (2.3.9). Moreover, the spectral density for $\{G_n\}$ is then

$$P_G(\omega) = P_U(\omega) + P_\epsilon(\omega) = \sigma^2 \left[1 + \frac{|\hat{K}_N(\omega)|^2}{\lambda \tilde{\omega}^{2p}}\right],$$

whence

$$P_G(\omega) = \sigma^2 \left(1 - Z_{q;\lambda}\right)^{-1}. \tag{2.4.3}$$

The statistical likelihood of any suggested values of σ^2 and λ may now be estimated from the data. Following Whittle [44], the logarithm of the likelihood function of P_G is given approximately by

$$\left.\begin{array}{l} \text{constant} - \frac{1}{2}\sum_{q=0}^{N-1} [\log P_G(qh) + I(qh)/P_G(qh)] \\ \text{(using the definition of joint distribution} \\ \text{of the random variables)} \end{array}\right\}, \tag{2.4.4}$$

where

$$I(\omega) = \left| \sum_{n=0}^{N-1} G_n \exp(-2\pi\, i\omega n) \right|^2 \tag{2.4.5}$$

is the periodogram of the data, with $I(qh) = |\hat{G}_{N,q}|^2$.

We now maximize (2.4.4) with respect to σ^2 and λ. The partial maximum with respect to σ^2 may be found exactly (in terms of λ with the maximizing value of σ^2 given by the maximum w.r.t λ may then be found) by minimizing

$$V_{ML}(\lambda) = \frac{1}{2} N \log \left[\sum_{q=1}^{N-1} |\hat{G}_{N,q}|^2 (1 - Z_{q;\lambda}) \right] - \frac{1}{2} \sum_{q=1}^{N-1} \log(1 - Z_{q;\lambda}). \tag{2.4.6}$$

Thus the optimal regularization parameter is given by the minimizer of a sample function of λ, depending on the known Fourier coefficients $\hat{G}_{N,q}$ and

$\hat{K}_{N,q}$. No prior knowledge of σ^2 is assumed but an *a posteriori* estimate is given by (2.4.5). (2.4.6) can be minimized with respect to λ and in order to minimize $V_{ML}(\lambda)$ in equation (2.4.6), we have used a subroutine which uses quadratic interpolation technique to obtain a minimum.

3. Operator Theoretic Method

Very few methods have approached the inversion of the Laplace transform from an operator theoretic point of view. This approach enables us to obtain highly accurate approximations with little effort.

In what follows, the Laplace transform is seen as an operator $\mathcal{L}$ acting in the separable Hilbert space $L^2_{dx}(0,\infty)$,

$$L^2_{dx}(0,\infty) \xrightarrow{\mathcal{L}} L^2_{dx}(0,\infty)\,,$$

defined in the L^2 sense by,

$$\mathcal{L}(y)(s) = \int_0^\infty e^{-sx} y(x)\,dx.$$

It is well known (see [45]) that the Laplace transform of a function of $L^2_{dx}(0,\infty)$ belongs to the Hardy space $\mathcal{H}^2$ (see [2]).
where,

$$\mathcal{H}^2 = \left\{ F(\sigma+i\tau) \text{ analytic for } \sigma > 0 \text{ and } \sup_{\sigma\geq 0} \int_{\Re} |F(\sigma+i\tau)|^2 d\tau < \infty \right\},$$

$$\mathcal{L}(y)(s) \in \mathcal{H}^2 \Leftrightarrow y \in L^2_{dx}(0,\infty)\,,$$

and thus the domain of $\mathcal{L}$ coincides with those traces of functions in $\mathcal{H}^2$ on the positive real line that are square integrable. Operators acting in the space of analytic functons have a very interesting feature. Indeed, bounded operators on the space of entire functions can be represented by differential operators of infinite order (see [23], [43]).

To employ these concept, we shall need to iterate the Laplace transform. The symmetry of the kernel of $\mathcal{L}$ suggest the construction of the convolution operator as done earlier. But first we need to iterate the Laplace transform, so as to obtain a simple convolution form giving rise to the direct employment of the infinite differential operator.

$$\mathcal{L}^2(y)(s) = \int_0^\infty \frac{y(x)}{s+x} dx\,, \text{ where } y \in L^2_{dx}(0,\infty).$$

To bring differential operators into our work, we recall that $-i\frac{d}{dx}$ is a self-adjoint operator acting on $L^2_{dx}(-\infty,\infty)$. Thus the space $L^2_{dx}(0,\infty)$ must be transformed into $L^2_{dx}(-\infty,\infty)$ with proper change of variable and this is achieved by the following unitary transformation operator V

$$V : L^2(0,\infty) \longrightarrow L^2(-\infty,\infty).$$

Following [45] we define V by

$$\begin{aligned} Vy(x) &= e^{\frac{x}{2}}\, y(e^x)\,, \\ V^{-1}g(x) &= \frac{1}{\sqrt{x}} \quad g(\mathrm{Ln}\, x)\,. \end{aligned}$$

It is readily seen that

$$VV^* = I \quad \text{and so} \quad V^{-1} = V^*\,,$$

where the adjoint V^* is defined by

$$\int_{-\infty}^{\infty} Vf(t)\,\overline{\Phi(t)}\,dt = \int_0^{\infty} f(t)\overline{V^*\Phi(t)}\,dt.$$

We now have the operator A acting on the desired space $L^2_{dx}(-\infty,\infty)$ defined by

$$A = V\mathcal{L}^2V^{-1}\,,$$

$$\begin{array}{ccc} L^2_{dx}(-\infty,\infty) & \xrightarrow{\;A\;} & L^2_{dx}(-\infty,\infty) \\ {\scriptstyle V^{-1}}\big\downarrow & & \big\uparrow \\ L^2_{dx}(0,\infty) & \xrightarrow{\;\mathcal{L}^2\;} & L^2_{dx}(0,\infty) \end{array}$$

It is immediate to see that A is an integral operator of the Carleman type defined by a convolution

$$Ay(x) = K * y(x) = \int_{-\infty}^{\infty} K(x-\eta)\, y(\eta)d\eta\,,$$

where

$$K(x) = \frac{1}{2\cosh\left(\frac{x}{2}\right)}$$

the Fourier transform $\wedge$ is first defined on the set of continuous functions with compact support and then extended by continuity to $L^2_{dx}(-\infty,\infty)$ that is by closing its graph.

Result. (see [45]) *Let* $g(s) \in \mathcal{H}^2 \bigcap L^2_{dx}(0,\infty)$. *Then* $\mathcal{L}^{-1}F = \dfrac{1}{\pi} V^{-1} \cos \pi D V \mathcal{L} g$, *where* $D = \dfrac{d}{dx}$.

The infinite differential operator above can act as a shift operator with some additional conditions imposed on g.

Result. (see [45]) *Let the following conditions hold:*
(a) $g(s)$ *is real for real* s.
(b) $f \in \mathcal{H}^2 \bigcap L^2_{dx}(0,\infty)$.
(c) $V\mathcal{L}g$ *has an analtic extension in the strip* $\Omega = \{|I_m z| \leq \pi\}$.
Then $\mathcal{L}^{-1}g(x) = \dfrac{1}{\pi} V^{-1} \operatorname{Re}[V\mathcal{L}g(x + i\pi)]$.

Remark. (The Numerical Approximation) We can simplify the above result to the form

$$f(x) = \mathcal{L}^{-1}g = \frac{-\operatorname{Im}(\mathcal{L}g)(-x)}{\pi}$$

The above formula gives an exact inversion formula provided g satisfies the condition in the previous result, and more importantly if $\mathcal{L}g$ can be obtained easily and/or taulated. In general $\mathcal{L}g$ is very difficult to evaluate and this brings about the necessity and strong need for the numerical aproximation for g, in such a way, the basis functions for the approximation are of the same nature of the original function g satisfying the condition in the inversion formula in the above result. For this, we employ the classical Laguerre polynomials with a proper change of variable. Consider $G(s) = e^{-cs} g\left(\frac{1}{s}\right)/s$, thus for any $g \in L^2_{dx}(0,\infty)$, G is also an $L^2_{dx}(0,\infty)$ and the approximation of G can be expanded in the form

$$G_{\text{approx}}(x) = \sum_{k=0}^{N} a_k L_k^{2\alpha}(x),$$

where $L_k^{2\alpha}$ are the classical Laguerre polynomial with weight function $e^{-\alpha x}$ and a_k ae defined by

$$a_k = \sqrt{2\alpha} \int_0^{\infty} e^{-\alpha x} L_k^{2\alpha}(x) e^{-cx} \frac{g\left(\frac{1}{x}\right)}{x} dx.$$

Here we have $\alpha > c$.

Now $g_{\text{approx}} = \sqrt{2\alpha} \sum_{k=0}^{N} a_k L_k^{2\alpha}\left(\dfrac{1}{x}\right) \dfrac{e^{-(\alpha - c)x}}{x}$.

The Laplace of g is readily tabulated and hence the theory is applicable and, our final approx of f

$$f_{\text{approx}}(x) = \frac{-Im\left[\sqrt{2\alpha}\sum_{k=0}^{N} a_k \mathcal{L}\left(L_k^{2\alpha}\left(\frac{1}{t}\right)\frac{e^{-(\alpha-c)/t}}{t}\right)\right](-x)}{\pi}.$$

The only numerical work to be done is to approximate the coefficients a_k with efficient integration quadrature with a moderate choice of the parameters α, c and the number N. For this *Maple V4* subroutine is attached.

4. Numerical Checks and Comparison

In this section we tabulate the results of the above method applied to test problems over the inteerval $[0, T]$. All data functions have the property $g(s) = 0(s^{-1})$, so we deal with severely ill-posed problems. Therefore, nonoise is added apart from machine rounding error. In all test problems we have taken $N = 256$ data points.

Problem 1. (Theocaris [41], case 5 p. 79)

$$\begin{aligned} f(t) &= e^{-at}, \quad a = 1.0\,, \\ g(s) &= \frac{1}{s-a}\,. \end{aligned}$$

The numerical calculations are given in Table 1.

Problem 2. (Theocaris [41], case 4 p. 79).

$$\begin{aligned} f(t) &= e^{-at}\sin bt, \quad a = 5.0, \quad b = 2.2\,, \\ g(s) &= \frac{U\sin V}{s^2 + 2US\cos V + U^2}\,, \quad U = \left(a^2+b^2\right)^{-\frac{1}{2}}\,, \\ V &= \tan^{-1}(b/a)\,. \end{aligned}$$

The numerical calculations are given in Table 1.

Problem 3. (Brianzi and Mcwhirter [9], [10], [31]).

$$\begin{aligned} f(t) &= t^a e^{-bt} \text{ for } a = 1.0, \ b = 1.0 \\ g(s) &= \frac{\Gamma(a)}{(s+b)^{a+1}} \end{aligned}$$

Problem	T The period	h The step size	β	λ The regularizing parameter	$V(\lambda)$	$\|\|f - f_\lambda\|\|_\infty$
1	12.50	0.04883	10.0	0.9901×10^{-8}	0.7852×10^{1}	0.0058
2	9.0	0.03516	5.0	0.121×10^{-10}	0.8365×10^{3}	0.009
3	12.01	0.04727	10.0	0.199×10^{-10}	0.1134×10^{2}	0.0012
4	14.50	0.0566	10.0	0.125×10^{-12}	0.1992×10^{1}	0.0019

Table 1: Regularization Method

Prob.	T Period	b	c	N Number of Polys	Error $= \|\|f - f_{\text{approx}}\|\|_{\infty.}$
1	12.5	4	1	50	2.3×10^{-4}
2	9	6	1	80	4.9×10^{-4}
3	12.01	3	1	40	2.2×10^{-4}
4	14.5	4	1	60	2.1×10^{-4}

Table 2: Operator Theoretic Method

The Numerical calculations are given in Table 1.

Problem 4. (Varah [42]).

$$\begin{aligned} f(t) &= t^a e^{-bt} \text{ for } a = 3.0,\ \ b = 1.0\,, \\ g(s) &= \frac{\Gamma(a+1)}{(s+b)^{a+1}}\,. \end{aligned}$$

The Numerical calculations are given in Table 1.

Table 2 illustrates the numerical computations of the operator theoretic method, where the errof is taking in the infinity norm, for the same examples discussed in the regularization method.

Concluding Remarks

Our methods worked very well over all the four test problems and results obtained yield a very good approximation to the true solutions, shown in Table 1 and Table 2.

Acknowledgements

1. The authors appreciate and acknowledge, the excellent research and computing facilities availed at King Fahd University of Petroleum and Minerals, Dhahran during the preparation of this paper.

2. This research has been supported by King Fahd Univrsity of Petroleum and Minerals Dhahran, under the project No. MS/ LAPLACE/210.

References

[1] H. Akaike, *Likelihood and Bayes Procedures in Bayesian Statistics* (Eds. J.M. Bernardo, M.H. Degroot, D.V. Linley, A.F.M. Smith), University Press Valencia (1980).

[2] A.A. Al-Shuaibi, *Journal of Mathematical Analysis and Applications*, **228** (1998), 16-36.

[3] U. Amato, W. Hughes, *Inverse Probl.*, **7** (1991), 793.

[4] R.S. Anderssen, P. Bloomfield, *Numer. Math.*, **22** (1974), 157.

[5] R.S. Anderssen, P. Bloomfield, *Technometrics*, **16** (1974), 69.

[6] D.S. Ang, et al, *Journal of Mathematics of Computation*, **53**, No. 188 (1989), 589.

[7] M. Bertero, E.R. Pike, *Inverse Problems*, **7** (1991), 1-20.

[8] W.e. Blass, et al, Academic Press, New York (1981).

[9] B. Brianzi, M. Frontini, *Inverse Probl.*, **7** (1991), 355.

[10] P. Brianzi, *Inverse Problems*, **10** (1994), 55-61.

[11] M.K. Charter, *Maximum Entropy and Bayesian Nethods* (Ed-s. C.-R. Smith, et al), Kluwer Academic Publishers, The Netherlands, Seattle (1991), 287.

[12] C. Cristina, et al, *Inverse Probl.*, **9** (1993), 57.

[13] C. Cristina, V. Famin, *Math. Computation*, **64**, No. 211 (1995), 1193-1198.

[14] B. Davies, B. Martin, *J. Comput. Physics*, **33**, No. 2 (1979).

[15] Heinz W. Engl, Martin Hanek, *Regularization of Inverse Problems*, Kluwer Dordrecht, (1996).

[16] W.A. Essah, L.M. Delves, *Inverse Probl.*, **4** (1988), 705.

[17] Gunter Fleischer, Bend Hofmann Dordrecht (1996).

[18] B. Gabuti, L. Sacripante, *J. Comput. Appl. Appl. Math.*, **34** (1991), 191-200.

[19] J.J. Good, MIT Press, Cambridge, MIT 1965).

[20] G. Guinta, et al, *Math.*, **54** (1988), 193.

[21] P.F. Gull, *Maximum Entropy and Bayesian Methods* (Ed. J. Skilling), Kluwer Academic Publishers, The Netherlands (1989), 53.

[22] J. Hadamard, Yale University Press, New Haven (1923).

[23] K. Hoffman, *Banach Spaces of Analytic Functions*, Dover, New York (1962).

[24] P.A. Jansen, Academic Press, New York (1984).

[25] Kitagawa, W. Gersch, *IEEE Transactions on Automatic Contrl*, **AC-30** (1985), 48.

[26] G. Klein, *J. Comp. Appl. Math.*, **5** (1979), 259.

[27] Gethard Landl, R.S. Anderssen, *Inverse Problems*, **12** (1996), 35-53.

[28] R.D. Levin, M. Tribus, MIT, Cambridge, M.A. 1979).

[29] B.M. Levitan, *Lie Theorems for Generalized Translations*, Amer. Math. Soc. Transl., Ser. 2, **105** (1975), 211-220.

[30] D.J.C. Mackay, *Neural Computation*, **4** (1992).

[31] M.A. Main, F.H. Ruyngart, *J. Comput. Control*, **IV**, 20 (1995), 259-267.

[32] J.G. McWhirtr, P.R. Pike, *J. Phys., A*, **11**, No. 9 (1978), 1729.

[33] L.R. Mead, N. Papanicolaou, *J. Math. Phys.*, **25**, No. 8 (1984), 2404.

[34] L.R. Mead, *J. Math. Phys.*, **27**, No. 12 (1986), 2903.

[35] R. Piessens, *J. Comp. Appl. Maths.*, **1** (1975), 115.

[36] R. Piessens, M. Branders, *Proc. IEEE*, **118** (1971), 1517.

[37] A. Popoulis, *Quarterly Applied Math.*, **14** (1956), 405.

[38] G. Rodriguez, S. Seatzu, *IMA J. of Numer. Anal.*, **13** (1993), 463.

[39] Y. Sakamoto, Kluwer Academic Publishers, The Netherlands (1991).

[40] J. Skilling, *Maximum Entropy and Bayessian Methods* (Ed-s. W.T. Grandy, Jr., L.H. Schick), Kulwer Academic Publishers, The Netherlands (1991), 267.

[41] A. Talbot, *J. Inst. Math. Appl.*, **23**, No. 1 (1979), 97.

[42] P. Theocaris, A.C. Chrysakis, *J. Math. Appl.*, **20** (1977), 73.

[43] T. Duc Van, S. Nho Hao, *Differential Operators of Infinite Order with Real Argument and Their Applications*, World Scientific, Singapore (1994).

[44] J.M. Varah, *SIAM J. Sci., Statist. Comput.*, **4**, No. 2 (1983), 164.

[45] P. Whittle, *Skand. Actuarietidkr*, **35** (1952), 48.