**International Journal of Pure and Applied Mathematics**

**Volume 14 No. 4 2004, 437-451**

# SEMI-$(E, F)$-CONVEX FUNCTIONS AND SEMI-$(E, F)$-CONVEX PROGRAMMING

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**Abstract:** In this paper, a new class of non-convex functions, called semi-$(E, F)$-convex (quasi-semi-$(E, F)$-convex, pseudo-semi-$(E, F)$-convex) functions are introduced, and some of their basic characters are discussed. At the same time, some sufficient conditions of optimality and duality theorems for the generalized convex programming are studied.

**AMS Subject Classification:** 26A33
**Key Words:** $(E, F)$-convex sets, $(E, F)$-convex functions, semi-$(E, F)$-convex functions, semi-$(E, F)$-convex programming, optimality conditions

## 1. Introduction

Convexity and generalized convexity play a central role in optimization theory, so the search on convexity and generalized convexity is one of the most important aspects in mathematical programming. During the past several decades, various significant generalizations of convexity are presented, see [1]-[12]. Youness [12] brought forward the concepts of $E$-convex sets, $E$-convex functions and $E$-convex programming, discussed some of their basic properties, and established some optimal results of $E$-convex programming. Jian [4], [5]

Received: April 19, 2004 

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introduced $(E,F)$-convex sets, $(E,F)$-convex functions and $(E,F)$-convex programming by extending the definitions of $E$-convex sets, $E$-convex functions and $E$-convex programming, discussed some of their properties and also gave some examples to show that some results in [12] are incorrect as well as [2], [11]. Chen [2] introduced a class of semi-$E$-convex function and also discuss its basic properties.

In this paper, basing on semi-$E$-convexity and $(E,F)$-convexity, we introduce a new class of semi-$(E,F)$-convex (quasi-semi-$(E,F)$-convex, pseudo-semi-$(E,F)$-convex) functions, discuss their basic characters and study some sufficient conditions of optimality and duality theorems for the generalized convex programming.

We always assume that $M$ is an nonempty subset in $R^n$ throughout this paper, we review some concepts as follows which have some relationships with this paper.

**Definition 1.1.** (see [12]) A set $M$ is said to be an $E$-convex set if there is a point-to-point map $E : M \to R^n$ such that

$$\lambda E(x) + (1-\lambda)E(y) \in M,\ \forall x, y \in M,\ \forall \lambda \in [0,1].$$

**Definition 1.2.** (see [12]) A function $f : M \to R$ is said to be $E$-convex on a set $M$ if there is a point-to-point map $E : M \to R^n$ such that $M$ is an $E$-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda)f(E(y)),\quad \forall x, y \in M, \lambda \in [0,1].$$

For sets $A, B \subseteq R^n$ and $\alpha, \beta \in R$, denote $2^A = \{C : \ C \subseteq A\}$, $\alpha A + \beta B = \{\alpha x + \beta y : x \in A, y \in B\}$.

**Definition 1.3.** (see [4]) A set $M$ is said to be an $(E,F)$-convex set if there are two point-to-set maps $E, F : M \to 2^{R^n}$ such that

$$\lambda E(x) + (1-\lambda)F(y) \subseteq M,\quad \forall x, y \in M,\ \forall \lambda \in [0,1].$$

**Definition 1.4.** (see [4]) A function $f : M \to R$ is said to be an $(E,F)$-convex function on a set $M$, if there are two point-to-set maps $E, F : M \to 2^{R^n}$ such that $M$ is an $(E,F)$-convex set and

$$f(\lambda \overline{x} + (1-\lambda)\overline{y}) \le \lambda f(\overline{x}) + (1-\lambda)f(\overline{y}),\quad \forall x, y \in M,\ \forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y),\ \forall \lambda \in [0,1].$$

**Definition 1.5.** (see [4]) The problem $\min\{f(x):\ x\in M\}$ is said to be an $(E,F)$-convex programming if there are two point-to-set maps $E,F:M\to 2^{R^n}$ such that the feasible set $M$ is $(E,F)$-convex and the objective function $f$ is $(E,F)$-convex on $M$.

**Definition 1.6.** (see [2]) A function $f:M\to R$ is said to be semi-$E$-convex on a set $M$ if there is a point-to-point map $E:M\to R^n$ such that $M$ is an $E$-convex set and

$$f(\lambda E(x)+(1-\lambda)E(y))\le\lambda f(x)+(1-\lambda)f(y),\quad \forall x,y\in M,\ \forall\lambda\in[0,1].$$

## 2. Semi-$(E,F)$-Convex Function

In this section, we present the definition of semi-$(E,F)$-convex function and discuss its main properties.

**Definition 2.1.** A function $f:M\to R$ is said to be semi-$(E,F)$-convex on $M$ if there are two point-to-set maps $E,F:M\to 2^{R^n}$ such that $M$ is an $(E,F)$-convex set and

$$f(\lambda\overline{x}+(1-\lambda)\overline{y})\le\lambda f(x)+(1-\lambda)f(y),$$
$$\forall x,y\in M,\ \forall\overline{x}\in E(x),\ \forall\overline{y}\in F(y),\ \forall\lambda\in[0,1].$$

Furthermore, $f$ is said to be strictly semi-$(E,F)$-convex function if the inequality is strict for $x\ne y$, and $\lambda\in(0,1)$.

**Remark 2.1.** Each convex function $f$ on $M$ is a semi-$(E,F)$-convex function. This conclusion can be proved by taking maps $E(x)=\{x\}$ and $F(y)=\{y\}$.

**Remark 2.2.** Every semi-$E$-convex function on an $E$-convex set is a semi-$(E',F')$-convex function. The proof is easy by taking maps defined as $E'(x)=\{E(x)\}$ and $F'(y)=\{E(y)\}$.

**Theorem 2.1.** *Let $M$ be an $(E,F)$-convex set and $f:M\to R$ be an $(E,F)$-convex function, then $f$ is a semi-$(E,F)$-convex function if and only if*

$$f(\overline{x})\le f(x),f(\overline{y})\le f(y),\ \forall x,y\in M,\ \forall\overline{x}\in E(x),\ \forall\overline{y}\in F(y).$$

*Proof.* Show the sufficiency. Since $f:M\to R$ is $(E,F)$-convex on set $M$, taking account the given conditions, for $\forall x,y\in M$, $\forall\overline{x}\in E(x)$, $\forall\overline{y}\in F(y)$, $\forall\lambda\in[0,1]$, we have $f(\lambda\overline{x}+(1-\lambda)\overline{y})\le\lambda f(\overline{x})+(1-\lambda)f(\overline{y})\le\lambda f(x)+(1-\lambda)f(y)$. Hence $f$ is semi-$(E,F)$-convex on $M$.

Conversely, suppose that the function $f : M \to R$ is semi-$(E, F)$-convex, then we get

$$f(\lambda\overline{x} + (1-\lambda)\overline{y}) \leq \lambda f(x) + (1-\lambda)f(y),$$
$$\forall x, y \in M,\ \forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y),\ \forall \lambda \in [0,1].$$

Letting $\lambda = 1$, $f(\overline{x}) \leq f(x)$ is obtained, and letting $\lambda = 0, f(\overline{y}) \leq f(y)$ is obtained. The proof is completed. □

**Proposition 2.1.** *If functions $f_i : R^n \to R$ is semi-$(E, F)$-convex and bounded on $(E, F)$-convex set $M$ for each $i \in I$, then the function $f(x) = \sup\{f_i(x) : i \in I\}$ is semi-$(E, F)$-convex on $M$.*

*Proof.* Since $f_i$ is semi-$(E, F)$-convex and bounded on $(E, F)$-convex set $M$, then for $\forall x, y \in M,\ \forall \lambda \in [0,1]$ and $\forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y)$, we have

$$\begin{aligned} f(\lambda\overline{x} + (1-\lambda)\overline{y}) &= \sup\{f_i(\lambda\overline{x} + (1-\lambda)\overline{y}) : i \in I\} \\ &\leq \sup\{\lambda f_i(x) + (1-\lambda)f_i(y) : i \in I\} \\ &\leq \lambda \sup\{f_i(x) : i \in I\} + (1-\lambda)\sup\{f_i(y) : i \in I\} \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

So, the function $f(x) = \sup\ \{f_i(x) : i \in I)\}$ is semi-$(E, F)$-convex on $M$. □

**Proposition 2.2.** *Let $M$ be an $(E, F)$-convex set and functions $f_i :\ M \to R$ be semi-$(E, F)$-convex on $M$ $(i = 1, 2, 3, \ldots, k)$, then $h(x) = \sum_{i=1}^{k} a_i f_i(x)(a_i \geq 0, i = 1, 2, \ldots, k)$ is a semi-$(E, F)$-convex function on $M$.*

The proof is easy by using the definition of semi-$(E, F)$-convex function.

**Proposition 2.3.** *Let $M$ be an $(E, F)$-convex set, function $f.M \to R$ be semi-$(E, F)$-convex on $M$, and let $\phi : R \to R$ be a monotonically nondecreasing convex function, then the composite function $\phi(f)$ is semi-$(E, F)$-convex on $M$.*

*Proof.* Since $f$ is semi-$(E, F)$-convex on $(E, F)$-convex set $M$, we have

$$f(\lambda\overline{x} + (1-\lambda)\overline{y}) \leq \lambda f(x) + (1-\lambda)f(y),$$
$$\forall x, y \in M,\ \forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y),\ \forall \lambda \in [0,1].$$

In view of $\phi : R \to R$ being nondecreasing convex function, one has

$$\phi\left(f(\lambda\overline{x} + (1-\lambda)\overline{y}\right) \leq \phi(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda\phi(f(x)) + (1-\lambda)\phi(f(y)).$$

This shows that $\phi(f)$ is semi-$(E, F)$-convex on $M$. □

**Proposition 2.4.** *Assume that a function $f : M \to R$ is semi-$(E,F)$-convex on an $(E,F)$-convex set $M$, then the level set $K_\alpha = \{x : x \in M, f(x) \leq \alpha\}$ is an $(E,F)$-convex set for all $\alpha \in R$.*

*Proof.* Assume that $x, y \in K_\alpha$ and $\lambda \in [0,1]$, from the given conditions, we have

$$f(x) \leq \alpha, f(y) \leq \alpha, \lambda E(x) + (1-\lambda)F(y) \subseteq M,$$

$$f(\lambda\overline{x} + (1-\lambda)\overline{y}) \leq \lambda f(x) + (1-\lambda)f(y) \leq \alpha, \ \forall \overline{x} \in E(x), \ \forall \overline{y} \in F(y)$$

So $\lambda\overline{x} + (1-\lambda)\overline{y} \in K_\alpha$, that is, $\lambda E(x) + (1-\lambda)F(y) \subseteq K_\alpha$. This follows that the set $K_\alpha$ is $(E,F)$-convex. □

**Remark 2.3.** The converse of Proposition 2.4 is not true.

For showing this, a counterexample is given as follows, it is a slight modification of Example 5 in [2]. Let $M = R$, maps $E, F : M \to 2^M$ and function $f : M \to R$ be defined as

$$E(x) = F(x) = \begin{cases} \{1\}, & \text{if } \ 1 \leq x \leq 4; \\ \{1 + \dfrac{2}{\pi}\arctan(1-x)\}, & \text{if } \ x < 1; \\ \{2 + \dfrac{4}{\pi}\arctan(x-4)\}, & \text{if } \ x > 4, \end{cases}$$

$$f(x) = \begin{cases} 2, & \text{if } \ x < 1 \text{ or } x > 4; \\ x - 3, & \text{if } \ 1 \leq x < 2; \\ 3 - x, & \text{if } \ 2 \leq x \leq 3; \\ x - 3, & \text{if } \ 3 < x \leq 4. \end{cases}$$

Obviously, the set $M$ is an $(E,F)$-convex set, and the level set

$$K_\alpha = \begin{cases} R, & \text{if } \ \alpha \geq 2; \\ [1,4], & \text{if } \ 1 < \alpha < 2; \\ [1,2] \cup [3-\alpha, 3+\alpha], & \text{if } \ 0 \leq \alpha \leq 1; \\ [1,2), & \text{if } \ \alpha < 0. \end{cases}$$

is always $(E,F)$-convex. But for points $1 \in E(1)$ and $3 \in F(5)$,

$$f(\frac{1}{2} \times 1 + \frac{1}{2} \times 3) = f(2) = 1 > \frac{1}{2}f(1) + \frac{1}{2}f(5) = 0.$$

so the function $f(x)$ is not semi-$(E,F)$-convex on the $(E,F)$-convex set $M$.

By quoting the notation of [4], we discuss the relationships between a semi-$(E,F)$-convex function $f$ and its epigraph $\operatorname{epi}(f)$.

For every map $E : M \to 2^{R^n}$, we define an extension map $E^+ : M \times R \to 2^{R^{n+1}}$ as $E^+(x,\alpha) = (E(x),\alpha)$, and define the epigraph $\mathrm{epi}\,(f)$ of $f$ on $M$ as follows: $\mathrm{epi}\,(f) = \{(x,\alpha) :\ x \in M, \alpha \in R, f(x) \le \alpha\}$.

**Theorem 2.2.** *A function $f : M \to R$ is semi-$(E,F)$-convex on an $(E,F)$-convex set $M$ if and only if the set $\mathrm{epi}\,(f)$ is $(E^+,F^+)$-convex on $R^{n+1}$.*

*Proof.* Assume that $f$ is semi-$(E,F)$-convex, and letting $(x,\alpha)$, $(y,\beta) \in \mathrm{epi}\,(f)$. Then we have $x,y \in M, f(x) \le \alpha$ and $f(y) \le \beta$. For $\forall\lambda \in [0,1]$, one gets

$$\lambda E^+(x,\alpha) + (1-\lambda)F^+(y,\beta) = (\lambda E(x) + (1-\lambda)F(y), \lambda\alpha + (1-\lambda)\beta),$$

$$f(\lambda\overline{x} + (1-\lambda)\overline{y}) \le \lambda f(x) + (1-\lambda)f(y) \le \lambda\alpha + (1-\lambda)\beta, \quad \forall\overline{x} \in E(x), \overline{y} \in F(y).$$

That is, $(\lambda\overline{x} + (1-\lambda)\overline{y}, \lambda\alpha + (1-\lambda)\beta) \in \mathrm{epi}\,(f)$, i.e., $\lambda E^+(x,\alpha) + (1-\lambda)F^+(y,\beta) \subseteq \mathrm{epi}\,(f)$. Hence $\mathrm{epi}\,(f)$ is $(E^+,F^+)$-convex on $R^{n+1}$.

Conversely, suppose that $\mathrm{epi}\,(f)$ is $(E^+,F^+)$-convex on $R^{n+1}$, then, for $\forall x,y \in M$, we have $(x,f(x)),(y,f(y)) \in \mathrm{epi}\,(f)$, and by the $(E^+,F^+)$-convexity of $\mathrm{epi}\,(f)$, we get for $\forall\lambda \in [0,1]$

$$\lambda E^+(x,f(x)) + (1-\lambda)F^+(y,f(y)) = (\lambda E(x) + (1-\lambda)F(y), \lambda f(x) + (1-\lambda)f(y)) \subseteq \mathrm{epi}\,(f).$$

Since $M$ is $(E,F)$-convex, then we get $\lambda E(x) + (1-\lambda)F(y) \subseteq M$. Hence

$$f(\lambda\overline{x} + (1-\lambda)\overline{y}) \le \lambda f(x) + (1-\lambda)f(y),\ \forall\overline{x} \in E(x),\ \forall\overline{y} \in F(y),\ \forall\lambda \in [0,1],$$

this shows that $f$ is semi-$(E,F)$-convex on $M$. □

## 3. Semi-$(E,F)$-Convex Programming

In this section, we consider the following nonlinear programming problem

$$(P) \qquad \min\,\{f(x) \quad x \in M\}.$$

**Definition 3.1.** The program $(P)$ is said to be a semi-$(E,F)$-convex program if there are two point-to-set maps $E,F : M \to 2^M$ such that the feasible set $M$ is $(E,F)$-convex and the objective function $f$ is semi-$(E,F)$-convex on $M$.

**Theorem 3.1.** *The set of optimal solutions of the semi-$(E,F)$-convex programming $(P)$ is an $(E,F)$-convex set.*

*Proof.* Let $\overline{x}$ be an optimal solution of $(P)$ and $\alpha = f(\overline{x})$, then the optimal solution set $X$ can be expressed as: $X = \{x \in M : f(x) \leq \alpha\} = K_\alpha$. So $X = K_\alpha$ is an $(E,F)$-convex set by Proposition 2.4. □

**Theorem 3.2.** *Suppose that a function $f : M \to R$ is differentiable and semi-$(E,F)$-convex on $(E,F)$-convex set $M$, $x^* \in M$ and $x^* \in E(x^*) \cup F(x^*)$. Then $x^*$ is an optimal solution of the semi-$(E,F)$-convex programming $(P)$ if and only if one of the following two statements holds*

*(i) If $x^* \in E(x^*)$, then $\nabla f(x^*)^T(\overline{y} - x^*) \geq 0,\ \forall y \in M,\ \forall \overline{y} \in F(y)$,*

*(ii) If $x^* \in F(x^*)$, then $\nabla f(x^*)^T(\overline{y} - x^*) \geq 0,\ \forall y \in M,\ \forall \overline{y} \in E(y)$.*

*Proof.* The proof of the necessary condition. Let $x^* \in E(x^*)$, noting that $M$ is $(E,F)$-convex, one has $(1-\lambda)E(x^*) + \lambda F(y) \subseteq M,\ \forall y \in M,\ \forall \lambda \in [0,1]$. Since $x^*$ is an optimal solution of the semi-$(E,F)$-convex programming $(P)$, for $\forall \overline{y} \in F(y) \subseteq M$, in view of $(1-\lambda)x^* + \lambda\overline{y} \in M$, by Taylor expansion, we have

$$f(x^*) \leq f((1-\lambda)x^* + \lambda\overline{y}) = f(x^*) + \lambda\nabla f(x^*)^T(\overline{y} - x^*) + o(\lambda), \lambda \in (0,1].$$

So $\lambda\nabla f(x^*)(\overline{y}-x^*)+o(\lambda) \geq 0$. Dividing this inequality by $\lambda$ and taking $\lambda \to +0$, we have $\nabla f(x^*)^T(\overline{y} - x^*) \geq 0$.

For the previous case, the proof is similar to the case of $x^* \in E(x^*)$.

The proof of the sufficiency. If $x^* \in E(x^*)$, we have from the $(E,F)$-convexity of the set $M$, $(1-\lambda)E(x^*) + \lambda F(y) \subseteq M, \quad \forall y \in M,\ \forall \lambda \in [0,1]$. Since $f$ is semi-$(E,F)$-convex on $M$, one has

$$f((1-\lambda)x^* + \lambda\overline{y}) \leq (1-\lambda)f(x^*) + \lambda f(y) = f(x^*) + \lambda(f(y) - f(x^*)), \qquad \forall \overline{y} \in F(y).$$

So, $f(y) - f(x^*) \geq \frac{f(x^*+\lambda(\overline{y}-x^*))-f(x^*)}{\lambda}$. Letting $\lambda \to +0$, we have $f(y) - f(x^*) \geq \bigtriangledown f(x^*)^T(\overline{y} - x^*) \geq 0$. This shows that $x^*$ is an optimal solution of $f$ on $M$. The case of $x^* \in F(x^*)$ is similar. □

**Theorem 3.3.** *Let $x^*$ be a local optimal solution of the semi-$(E,F)$-convex programming $(P)$, if $x^* \in E(x^*) \cup F(x^*)$, then $x^*$ is a global optimal solution of $(P)$.*

*Proof.* Suppose, by contradiction, that $x^*$ is not a global optimal solution, so $f(y) < f(x^*)$ for some $y \in M$. Let $x^* \in E(x^*)$ (the case of $x^* \in F(x^*)$ is similar), since $f$ is semi-$(E,F)$-convex on the $(E,F)$-convex set $M$, for $\forall \overline{y} \in$

$F(y)$ and $\forall\lambda \in (0,1)$, we have

$$f(x^* + \lambda(\overline{y} - x^*)) = f(\lambda\overline{y} + (1-\lambda)x^*) \leq \lambda f(y) + (1-\lambda)f(x^*)) < f(x^*).$$

But for $\lambda > 0$ small enough, this strict inequality contradicts the fact that $x^* + \lambda(\overline{y} - x^*) \in M \bigcap N_\epsilon(x^*)$ and $x^*$ being a local optimal solution, and the proof is completed. □

**Theorem 3.4.** *Suppose that function $f : M \to R$ is strictly semi-$(E,F)$-convex on an $(E,F)$-convex $M$, then the global optimal solution of semi-$(E,F)$-convex programming $(P)$ is unique.*

*Proof.* By contradiction, suppose that $x_1, x_2 \in M$ are two global optimal solutions of $(P)$ and $x_1 \neq x_2$, since $f$ is strictly semi-$(E,F)$-convex on the $(E,F)$-convex set $M$, for $\forall\overline{x}_1 \in E(x_1)$, $\forall\overline{x}_2 \in F(x_2)$, $\forall\lambda \in (0,1)$, we have

$$\begin{aligned}\lambda E(x_1) + (1-\lambda)F(x_2) \subseteq M, f(\lambda\overline{x}_1 + (1-\lambda)\overline{x}_2)&\\ < \lambda f(x_1) + (1-\lambda)f(x_2) = f(x_1).&\end{aligned}$$

This contradicts that $x_1$ is a global solution of $(P)$. Hence the global solution of $(P)$ is unique. □

In the subsequent discussions of this section, we apply the associated results on semi-$(E,F)$-convex programming $(P)$ to nonlinear programming problems with inequality constraints, consider

$$(P_g) \qquad \begin{array}{ll}\min & f(x)\\ \text{s.t.} & g_i(x) \leq 0, \quad i = 1,2,...,m,\\ & x \in R^n.\end{array}$$

Denote the feasible set of $(P_g)$ by $M_g = \{x \in R^n |\ g_i(x) \leq 0, \quad i = 1,2,\cdots,m\}$.

**Theorem 3.5.** *Suppose that there are two point-to-set maps $E, F : R^n \to 2^{R^n}$ such that $f, g_i, (i = 1,2,...,m)$ are all semi-$(E,F)$-convex functions on $R^n$. Then:*

*(i) The feasible set $M_g$ of problem $(P_g)$ is $(E,F)$-convex, moreover, $(P_g)$ is an semi-$(E,F)$-convex programming.*

*(ii) The optimal solution set of $(P_g)$ is $(E,F)$-convex.*

*(iii) Let $x^*$ be a local optimal solution of $(P_g)$, if $x^* \in E(x^*) \cup F(x^*)$, then $x^*$ is a global solution of $(P_g)$.*

*(iv) If $f$ is strictly semi-$(E,F)$-convex, then the global solution of $(P_g)$ is unique.*

*Proof.* (i) Since $g_i$ is semi-$(E,F)$-convex on $R^n$ for each $i$ and $M_g \subseteq R^n$, for $\forall x_1, x_2 \in M_g,\ \forall \overline{x}_1 \in E(x_1),\ \forall \overline{x}_2 \in F(x_2),\ \forall \lambda \in [0,1]$, one has

$$g_i(x_1) \le 0, \quad g_i(x_2) \le 0,$$

$$g_i(\lambda \overline{x}_1 + (1-\lambda)\overline{x}_2) \le \lambda g_i(x_1) + (1-\lambda) g_i(x_2) \le 0, i = 1, 2, \cdots, m.$$

So $\lambda \overline{x}_1 + (1-\lambda)\overline{x}_2 \in M_g$, that is, $\lambda E(x_1) + (1-\lambda)F(x_2) \subseteq M_g$. This shows that $M_g$ is an $(E,F)$-convex set and $(P_g)$ is a semi-$(E,F)$-convex programming.

Conclusions (ii), (iii) and (iv) are direct corollaries of Theorems 3.2, 3.4 and 3.5. □

## 4. Optimality Conditions

At first of this section, we further extend the concept of semi-$(E,F)$-convex function, then discuss the optimality conditions of the corresponding programming.

**Definition 4.1.** A function $f : M \to R$ is said to be quasi-semi-$(E,F)$-convex on an $(E,F)$-convex set $M$, if

$$f(\lambda \overline{x} + (1-\lambda)\overline{y}) \le \max\{f(x), f(y)\},$$
$$\forall x, y \in M,\ \forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y),\ \forall \lambda \in [0,1].$$

**Remark 4.1.** Each semi-$(E,F)$-convex function on an $(E,F)$-convex set $M$ is quasi-semi-$(E,F)$-convex on $M$.

**Definition 4.2.** A function $f : M \to R$ is said to be pseudo-semi-$(E,F)$-convex on an $(E,F)$-convex set $M$, if there is a function $b(x,y) : M \times M \to R$ such that

$$f(x) < f(y) \Longrightarrow f(\lambda \overline{x} + (1-\lambda)\overline{y}) \le f(y) + \lambda(\lambda - 1) b(x,y), b(x,y) > 0,$$

holds for $\forall x, y \in M,\ \forall \overline{x} \in E(x),\ \forall \overline{y} \in F(y),\ \forall \lambda \in (0,1)$.

**Remark 4.2.** If $f : M \to R$ is a semi-$(E,F)$-convex function on an $(E,F)$-convex set $M$, then $f$ is pseudo-semi-$(E,F)$-convex on $M$.

In fact, for semi-$(E,F)$-convex function $f$ and $f(x) < f(y)$ as well as $\lambda \in (0,1)$, one has

$$f(\lambda \overline{x} + (1-\lambda)\overline{y}) \le \lambda f(x) + (1-\lambda) f(y) = f(y) + \lambda(f(x) - f(y))$$
$$\le f(y) + \lambda(1-\lambda)(f(x) - f(y)) = f(y) + \lambda(\lambda - 1)(f(y) - f(x)).$$

So the required result is easily obtained by defining $b(x, y) = f(y) - f(x) > 0$.

The following lemma is useful in the subsequent discussions.

**Lemma 4.1** *Let $f(x)$ be a differentiable and pseudo-semi-$(E, F)$-convex function on $M$, if $x^* \in F(x^*)$, then*

$$f(x) < f(x^*) \Longrightarrow \nabla f(x^*)^T(\overline{x} - x^*) < 0, \quad \forall \overline{x} \in E(x).$$

*Proof.* Since $f(x)$ is pseudo-semi-$(E, F)$-convex on $M$, noticing that $x^* \in F(x^*)$ and $f(x) < f(x^*)$, we have from Definition 4.2, for any $\lambda \in (0, 1)$

$$f(\lambda\overline{x} + (1-\lambda)x^*) \leq f(x^*) + \lambda(\lambda - 1)b(x, x^*), \ b(x, x^*) > 0, \qquad \forall x \in M, \ \forall \overline{x} \in E(x).$$

Again, one has from the differentiability of $f$

$$\begin{aligned} &f(\lambda\overline{x} + (1-\lambda)x^*) \\ &\qquad = f(x^*) + \lambda\nabla f(x^*)^T(\overline{x} - x^*) + o(\lambda) \leq f(x^*) + \lambda(\lambda - 1)b(x, x^*). \end{aligned}$$

That is, $\lambda\nabla f(x^*)^T(\overline{x} - x^*) + o(\lambda) \leq \lambda(\lambda - 1)b(x, x^*)$. Dividing this inequality by $\lambda$ and taking $\lambda \to +0$, we have $\nabla f(x^*)^T(\overline{x} - x^*) \leq -b(x, x^*) < 0$. □

**Theorem 4.1.** (Karush-Kuhn-Tucker Sufficient Conditions) *Assume that the function $f(x)$ is differentiable and pseudo-semi-$(E, F)$-convex on $R^n$, $g_i(x) (i = 1, 2, \ldots, m)$ are differentiable and quasi-semi-$(E, F)$-convex functions on $R^n$, $x^*$ is a Kuhn-Tucker point of $(P_g)$, i.e., there exist multipliers $u_i (i = 1, 2, \ldots, m)$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = 0; u_i \geq 0, g_i(x^*) \leq 0, u_i g_i(x^*) = 0, \qquad i = 1, 2, \ldots, m.$$

*If $x^* \in F(x^*)$, then $x^*$ is an optimal solution of the problem $(P_g)$.*

*Proof.* For any $x \in M_g$, we have $g_i(x) \leq 0 = g_i(x^*), i \in I(x^*) = \{i : g_i(x^*) = 0\}$. Therefore, from the quasi-semi-$(E, F)$-convexity of $g_i$ and $x^* \in F(x^*)$, we obtain $g_i(\lambda\overline{x} + (1-\lambda)x^*) \leq \max\{g_i(x), g_i(x^*)\} = 0$, $\forall \overline{x} \in E(x)$, $\forall \lambda \in (0, 1)$.

On the other hand, by the differentiability of $g_i(x)$, one gets

$$0 \geq g_i(\lambda\overline{x} + (1-\lambda)x^*) = g_i(x^*) + \lambda\nabla g_i(x^*)(\overline{x} - x^*) + o(\lambda).$$

Noting that $g_i(x^*) = 0$ for $i \in I(x^*)$, one has $\lambda \nabla g_i(x^*)(\overline{x} - x^*) + o(\lambda) \leq 0$. Dividing the above inequality by $\lambda$ and taking $\lambda \to +0$, we have

$$\nabla g_i(x^*)(\overline{x} - x^*) \leq 0, \quad \forall i \in I(x^*), \ \forall x \in M_g.$$

Thus, using the KKT conditions and multipliers $u_i \geq 0$, one has

$$\nabla f(x^*)^T(\overline{x} - x^*) = -\sum_{i=1}^{m} u_i \nabla g_i(x^*)^T(\overline{x} - x^*) = -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(\overline{x} - x^*) \geq 0.$$

Hence, from Lemma 4.1, one can conclude $f(x) \geq f(x^*)$, therefore $x^*$ is an optimal solution of $(P_g)$. □

**Corollary 4.1.** *Suppose that $f(x), g_i(x)( \ i = 1, 2, \ldots, m)$ are all differentiable and semi-$(E,F)$-convex functions on $R^n$, and $x^*$ be a Karush-Kuhn-Tucker point of $(P_g)$. If $x^* \in E(x^*) \cup F(x^*)$, then $x^*$ is an optimal solution of $(P_g)$.*

*Proof.* If $x^* \in F(x^*)$, noting that $f(x)$ is also pseudo-semi-$(E,F)$-convex on $R^n$ and $g_i(x)$ $(i = 1, 2, \ldots, m)$ are also quasi-semi-$(E,F)$-convex on $R^n$ (see Remark 4.1 and Remark 4.2), the required result holds from Theorem 4.1.

Now we discuss the case of $x^* \in E(x^*)$. For each $x \in M_g$, we get $g_i(x) \leq 0 = g_i(x^*), \ i \in I(x^*) = \{i : \ \ g_i(x^*) = 0\}$.

Therefore, using the quasi-semi-$(E,F)$-convexity of functions $g_i$, similar to the proof of Theorem 4.1, we have

$$\nabla g_i(x^*)^T(\overline{x} - x^*) \leq 0, \quad \forall i \in I(x^*), \quad \forall x \in M_g, \ \forall \overline{x} \in F(x).$$

Using the Karush-Kuhn-Tucker conditions and multipliers $u_i \geq 0$, we have

$$\nabla f(x^*)^T(\overline{x} - x^*) = -\sum_{i=1}^{m} u_i \nabla g_i(x^*)^T(\overline{x} - x^*) = -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(\overline{x} - x^*) \geq 0\,.$$

Again, in view of the differentiability of $f(x)$, one has

$$\lambda f(x) + (1-\lambda) f(x^*) \geq f(\lambda \overline{x} + (1-\lambda)x^*) = f(x^*) + \lambda \nabla f(x^*)^T(\overline{x} - x^*) + o(\lambda),$$

that is, $\lambda \nabla f(x^*)^T(\overline{x} - x^*) + o(\lambda) \leq \lambda(f(x) - f(x^*))$. Dividing this inequality by $\lambda$ and taking $\lambda \to +0$, we have

$$0 \leq \nabla f(x^*)^T(\overline{x} - x^*) \leq f(x) - f(x^*),\ f(x) \geq f(x^*),\quad \forall x \in M_g.$$

So $x^*$ is an optimal solution of $(P_g)$. □

## 5. Duality Theorems

In this section, we will study the Wolfe Duality Theorems of $(P_g)$ under the semi-$(E, F)$-convexity. If functions $f$ and $g_i$ are all differentiable on $R^n$, then the Wolfe Dual Problem of $(P_g)$ can be expressed as (see [9])

$$(D_g) \quad \begin{array}{ll} \max & f(z) + u^T g(z) \\ \text{s.t.} & \nabla f(z) + \nabla g(z) u = 0, \\ & u \geq 0, \end{array}$$

where $u = (u_1, u_2, \ldots, u_m)^T$, $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))^T$. For convenience, we denote the feasible set of $(D_g)$ by $\overline{M}_g = \{(z, u) : \nabla f(z) + \nabla g(z) u = 0, u \geq 0\}$.

**Theorem 5.1.** (Weak Duality Theorem) *Suppose that $f(x), g_i(x)$ $(i = 1, 2, \ldots, m)$ are all differentiable and semi-$(E, F)$-convex functions on $R^n$. If $z \in F(z) \cup E(z)$ holds for each $(z, u) \in \overline{M}_g$, then $f(x) \geq f(z) + u^T g(z)$ holds for each feasible point $x$ of $(P_g)$ and each feasible point $(z, u)$ of $(D_g)$, i.e.,*

$$f(x) \geq f(z) + u^T g(z),\quad \forall x \in M_g,\quad \forall (z, u) \in \overline{M}_g.$$

*Proof.* Suppose that $x$ is a feasible solution of $(P_g)$ and $(z, u)$ is a feasible solution of $(D_g)$, and assume $z \in F(z)$ (the proof in the case of $z \in E(z)$ is similar), since $f(x)$ is semi-$(E, F)$-convex on $R^n$, we have

$$f(\lambda\overline{x} + (1 - \lambda)z) \leq \lambda f(x) + (1 - \lambda) f(z),\ \forall \overline{x} \in E(x),\ \forall \lambda \in (0, 1].$$

Again, one has by Taylor expansion

$$f(\lambda\overline{x} + (1 - \lambda)z) = f(z) + \lambda \nabla f(z)^T(\overline{x} - z) + o(\lambda) \leq \lambda f(x) + (1 - \lambda) f(z),$$

that is, $\lambda \nabla f(z)^T(\overline{x} - z) + o(\lambda) \leq \lambda(f(x) - f(z))$. Dividing this inequality by $\lambda$ and taking $\lambda \to +0$, one has $\nabla f(z)^T(\overline{x} - z) \leq f(x) - f(z)$.

Similarly, we have $\nabla g(z)^T(\overline{x} - z) \leq g(x) - g(z)$.

Also, by the dual feasibility, we have $\nabla f(z)^T(\overline{x}-z) = -(\nabla g(z)u)^T(\overline{x}-z) = -u^T\nabla g(z)^T(\overline{x}-z)$. Taking into account $u \geq 0$ and $g(x) \leq 0$, we get

$$\begin{aligned} f(x)-f(z) \geq \nabla f(z)^T(\overline{x}-z) &= -u^T\nabla g(z)^T(\overline{x}-z) \\ &\geq u^T(g(z)-g(x)) = u^Tg(z) - u^Tg(x) \geq u^Tg(z), \end{aligned}$$

This shows that $f(x) \geq f(z) + u^Tg(z)$. □

We obtain four corollaries from Theorem 5.1, and their proofs are easily completed.

**Corollary 5.1.** *Suppose that the conditions of Theorem 5.1 are satisfied, then*

$$\inf\{f(x) : x \in M_g\} \geq \sup\{f(z) + u^Tg(z) : (z,u) \in \overline{M}_g\}.$$

**Corollary 5.2.** *Suppose that the conditions of Theorem 5.1 are satisfied. If there exist a feasible point $\overline{x}$ of $(P_g)$ and a feasible point $(\overline{z},\overline{u})$ of $(D_g)$ such that $f(\overline{x}) = f(\overline{z}) + \overline{u}^Tg(\overline{z})$, then $\overline{x}$ and $(\overline{z},\overline{u})$ solve the problem $(P_g)$and the problem $(D_g)$, respectively.*

**Corollary 5.3.** *Under the conditions of Theorem 5.1, if the optimal value of $(P_g)$ equals $-\infty$, then the optimal value of $(D_g)$ equals $-\infty$, too.*

**Corollary 5.4.** *Under the conditions of Theorem 5.1, if the optimal value of $(D_g)$ is $+\infty$, then $(P_g)$ has no feasible solution.*

**Theorem 5.2.** (Strong Duality Theorem) *Suppose that $f(x), g_i(x)$ $(i = 1, 2, \ldots, m)$ are all differentiable and semi-$(E,F)$-convex functions on $R^n$, $(x^*, u^*)$ is a KKT pair of $(P_g)$ and $x^* \in E(x^*) \cup F(x^*)$ (so $x^*$ is an optimal solution of $(P_g)$ from Corollary 4.1). If $z \in F(z) \cup E(z)$ holds for each feasible point $(z,u)$ of $(D_g)$, then $(x^*, u^*)$ is an optimal solution of $(D_g)$ and the optimal values of $(P_g)$ and $(D_g)$ are equal.*

*Proof.* Let $(z,u) \in \overline{M}_g$, and assume that $z \in F(z)$ (the discussions for $z \in E(z)$ is similar), noting that $f(x)$ is semi-$(E,F)$-convex on $R^n$ and $(x^*, u^*)$ is a feasible solution of $(D_g)$, we have

$$f(\lambda\overline{x^*} + (1-\lambda)z) \leq \lambda f(x^*) + (1-\lambda)f(z), \quad \forall \overline{x^*} \in E(x^*),\ \forall\lambda \in (0,1].$$

Since $f(x)$ is differentiable, one has

$$f(\lambda\overline{x^*} + (1-\lambda)z) = f(z) + \lambda\nabla f(z)^T(\overline{x^*}-z) + o(\lambda) \leq \lambda f(x^*) + (1-\lambda)f(z),$$

that is, $\lambda\nabla f(z)^T(\overline{x^*}-z) + o(\lambda) \leq \lambda(f(x^*)-f(z))$. Dividing this inequality by $\lambda$ and taking $\lambda \to +0$, we have $\nabla f(z)^T(\overline{x^*}-z) \leq f(x^*) - f(z)$.

Similarly, we have $\nabla g(z)^T(\overline{x^*}-z) \leq g(x^*)-g(z)$. Therefore, taking into account $u \geq 0, g(x^*) \leq 0, g(x^*)^T u^* = 0$, we have

$$\begin{aligned} f(x^*) + u^{*T} g(x^*) - f(z) - u^T g(z) &= f(x^*) - f(z) - u^T g(z) \\ &\geq \nabla f(z)^T(\overline{x^*} - z) - u^T g(z) \\ &= -u^T \nabla g(z)^T(\overline{x^*} - z) - u^T g(z) \\ &= -u^T(\nabla g(z)^T(\overline{x^*} - z) + g(z)) \\ &\geq -u^T g(x^*) \geq 0. \end{aligned}$$

This shows that $(x^*, u^*)$ is an optimal solution of $(D_g)$. Noting that $u^{*T} g(x^*) = 0$, one knows the optimal values of $(P_g)$ and $(D_g)$ are equal. □

## Acknowledgements

Project supported by the National Natural Science Foundation (No. 10261001) and Guangxi Science Foundation (No. 0236001, 0249003) of P.R. China.

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