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CHARACTERIZATIONS OF FIXED POINTS FOR CONTINUOUS MAPPINGS ON COMPACT HAUSDORFF SPACES

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Abstract: Necessary and sufficient conditions for the existence of fixed points of continuous self mappings over compact Hausdorff spaces are given. Our results extend some results of Jungck [1], [2], [3], Liu [7], [8] and Liu, Zhang and Kang [10].

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1. Introduction and Preliminaries

Jungck [1], [2], [3] first gave some necessary and sufficient conditions for the existence of fixed points of continuous self mappings on complete metric spaces. Afterwards, Khan and Fisher [4] established a few theorems similar to that of Jungck [1], [2], [3].

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Liu [7], [8], [5], [9], [6] and Liu, Zhang and Kang [10] obtained some sufficient conditions or necessary and sufficient conditions for the existence of fixed points of any self mapping on compact metric spaces and compact Hausdorff spaces, respectively. The main aim of this paper is to offer some characterizations for the existence of fixed points of continuous self mappings on compact Hausdorff spaces. We also establish some common fixed points theorems for a pair of continuous mappings. The results presented in this paper extend, improve and unify the corresponding results due to Jungck [1], [2], [3] , Liu [7], [8] and Liu, Zhang and Kang [10].

Throughout this paper, we assume that (X,τ) is a compact Hausdorff spaces, F denotes a continuous function from $X \times X$ into $[0,+\infty)$ such that $F(x,y)=0$ if and only if $x=y$, ω and N denote the sets of nonnegative integers and positive integers, respectively. For $B \subseteq X$, $x,y \in X$ and self mappings f,g on X, define

$$\begin{aligned}
&\delta_F(B) = \sup\{F(a,b) : a,b \in B\},\\
&C_f = \{g : g : X \to X \text{ and } gf = fg\},\\
&A_f = \{h : h : X \to X \text{ and } h \cap_{n\in N} f^n X = \cap_{n\in N} f^n X\},\\
&H_f = \{h : h : X \to X \text{ and } h \cap_{n\in N} f^n X = \cap_{n\in N} f^n X\},\\
&O_{f,g}(x) = \{f^i g^j x : i,j \in \omega\},\\
&O_{f,g}(x,y) = O_{f,g}(x) \cup O_{f,g}(y).
\end{aligned}$$

It is clear that $H_f \supseteq A_f \cup C_f$ and $\{f^n : n \in \omega\} \subseteq C_f$.

Lemma 1.1. (see [8]) *Let f,g be self mappings on (X,τ) such that fg is continuous and $g \in C_f$. Let $A = \cap_{n\in N}(fg)^n X$. Then:*

(a) *$hA \subseteq A$ for $h \in C_{fg}$;*

(b) *$A = fgA = fA = gA \neq \emptyset$;*

(c) *A is compact.*

2. Main Results

Theorem 2.1. *Let f be a continuous self mapping of (X,τ) and F be a symmetric function. Then the following statements are equivalent:*

$$f \text{ has a fixed point;} \tag{2.1}$$

There exists $h \in C_f$ such that

$$F(fx,fy) > F(hx,hy), \tag{2.2}$$

for all $x, y \in X$ *with* $fx \neq fy$*;*

For each $x, y \in X$ *with* $fx \neq fy$*, there exists* $h \in C_f$ *such that*

$$F(fx, fy) > F(hx, hy); \tag{2.3}$$

For each $x, y \in X$ *with* $fx \neq fy$*, there exists* $h \in C_f$ *such that*

$$F(fx, fy) > \min\{F(hx, hy), F(x, fx), F(y, fy)\}. \tag{2.4}$$

For each $x, y \in X$ *with* $fx \neq fy$*, there exists* $h \in C_f$ *such that*

$$\begin{aligned} F(fx, fy) > \min\Big\{&F(hx, hy), F(x, fx), F(y, fy), \\ &\frac{F(x, fx)F(y, fy)}{F(fx, fy)}, \frac{F(hx, hy)F(y, fy)}{F(fx, fy)}, \\ &\frac{F(hx, hy)F(x, fx)}{F(fx, fy)}, \frac{F^2(x, fx)}{F(fx, fy)}, \frac{F^2(y, fy)}{F(fx, fy)}, \frac{F^2(hx, hy)}{F(fx, fy)}\Big\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &d(fx, fy) \\ &\quad > \inf\{F(hx, hy), F(f^k x, f^{k+1}x), F(f^k y, f^{k+1}y) : k \in \omega, h \in C_f\}, \end{aligned} \tag{2.6}$$

for all $x, y \in X$ *with* $fx \neq fy$*;*

$$\begin{aligned} F(fx, fy) > \inf\Big\{&F(f^k x, f^{k+1}x), F(f^k y, f^{k+1}y), F(hx, hy), \\ &\frac{F(f^k x, f^{k+1}x)F(f^m y, f^{m+1}y)}{F(fx, fy)}, \frac{F(f^k x, f^{k+1}x)F(hx, hy)}{F(fx, fy)}, \\ &\frac{F(f^k y, f^{k+1}y)F(hx, hy)}{F(fx, fy)}, \frac{F(hx, hy)F(tx, ty))}{F(fx, fy)}, \\ &\frac{F(f^k x, f^{k+1}x)F(f^m x, f^{m+1}x)}{F(fx, fy)}, \frac{F(f^k y, f^{k+1}y)F(f^m y, f^{m+1}y)}{F(fx, fy)} \\ &: k, m \in \omega,\ h, t \in C_f\Big\}, \end{aligned} \tag{2.7}$$

for all $x, y \in X$ *with* $fx \neq fy$*;*

$$F(fx, fy) > \inf\{F(hx, hy), F(hx, fhx), F(hy, fhy) : h \in C_f\} \tag{2.8}$$

for all $x, y \in X$ *with* $fx \neq fy$*;*

$$F(fx, fy) > \inf\Big\{F(hx, fhx), F(hy, fhy), F(hx, hy), \frac{F(hx, fhx)F(ty, fty)}{F(fx, fy)}, \frac{F(hx, fhx)F(tx, ty)}{F(fx, fy)}, \frac{F(hy, fhy)F(tx, ty)}{F(fx, fy)}, \frac{F(hx, fhx)F(tx, ftx)}{F(fx, fy)}, \frac{F(hy, fhy)F(ty, fty)}{F(fx, fy)}, \frac{F(hx, hy)F(tx, ty)}{F(fx, fy)} : h, t \in C_f\Big\}, \quad (2.9)$$

for all $x, y \in X$ *with* $fx \neq fy$*;*

There exists $h \in C_f$ *such that*

$$F(x, y) > F(hx, hy), \quad (2.10)$$

for any $x, y \in X$ *with* $x \neq y$*;*

For each $x, y \in X$ *with* $x \neq y$*, there exists* $h \in C_f$ *such that*

$$F(x, y) > F(hx, hy). \quad (2.11)$$

For each $x, y \in X$ *with* $x \neq y$*, there exists* $h \in C_f$ *such that*

$$F(x, y) > \min\{F(hx, hy), F(x, fx), F(y, fy)\}. \quad (2.12)$$

For each $x, y \in X$ *with* $x \neq y$*, there exists* $h \in C_f$ *such that*

$$F(x, y) > \min\Big\{F(hx, hy), F(x, fx), F(y, fy), \frac{F(hx, hy)F(x, fx)}{F(x, y)}, \frac{F(hx, hy)F(y, fy)}{F(x, y)}, \frac{F(x, fx)F(y, fy)}{F(x, y)}, \frac{F^2(hx, hy)}{F(x, y)}, \frac{F^2(y, fy)}{F(x, y)}, \frac{F^2(x, fx)}{F(x, y)}\Big\}, \quad (2.13)$$

$$F(x, y) > \inf\{F(f^k x, f^{k+1} x), F(f^k y, f^{k+1} y), F(hx, hy) : k \in \omega,\ h \in C_f\}, \quad (2.14)$$

for any $x, y \in X$ *with* $x \neq y$*;*

$$F(x, y) > \inf\Big\{F(f^k x, f^{k+1} x), F(f^k y, f^{k+1} y), F(hx, hy), \frac{F(f^k x, f^{k+1} x)F(f^m y, f^{m+1} y)}{F(x, y)}, \frac{F(f^k x, f^{k+1} x)F(hx, hy)}{F(x, y)},$$

$$\frac{F(f^ky, f^{k+1}y)F(hx, hy)}{F(x, y)}, \frac{F(hx, hy)F(tx, ty)}{F(x, y)},$$
$$\frac{F(f^kx, f^{k+1}x)F(f^mx, f^{m+1}x)}{F(x, y)}, \frac{F(f^ky, f^{k+1}y)F(f^my, f^{m+1}y)}{F(x, y)}$$
$$: k, m \in \omega,\ h, t \in C_f\Big\}, \quad (2.15)$$

for any $x, y \in X$ *with* $x \neq y$;

$$F(x, y) > \inf\{F(hx, fhx), F(hy, fhy), F(hx, hy) : h \in C_f\}, \quad (2.16)$$

for any $x, y \in X$ *with* $x \neq y$;

$$F(x, y) > \inf\Big\{F(hx, fhx), F(hy, fhy), F(hx, hy),$$
$$\frac{F(hx, fhx)F(ty, fty)}{F(x, y)}, \frac{F(hx, fhx)F(tx, ty)}{F(x, y)},$$
$$\frac{F(hy, fhy)F(tx, ty)}{F(x, y)}, \frac{F(hx, fhx)F(tx, ftx)}{F(x, y)}, \frac{F(hy, fhy)F(ty, fty)}{F(x, y)},$$
$$\frac{F(hx, hy)F(tx, ty)}{F(x, y)} : h, t \in C_f\Big\}, \quad (2.17)$$

for any $x, y \in X$ *with* $x \neq y$;

There exists a continuous mapping $h \in C_f$ *satisfying*

$$F(hx, hy) < \max\{F(fx, fy), F(fx, hx), F(fy, hy),$$
$$\frac{F(fx, hx)F(fy, hy)}{F(fx, fy)}, \frac{F(hx, hy)F(fx, hx)}{F(fx, fy)},$$
$$\frac{F(fx, hy)F(hx, fy)}{F(fx, fy)}, \frac{F^2(fx, hx)}{F(fx, fy)}\Big\}, \quad (2.18)$$

for all $x, y \in X$ *with* $fx \neq fy$;

There exist $m, n \in N$ *and a continuous mapping* $h \in C_f$ *such that*

$$F(h^mx, h^ny) < F(fx, fy), \quad (2.19)$$

for $x, y \in X$ *with* $h^mx \neq h^ny$;

There exist $m, n \in N$ *and a continuous mapping* $g \in C_f$ *such that*

$$F(g^mx, g^ny) < \delta_F(\{hz : z \in O_{f,g}(x, y) \text{ and } h \in C_{gf}\}), \quad (2.20)$$

for $x, y \in X$ *with* $g^mx \neq g^ny$;

There exist $m, n \in N$ and a continuous mapping $g \in C_f$ such that

$$F(g^m x, g^n y) < \delta_F(\{hz : z \in O_{f,g}(x,y) \text{ and } h \in H_{gf}\}), \tag{2.21}$$

for $x, y \in X$ with $g^m x \neq g^n y$;

There exist $m, n \in N$ and a continuous mapping $g : X \to X$ such that $f \in A_{gf}$ and

$$F(g^m x, g^n y) < \delta_F(\{hz : z \in O_{f,g}(x,y) \text{ and } h \in H_{gf}\}), \tag{2.22}$$

for $x, y \in X$ with $g^m x \neq g^n y$.

Proof. Let w be a fixed point of f in X. Set $gx = hx = w$ for any $x \in X$. It is easy to verify that $g, h \in C_f$ and (2.2), (2.10), (2.18), (2.19), (2.21) and (2.22) hold. (2.2) $\Rightarrow$ (2.3) $\Rightarrow$ (2.4) $\Rightarrow$ (2.5) $\Rightarrow$ (2.7) $\Rightarrow$ (2.9), (2.4) $\Rightarrow$ (2.6) $\Rightarrow$ (2.8) $\Rightarrow$ (2.9), (2.10) $\Rightarrow$ (2.11) $\Rightarrow$ (2.12) $\Rightarrow$ (2.13) $\Rightarrow$ (2.15) $\Rightarrow$ (2.17), (2.12) $\Rightarrow$ (2.14) $\Rightarrow$ (2.16) $\Rightarrow$ (2.17) and (2.19) $\Rightarrow$ (2.20) $\Rightarrow$ (2.21) are clear.

Now we shall verify the following implications: (2.9) $\Rightarrow$ (2.1), (2.17) $\Rightarrow$ (2.1), (2.18) $\Rightarrow$ (2.1), (2.21) $\Rightarrow$ (2.1) and (2.22) $\Rightarrow$ (2.1).

(2.9) $\Rightarrow$ (2.1) Let $A = \cap_{n \in N} f^n X$. It follows from Lemma 1.1 that A is a nonempty compact subset, $A = fA$ and $hA \subseteq A$ for $h \in C_f$. Since f and F are continuous and A is compact, there exists $a \in A$ such that $F(a, fa) = \inf\{F(x, fx) : x \in A\}$. Since $A = fA$, there exists $w \in A$ such that $a = fw$. Suppose that $a \neq fa$. That is, $fw \neq ffw$. By (2.9) we obtain that

$$\begin{aligned}
F(a, fa) &= F(fw, ffw) \\
&> \inf\Big\{F(hw, fhw), F(hfw, fhfw), F(hw, hfw), \\
&\quad \frac{F(hw, fhw)F(tfw, ftfw)}{F(fw, ffw)}, \frac{F(hw, fhw)F(tw, tfw)}{F(fw, ffw)}, \\
&\quad \frac{F(hfw, fhfw)F(tw, tfw)}{F(fw, ffw)}, \frac{F(hw, fhw)F(tw, tfw)}{F(fw, ffw)}, \\
&\quad \frac{F(hfw, fhfw)F(tfw, ftfw)}{F(fw, ffw)}, \frac{F(hw, hfw)F(tw, tfw)}{F(fw, ffw)} : h, t \in C_f\Big\} \\
&\geq \inf\{F(hw, fhw), F(hfw, fhfw), \frac{F(hw, fhw)F(tfw, ftfw)}{F(fw, ffw)}, \\
&\quad \frac{F(hw, fhw)F(tw, ftw)}{F(fw, ffw)}, \frac{F(hfw, fhfw)F(tfw, ftfw)}{F(fw, ffw)}, \\
&\quad \frac{F(hfw, fhfw)F(tw, ftw)}{F(fw, ffw)} : h, t \in C_f\Big\} = F(a, fa),
\end{aligned}$$

which is a contradiction. Hence a is a fixed point of f. That is, (2.1) holds.

(2.17) $\Rightarrow$ (2.1) Since f and F are continuous and X is compact, there exists a point $w \in X$ such that $F(w, fw) = \inf\{F(x, fx) : x \in X\}$. If $w \neq fw$, then

$$\begin{aligned} F(w, fw) > \inf \Big\{ & F(hw, fhw), F(hfw, fhfw), F(hw, hfw), \\ & \frac{F(hw, fhw)F(tfw, ftfw)}{F(w, fw)}, \frac{F(hw, fhw)F(tw, tfw)}{F(w, fw)}, \\ & \frac{F(hfw, fhfw)F(tw, tfw)}{F(w, fw)}, \frac{F(hw, fhw)F(tw, tfw)}{F(w, fw)}, \\ & \frac{F(hfw, fhfw)F(tfw, ftfw)}{F(w, fw)}, \frac{F(hw, fhw)F(tw, tfw)}{F(w, fw)} : h, t \in C_f \Big\} \\ \geq \inf\{ & F(hw, fhw), F(hfw, fhfw), \frac{F(hw, fhw)F(tfw, ftfw)}{F(w, fw)}, \\ & \frac{F(hw, fhw)F(tw, ftw)}{F(w, fw)}, \frac{F(hfw, fhfw)F(tfw, ftfw)}{F(w, fw)}, \\ & \frac{F(hfw, fhfw)F(tw, ftw)}{F(w, fw)} : h, t \in C_f \Big\} = F(w, fw), \end{aligned}$$

which is impossible. Hence $w = fw$.

(2.18) $\Rightarrow$ (2.1) Let $A = \cap_{n \in N} f^n X$. From Lemma 1.1, we infer that A is compact and $fA = A$. Since h is continuous and $h \in C_f$, we deduce that $hA = h \cap_{n\in N} f^n X \subseteq \cap_{n\in N} f^n hX \subseteq A = fA$. Define a function $\Phi(x)$ by $\Phi(x) = F(fx, hx)$ for all $x \in A$. It is clear that $\Phi(x)$ is continuous on A and attains its minimum value at some $w \in A$. We claim that $fw = hw$. If not, then there exists $p \in A$ satisfying $hw = fp \neq fw$. Using (2.18), we conclude that

$$\begin{aligned} \Phi(p) &= F(fp, hp) = F(hw, hp) \\ &< \max \Big\{ F(fw, fp), F(fw, hw), F(fp, hp), \frac{F(fw, hw)F(fp, hp)}{F(fw, fp)}, \\ &\quad \frac{F(hw, hp)F(fw, hw)}{F(fw, fp)}, \frac{F(fw, hp)F(hw, fp)}{F(fw, fp)}, \frac{F^2(fw, hw)}{F(fw, fp)} \Big\} \\ &= \max\{F(fw, hw), F(fp, hp)\} = \max\{\Phi(w), \Phi(p)\} = \Phi(p), \end{aligned}$$

which is a contradiction. Hence $fw = hw$. By virtue of $h \in C_f$, we derive that

$$hfw = fhw = ffw. \tag{2.23}$$

Now suppose that $ffw \neq fw$. By (2.18) and (2.23), we get that

$$\begin{aligned} F(ffw, fw) &= F(hfw, hw) \\ &< \max \Big\{ F(ffw, fw), F(ffw, hfw), F(fw, hw), \\ &\quad \frac{F(ffw, hfw)F(fw, hw)}{F(ffw, fw)}, \frac{F(hfw, hw)F(ffw, hfw)}{F(ffw, fw)}, \\ &\quad \frac{F(ffw, hw)F(hfw, fw)}{F(ffw, fw)}, \frac{F^2(ffw, hw)}{F(ffw, fw)} \Big\} = F(ffw, fw), \end{aligned}$$

which is a contradiction. Thus $ffw = fw$, that is, fw is a fixed point of f.

(2.22) $\Rightarrow$ (2.1) Let $B = \cap_{n \in N} (gf)^n X$. By Lemma 1.1 we have $gfB = B \neq \emptyset$. Note that $f \in A_{gf}$. Then $fB = B$, $B = gfB = gB$. We now assert that B is a singleton. Otherwise $\delta_F(B) > 0$. Since B is compact and F is continuous, there exist $a, b \in B$ such that $F(a, b) = \delta_F(B)$. From $gB = fB = B$, we can find $x, y \in B$ such that $g^m x = a$, $g^n y = b$. Consequently,

$$\delta_F(B) = F(g^m x, g^n y) < \delta_F(\{hz : z \in O_{f,g}(x, y) \text{ and } h \in H_{gf}\}) \leq \delta_F(B),$$

which is a contradiction. Hence $\delta_F(B) = 0$. This implies that $B = \{w\}$ for some $w \in X$. Clearly w is a fixed point of f. Similarly we can prove that (2.21) implies (2.1). This completes the proof. □

Remark 2.1. Theorem 2.1 generalizes the Theorem of Jungck [1], Corollary 2.3 of Jungck [2], Corollary 2 of Liu [7] and Theorem 4 of Liu [8].

As the immediate consequences of Theorem 2.1, we have the following results.

Theorem 2.2. *Let f be a continuous self mapping of (X, τ) satisfying*

$$\begin{aligned} F(fx, fy) > \inf \Big\{ & F(hx, fhx), F(hy, fhy), F(hx, hy), \\ & \frac{F(hx, fhx)F(tx, fty)}{F(fx, fy)}, \frac{F(hx, fhx), F(tx, ty)}{F(fx, fy)}, \\ & \frac{F(hy, fhy)F(tx, ty)}{F(fx, fy)}, \frac{F(hx, fhx)F(tx, ftx)}{F(fx, fy)}, \\ & \frac{F(hy, fhy)F(ty, fty)}{F(fx, fy)}, \frac{F(hx, hy)F(tx, ty)}{F(fx, fy)} : h, t \in C_f \Big\}, \end{aligned}$$

for all $x, y \in X$ with $fx \neq fy$. Then f has a fixed point.

Remark 2.2. Theorem 2.2 extends Theorem 4 of Liu [8].

Theorem 2.3. *Let f and g be continuous self mappings of (X,τ) with $f \in A_{gf}$. If there exist $m, n \in N$ such that*

$$F(g^m x, g^n y) < \delta_F(\{hz : z \in O_{f,g}(x,y) \text{ and } h \in H_{gf}\}),$$

for $x, y \in X$ with $g^m x \neq g^n y$. Then f and $g$have a unique common fixed point.

Remark 2.3. Corollary 4.3 of Jungck [3] and Corollary 1 of Liu [7] is a special case of Theorem 2.3.

Theorem 2.4. *Let f and g be continuous self mappings of (X,τ) with $g \in C_f$. If there exist a continuous mapping $h : X \to X$ satisfying*

$$F(hx,hy) < \max\Big\{F(fx,fy), F(fx,hx), F(fy,hy), \frac{F(fx,hx)F(fy,hy)}{F(fx,fy)}, \frac{F(hx,hy)F(fx,hx)}{F(fx,fy)}, \frac{F(fx,hy)F(hx,fy)}{F(fx,fy)}, \frac{F^2(fx,hx)}{F(fx,fy)}\Big\},$$

for all $x, y \in X$ with $fx \neq fy$.Then f has a fixed point. Indeed, f and g have a unique common fixed point.

Remark 2.4. Theorem 2.4 improves Theorem 2.2 of Liu, Zhang and Kang [10].

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References

[1] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly*, **83** (1976), 261-263.

[2] G. Jungck, Periodic and fixed points, and commuting mappings, *Proc. Amer. Math. Soc.*, **76** (1979), 333-338.

[3] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, **103** (1988), 977-983.

[4] M. S. Khan, B. Fisher, Some fixed point theorems for commuting mappings, *Math. Nachr.*, **106** (1982), 323-326.

[5] Z. Liu, A note on fixed point in compact metric spaces, *Indian J. Math.*, **34** (1992), 173-176.

[6] Z. Liu, A fixed point theorem in compact metric spaces, *Bull. Malaysian Math. Soc.*, **15** (1992), 69-71.

[7] Z. Liu, Characterizations of fixed point and common fixed points, *Acta Sci. Math.*, **59** (1994), 579-584

[8] Z. Liu, Some fixed point theorems in compact Hausdorff spaces, *Indian J. Math.*, **36** (1994), 235-239.

[9] Z. Liu, Fixed points on pseudocompact Tichonov spaces, *Soochow J. Math.*, **20** (1994), 393-399.

[10] Z. Liu, L. Zhang, S. M. Kang, On charactericatiοns of fixed points, *Internat. J. Math. and Math. Sci.*, **27** (2001), 391-397.