International Journal of Pure and Applied Mathematics
Volume 14 No. 4 2004, 505-514

OSCILLATION CRITERIA FOR IMPULSIVE PARABOLIC DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

Peiguang Wang[1 §], Yonghong Wu[2]
[1]College of Electronic and Information Engineering
Hebei University
Baoding 071002, P.R. CHINA
e-mail: pgwang@mail.hbu.edu.cn
[2]Department of Mathematics and Statistics
Curtin University of Technology
G.P.O. Box U 1987, Perth, WA 6845, AUSTRALIA
e-mail: yhwu@math.curtin.edu.au

Abstract: Oscillation properties of the solutions of impulsive parabolic differential equations of neutral type are investigated in this paper via the method of impulsive differential inequalities. The criteria for oscillatory solutions of the equations with two different kinds of boundary conditions are obtained.

AMS Subject Classification: 35B05, 35R12
Key Words: oscillation, parabolic equation, delay, impulse

1. Introduction

The problem of oscillation and nonoscillation of solutions of impulsive partial differential equations has attracted a great deal of attention over the last few years. Several papers concerning impulsive parabolic differential equations have appeared recently (For example, see [1-3]). However, very few attention has

Received: May 26, 2004

§Correspondence author

been given to impulsive parabolic differential equations with delay (see [4,5]), especially impulsive parabolic differential equations of neutral type. In this paper, we consider a class of impulsive parabolic differential equations of neutral type in the form

$$\begin{cases} \dfrac{\partial}{\partial t}[u+\lambda u(t-\tau,x)] = a(t)\triangle u - p(t,x)u + f(t,x), \\ \hfill t \neq t_k, \\ u(t_k^+,x) - u(t_k^-,x) = b_k u(t_k,x), \quad k=1,2,\ldots, \end{cases} \tag{E}$$

where λ, τ and b_k are constants, $\tau > 0$ and $b_k > -1$ for $k = 1, 2, \ldots$; $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_{k+1} - t_k \equiv \tau$ and $\lim\limits_{t\to\infty} t_k = \infty$; $u = u(t,x)$ for $(t,x) \in R_+ \times \Omega \equiv G$, in which Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $R_+ = [0,\infty)$; $a \in PC[R_+, R_+]$, $p \in PC[R_+ \times \overline{\Omega}, R_+]$, $f \in PC[R_+ \times \overline{\Omega}, R]$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \ldots,$ and left continuous at $t = t_k$; $\triangle$ is the Laplacian in the Euclidean n-space R^n.

We consider the following boundary value conditions

$$\frac{\partial u}{\partial N} + \mu(t,x)u = \psi(t,x), \quad (t,x) \in R_+ \times \partial\Omega\,, \tag{B_1}$$

$$u = \varphi(t,x), \quad (t,x) \in R_+ \times \partial\Omega, \tag{B_2}$$

where $\mu \in PC[R_+ \times \partial\Omega, R_+]$, ψ, $\varphi \in PC[R_+ \times \partial\Omega, R]$, N denotes the unit exterior normal vector to $\partial\Omega$.

The aim of this paper is to establish oscillation criteria for the boundary value problem $(E),(B_1)$ or $(E),(B_2)$. Our approach is to reduce the multi-dimensional problem in question to a one-dimensional oscillation problem for ordinary differential equations or inequalities with impulse.

The solutions $u(t,x)$ of the boundary value problem $(E),(B_1)$ or $(E),(B_2)$ are piecewise continuous functions with only discontinuity points of the first kind at $t = t_k$, $k = 1, 2, \ldots$. As a convention, we shall assume that these are left continuous, that is, at the moments of impulse the following relations are satisfied:

$$u(t_k^-,x) = u(t_k,x) \text{ and } u(t_k^+,x) = (1+b_k)u(t_k,x), \quad k = 1,2,\ldots .$$

As is customary, a nonzero solution $u(t,x)$ of the problem $(E),(B_1)$ or $(E),(B_2)$ is said to be nonoscillatory in the domain G if for each positive number t_μ, $u(t,x)$ has a constant sign for $(t,x) \in [t_\mu,\infty) \times \Omega$. Otherwise, it is said to be oscillatory.

2. Oscillation Criteria for the Boundary Value Problem (*E*) and (*B*$_1$)

For each solution $u(x,t)$ of the boundary value problem $(E),(B_1)$, we associate a function $U(t)$ defined by

$$U(t)=\int_\Omega u(t,x)dx, \quad t>0. \tag{1}$$

Theorem 1. *If the following impulsive differential inequalities*

$$\begin{cases} \dfrac{d}{dt}\left[Y(t)+\lambda Y(t-\tau)\right]+P(t)Y(t)\le F_1(t), & t\ne t_k,\\ Y(t_k^+)=(1+b_k)Y(t_k), & k=1,2,\dots, \end{cases} \tag{2}$$

and

$$\begin{cases} \dfrac{d}{dt}\left[Y(t)+\lambda Y(t-\tau)\right]+P(t)Y(t)\le -F_1(t), & t\ne t_k,\\ Y(t_k^+)=(1+b_k)Y(t_k), & k=1,2,\dots, \end{cases} \tag{3}$$

have no eventually positive solutions, then each nonzero solution of the boundary value problem defined by (E) *and* (B_1) *is oscillatory in the domain* G*, in which*

$$P(t)=\min_{x\in\overline{\Omega}}\{p(t,x)\},\quad F_1(t)=a(t)\int_{\partial\Omega}\psi(t,x)d\omega+\int_\Omega f(t,x)dx.$$

Proof. Assume the contrary that $u(t,x)$ is a nonzero solution of the problem $(E),(B_1)$ which has a constant sign in the domain $[t_\mu,\infty)\times\Omega$ for some $t\ge 0$. Let $u(t,x)>0$ for $(t,x)\in[t_\mu,\infty)\times\Omega$.

For $t\ne t_k$, integrating equation (E) with respect to x over the domain Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega udx+\lambda\int_\Omega u(t-\tau,x)dx\right]&+\int_\Omega p(t,x)udx\\ &=a(t)\int_\Omega \triangle udx+\int_\Omega f(t,x)dx. \end{aligned} \tag{4}$$

Using the divergence theorem and (B_1), we have

$$\begin{aligned} \int_\Omega \triangle udx &=\int_{\partial\Omega}\frac{\partial u}{\partial N}d\omega\\ &=\int_{\partial\Omega}[\psi(t,x)-\mu(t,x)u]d\omega\le\int_{\partial\Omega}\psi(t,x)d\omega. \end{aligned} \tag{5}$$

Moreover

$$\int_\Omega p(t,x)u dx \geq P(t)\int_\Omega u dx. \tag{6}$$

Thus, from (1) and (4)-(6), we have

$$\frac{d}{dt}[U(t)+\lambda U(t-\tau)] + P(t)U(t) \leq F(t), \quad t \neq t_k. \tag{7}$$

For $t = t_k$, it follows from equation (E) that

$$\int_\Omega u(t_k^+, x)dx = (1+b_k)\int_\Omega u(t_k, x)dx, \quad k = 1, 2, \ldots,$$

that is

$$U(t_k^+) = (1+b_k)U(t_k), \quad k = 1, 2, \ldots, \tag{8}$$

which, together with (7) and (8), implies that impulsive differential inequality (2) has an eventually positive solution $U(t)$. This contradicts the condition of the theorem.

If $u(t,x) < 0$ for $(t,x) \in [t_\mu, \infty) \times \Omega$, let $\overline{u}(t,x) = -u(t,x)$, then $\overline{u}(t,x)$ is an eventually positive solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}[u + \lambda u(t-\tau, x)] = a(t)\triangle u - p(t,x)u - f(t,x), \\ \hfill t \neq t_k, \\ u(t_k^+, x) - u(t_k^-, x) = b_k u(t_k, x), \quad k = 1, 2, \ldots, \end{cases} \tag{E'}$$

$$\frac{\partial u}{\partial N} + \mu(t,x)u = -\psi(t,x), \qquad (t,x) \in R_+ \times \partial\Omega \tag{B_1'}$$

and satisfies

$$\frac{d}{dt}\left[\int_\Omega \overline{u}dx + \lambda\int_\Omega \overline{u}(t-\tau,x)dx\right] + \int_\Omega p(t,x)\overline{u}dx$$
$$= a(t)\int_\Omega \triangle\overline{u}dx + \int_\Omega f(t,x)dx. \tag{9}$$

Consequently,

$$\overline{U}(t) = \int_\Omega \overline{u}(t,x)dx$$

is an eventually positive solution of the impulsive differential inequality (3), which contradicts the condition of Theorem 1. This completes the proof of Theorem 1. □

In Theorem 1, by choosing $\psi(t,x) \equiv 0$, we have the following result.

Corollary 1. *If the following impulsive differential inequalities*

$$\begin{cases} \dfrac{d}{dt}\left[Y(t)+\lambda Y(t-\tau)\right]+P(t)Y(t) \le F_2(t), & t \ne t_k, \\ Y(t_k^+) = (1+b_k)Y(t_k), & k=1,2,\dots, \end{cases} \tag{10}$$

and

$$\begin{cases} \dfrac{d}{dt}\left[Y(t)+\lambda Y(t-\tau)\right]+P(t)Y(t) \le -F_2(t), & t \ne t_k, \\ Y(t_k^+) = (1+b_k)Y(t_k), & k=1,2,\dots, \end{cases} \tag{11}$$

have no eventually positive solutions, then each nonzero solution of the following boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left[u+\lambda u(t-\tau,x)\right] = a(t)\triangle u - p(t,x)u + f(t,x), & t \ne t_k, \\ u(t_k^+,x)-u(t_k^-,x) = b_k u(t_k,x), & k=1,2,\dots, \end{cases} \tag{E}$$

$$\frac{\partial u}{\partial N} + \mu(t,x)u = 0, \qquad (t,x) \in R_+ \times \partial\Omega \tag{B_1^*}$$

is oscillatory in the domain G, in which $F_2(t) = \displaystyle\int_\Omega f(t,x)dx$.

Now from the proof of the above theorem, the problem of establishing the oscillation criteria for the boundary value problem $(E),(B_1^*)$ can be reduced to an investigation of the properties of the impulsive differential inequality.

Lemma 1. (see [6]) *Assume that*

$$m'(t) \le q(t), \quad t \ne t_k, \ \ t \ge t_0, \tag{12}$$

$$m(t_k^+) \le (1+b_k)m(t_k), \qquad k=1,2,\dots, \tag{13}$$

where $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{t\to\infty} t_k = \infty$; $m \in PC'[R_+,R]$, $q \in PC[R_+,R]$ and b_k are constants. Then

$$m(t) \le \prod_{t_0<t_k<t}(1+b_k)m(t_0) + \int_{t_0}^{t} \prod_{s<t_k<t}(1+b_k)q(s)ds, \quad t \ge t_0.$$

Theorem 2. *Assume that $\lambda > 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, $k=1,2,\dots$. If*

$$\liminf_{t\to\infty} \int_T^t \prod_{s<t_k<t}(1+b_k)F_2(s)ds = -\infty, \tag{14}$$

$$\limsup_{t\to\infty}\int_T^t \prod_{s<t_k<t}(1+b_k)F_2(s)ds=\infty, \tag{15}$$

then each nonzero solution of the boundary value problem defined by (E) *and* (B_1^*) *is oscillatory in the domain* G.

Proof. Assume the contrary that $u(t,x)$ is a nonzero solution of the boundary value problem $(E),(B_1^*)$ which has a constant sign in the domain $[t_\mu,\infty)\times\Omega$ for some $t\geq 0$. Let $u(t,x)>0$ for $(t,x)\in[t_\mu,\infty)\times\Omega$ (The case of $u(t,x)<0$, for $(t,x)\in[t_\mu,\infty)\times\Omega$, can be considered by the same method as that in Theorem 1). From Theorem 1 and (1), $U(t)$ is an eventually positive solution and

$$\begin{cases}\dfrac{d}{dt}\left[U(t)+\lambda U(t-\tau)\right]+P(t)U(t)\leq F_2(t), & t\neq t_k,\\ U(t_k^+)=(1+b_k)U(t_k), & k=1,2,\ldots.\end{cases} \tag{16}$$

Let

$$W(t)=U(t)+\lambda U(t-\tau), \tag{17}$$

then, for $t\neq t_k$,

$$W(t)>0,\ \ W'(t)\leq F_2(t), \tag{18}$$

and for $t=t_k$, $k=1,2,\ldots$, we have

$$\begin{aligned}W(t_k^+)-W(t_k^-)&=U(t_k^+)+\lambda U(t_k^+-\tau)-U(t_k^-)-\lambda U(t_k^--\tau)\\&=b_kW(t_k).\end{aligned} \tag{19}$$

From Lemma 1, we have

$$W(t)\leq\prod_{t_0<t_k<t}(1+b_k)W(t_0)+\int_{t_0}^t\prod_{s<t_k<t}(1+b_k)F_2(s)ds,\quad t\geq t_0. \tag{20}$$

In view of (14), (20) implies that $W(t)\to-\infty$ as $t\to\infty$, which contradicts $W(t)>0$. This completes the proof of Theorem 2. □

3. Oscillation Criteria for the Boundary Value Problem (E) and (B_2)

The following Lemma is useful to the proof of Theorem 3.

Lemma 2. (see [7]) *The least eigenvalue* α_0 *of the following Dirichlet problem*

$$\begin{cases}\triangle u+\alpha u=0, & (x,t)\in\Omega\times R_+,\\ u=0, & (x,t)\in\partial\Omega\times R_+\end{cases} \tag{21}$$

is positive and the corresponding eigenfunction $\phi(x)$ *is positive on* $x \in \Omega$*.*

For each solution $u(x,t)$ of the boundary value problem defined by (E) and (B_2), we associate a function $V(t)$ defined by

$$V(t) = \int_\Omega u(x,t)\phi(x)dx, \quad t > 0. \tag{22}$$

Theorem 3. *If the following impulsive differential inequalities*

$$\begin{cases} \dfrac{d}{dt}\left[Z(t) + \lambda Z(t-\tau)\right] + [\alpha_0 a(t) + P(t)]Z(t) \le Q_1(t), & t \ne t_k, \\ Z(t_k^+) = (1+b_k)Z(t_k), & k = 1, 2, \dots, \end{cases} \tag{23}$$

and

$$\begin{cases} \dfrac{d}{dt}\left[Z(t) + \lambda Z(t-\tau)\right] + [\alpha_0 a(t) + P(t)]Z(t) \le -Q_1(t), & t \ne t_k, \\ Z(t_k^+) = (1+b_k)Z(t_k), & k = 1, 2, \dots, \end{cases} \tag{24}$$

have no eventually positive solutions, then each nonzero solution of the boundary value problem defined by (E) *and* (B_2) *is oscillatory in the domain* G*, in which*

$$P(t) = \min_{x\in\overline{\Omega}}\{p(t,x)\},$$
$$Q_1(t) = a(t)\int_{\partial\Omega} \varphi(t,x)\frac{\partial\phi(x)}{\partial N}d\omega + \int_\Omega f(t,x)\phi(x)dx.$$

Proof. Assume the contrary that $u(t,x)$ is a nonzero solution of the boundary value problem $(E), (B_2)$ which has a constant sign in the domain $[t_\mu, \infty)\times\Omega$ for some $t \ge 0$. Let $u(t,x) > 0$ for $(t,x) \in [t_\mu, \infty) \times \Omega$ (The case of $u(t,x) < 0$ for $(t,x) \in [t_\mu, \infty) \times \Omega$ can be considered by the same method as that in Theorem 1).

For $t \ne t_k$, multiplying both sides of equation (E) by $\phi(x)$ and integrating with respect to x over the domain Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\Big[\int_\Omega u\phi(x)dx + \lambda\int_\Omega u(t-\tau,x)\phi(x)dx\Big] + \int_\Omega p(t,x)u\phi(x)dx \\ = a(t)\int_\Omega \triangle u\phi(x)dx + \int_\Omega f(t,x)\phi(x)dx. \end{aligned} \tag{25}$$

Using Green's formula, we have

$$\begin{aligned}\int_\Omega \triangle u\phi(x)dx &= \int_{\partial\Omega}\left(\phi(x)\frac{\partial u}{\partial N} - u\frac{\partial\phi(x)}{\partial N}\right)d\omega + \int_\Omega u\triangle\phi(x)dx \\ &= -\int_{\partial\Omega}\varphi(t,x)\frac{\partial\phi(x)}{\partial N}d\omega - \alpha_0\int_\Omega u\phi(x)dx.\end{aligned} \tag{26}$$

Moreover

$$\int_\Omega p(t,x)u\phi(x)dx \geq P(t)\int_\Omega u\phi(x)dx. \tag{27}$$

Thus, from (22), (25)-(27) and using (21), we have

$$\frac{d}{dt}\left[V(t) + \lambda V(t-\tau)\right] + \left[\alpha_0 a(t) + P(t)\right]V(t) \leq Q_1(t), \quad t \neq t_k. \tag{28}$$

For $t = t_k$, it follows from equation (E) that

$$\int_\Omega u(t_k^+, x)\phi(x)dx = (1+b_k)\int_\Omega u(t_k, x)\phi(x)dx, \quad k = 1, 2, \ldots,$$

that is

$$V(t_k^+) = (1+b_k)V(t_k), \quad k = 1, 2, \ldots . \tag{29}$$

Clearly, (28) and (29) imply that impulsive differential inequality (23) has an eventually positive solution $V(t)$, which contradicts the condition of the theorem. This completes the proof of Theorem 3. □

In Theorem 3, if choosing $\varphi(t,x) \equiv 0$, we have the following result.

Corollary 2. *If the following impulsive differential inequalities*

$$\begin{cases}\dfrac{d}{dt}\left[Z(t) + \lambda Z(t-\tau)\right] + \left[\alpha_0 a(t) + P(t)\right]Z(t) \leq Q_2(t), & t \neq t_k, \\ Z(t_k^+) = (1+b_k)Z(t_k), & k = 1, 2, \ldots,\end{cases} \tag{23$'$}$$

and

$$\begin{cases}\dfrac{d}{dt}\left[Z(t) + \lambda Z(t-\tau)\right] + \left[\alpha_0 a(t) + P(t)\right]Z(t) \leq -Q_2(t), & t \neq t_k, \\ Z(t_k^+) = (1+b_k)Z(t_k), & k = 1, 2, \ldots,\end{cases} \tag{24$'$}$$

have no eventually positive solutions, then each nonzero solution of the following boundary value problem

$$\begin{cases}\dfrac{\partial}{\partial t}\left[u + \lambda u(t-\tau, x)\right] = a(t)\triangle u - p(t,x)u + f(t,x), & t \neq t_k, \\ u(t_k^+, x) - u(t_k^-, x) = b_k u(t_k, x), & k = 1, 2, \ldots,\end{cases} \tag{E}$$

$$u = 0, \quad (t, x) \in R_+ \times \partial\Omega \tag{B_2^*}$$

is oscillatory in the domain G, in which $Q_2(t) = \int_\Omega f(t,x)\phi(x)dx$.

Theorem 4. *Assume that $\lambda > 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, $k = 1, 2, \ldots$. If*

$$\liminf_{t\to\infty} \int_T^t \prod_{s<t_k<t} (1 + b_k) Q_2(s) ds = -\infty, \tag{30}$$

$$\limsup_{t\to\infty} \int_T^t \prod_{s<t_k<t} (1 + b_k) Q_2(s) ds = \infty, \tag{31}$$

then each nonzero solution of the boundary value problem defined by (E) and (B_2^*) is oscillatory in the domain G.

The proof is similar to that for Theorem 2 and thus it is omitted here.

Acknowledgements

This work was supported by the Natural Science Foundation of Hebei Province (A2004000089).

References

[1] D. Bainov, E. Minchev, K. Nakagawa, Asymptotic behaviour of solutions of impulsive semilinear parabolic equations, *Nonlinear Analysis*, **30** (1997), 2725-2734.

[2] D. Bainov, E. Minchev, Forced oscillations of solutions of impulsive nonlinear parabolic equations, *Comm. Appl. Nonlinear Anal.*, **5**, No. 1 (1998), 107-116.

[3] D. Bainov, E. Minchev, Oscillation of solutions of impulsive nonlinear parabolic differential-difference equations.

[4] X.L. Fu, X.Z. Liu, S. Sivaloganathan, Oscillation criteria for impulsive parabolic differential equations with delay, *J. Math. Anal. Appl.*, **268** (2002), 647-664.

[5] R. Pirapikaran, Impulsive integral inequalities of Gronwall-Bihari type, In: *Diff. Equs. Appl.* (Ed. R. Aftabizadeh)), Ohio University Press, **2** (1989), 296-308; *Internat. J. Theoret. Phys.*, **35**, No. 1 (1996), 207-215.

[6] V.S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow (1981).

[7] L.Q. Zhang, Oscillation criteria for certain impulsive parabolic partial differential equations, *Academic Periodical Abstracts China*, **5** (1999), 492-493.