International Journal of Pure and Applied Mathematics
Volume 14 No. 4 2004, 515-524

# FIXED POINTS AND GAME THEORY

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**Abstract:** The purpose of this paper is to report some topological properties of the fixed point set of multivalued operators. Existence of Nash equilibrium point for two-person games and a topological property of the Nash equilibrium point set are also obtained.

**AMS Subject Classification:** 47H10, 54H25
**Key Words:** fixed point, selection, absolute retract, Nash equilibrium point

## 1. Preliminaries

Let $\mathcal{M}$ be the family of all metric spaces. Let $X \in \mathcal{M}$. The space $X$ is called an absolute retract for metric spaces, briefly $X \in AR(\mathcal{M})$ if for any $Y \in \mathcal{M}$ and any nonempty closed subset $Y_0$ of $Y$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension $f : Y \to X$ over $Y$.

If $E$ is a Banach space then a nonempty subset $X$ of $E$ is said to be a retract of $E$ if there exists a continuous mapping $r : E \to X$, such that $r(x) = x$, for each $x \in X$.

Received: May 31, 2004 

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It is known that $X$ is a retract of a Banach space if and only if $X$ is an absolute retract and it is closed. Also, any absolute retract is arcwise connected and the continuous image of an absolute retract is arcwise connected.

If $(T, \mathcal{A}, \mu)$ is a complete $\sigma$-finite nonatomic measure space and $E$ is a Banach space, let us denote by $L^1(T, E)$ the Banach space of all measurable functions $u : T \to E$ which are Bochner $\mu$-integrable. Recall that a set $K \subset L^1(T, E)$ is said to be decomposable if for all $u, v \in K$ and each $A \in \mathcal{A}$:

$$u\chi_A + v\chi_{T\setminus A} \in K, \tag{1}$$

where $\chi_A$ stands for the characteristic function of the set $A$.

Let $X$ be a metric space. Throughout this paper we use the following notations and symbols:

$$\mathcal{P}(X) = \{Z | Z \subset X\}, \quad P(X) = \{Z \in \mathcal{P}(X) | Z \text{ nonempty}\},$$

$$P_p(X) = \{Z \in P(X) | \ Z \text{ has the property "}p\text{"}\},$$

where "$p$" could be: $cl$ = closed, $b$ = bounded, $cp$ = compact, or in normed spaces $cv$ = convex, $dec$ = decomposable, etc.

The following (generalized) functionals are used in the main section of the paper.

### The Gap Functional

$$(1) \ \ D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$D(A, B) = \begin{cases} \inf\{d(a, b) | \ a \in A, \ b \in B\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise}. \end{cases}$$

### Pompeiu-Hausdorff Generalized Functional

$$(2) \ \ H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H(A, B) = \begin{cases} \max\{\sup\limits_{a \in A} D(a, B), \sup\limits_{b \in B} D(A, b)\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{othewise} \ . \end{cases}$$

If $A$ is a nonempty subset of a metric space $X$ then $V(A,\epsilon) := \{x \in E|\ D(x,A) \leq \epsilon\}$, where $\epsilon > 0$.

It is well-known that the pair $(P_{b,cl}(X), H)$ is a complete metric space, provided $X$ is a complete space. Several basic notions are considered in the following definitions.

**Definition 1.1.** Let $(X,d)$ be a metric space. Then $F : X \to P(E)$ is called:

a) lower semi-continuous (briefly l.s.c.) if for each closed $C \subset E$ the set $\{x \in X|\ F(x) \subset C\}$ is closed.

b) upper semi-continuous (briefly u.s.c.) if for each closed $C \subset E$ the set $\{x \in X|\ F(x) \cap C\}$ is closed.

c) continuous if is l.s.c. and u.s.c.

d) $H$-u.s.c. if for each $\epsilon > 0$ there is $\delta > 0$ such that $F(B(x_0;\delta)) \subset V(F(x_0);\epsilon)$.

e) $a$-Lipschitz multi-function if $a \geq 0$ and $H(F(x),F(y)) \leq \text{ ad}\,(x,y)$, for each $x,y \in X$. The number $a$ is called the Lipschitz constant of $F$.

f) multi-valued $a$-contraction if $a \in [0,1[$ and $H(F(x),F(y)) \leq \text{ ad}\,(x,y)$, for each $x,y \in X$.

g) multi-valued Reich type operator if there exist three positive numbers $a,b,c$ with $a+b+c \in [0,1[$ such that $H(F(x),F(y)) \leq \text{ ad}\,(x,y)+bD(x.F(x))+cD(y,F(y))$, for each $x,y \in X$.

**Definition 1.2.** Let $X,Y$ be two nonempty sets.

i) Let $F : X \to P(Y)$ be a multi-valued operator. A single-valued operator $f : X \to Y$ is a selection for $F$ if and only if $f(x) \in F(x)$ for every $x \in X$.

ii) If $F : X \to P(X)$, then $x^*$ is a fixed point for $F$ if and only if $x^* \in F(x^*)$. Denote by $\text{Fix}\,F$ the fixed point set for $F$.

iii) Let $F : X \to P(Y)$ be a multi-valued operator. Then the graphic of $F$ is $\text{Graf}F := \{(x,y) \in X \times Y | y \in F(x)\}$.

## 2. Main Results

Let us start by recalling a classical result.

**Theorem 2.1.** (Michael' Selection Theorem) *Let $(X,d)$ be a metric space, $E$ be a Banach space and $F : X \to P_{cl,cv}(E)$ be l.s.c. on $X$. Then there exists $f : X \to E$ a continuous selection of $F$.*

A "decomposable" version of Michael's Selection Theorem for l.s.c. multi-functions with convex values is the following.

**Theorem 2.2.** (see Fryszkowski [4], Bressan-Colombo [2]) *Let* $(X, d)$ *be a separable metric space,* $(T, \mathcal{A}, \mu)$ *be a complete* $\sigma$*-finite nonatomic measure space,* $E$ *a separable Banach space and let* $F : X \to P_{cl,dec}(L^1(T, E))$ *be a l.s.c. multi-valued operator. Then* $F$ *has a continuous selection.*

For the $H$-u.s.c. case we have the following result.

**Theorem 2.3.** (Bressan-Colombo [2]) *Let* $(X, d)$ *be a separable metric space,* $(T, \mathcal{A}, \mu)$ *be a complete* $\sigma$*-finite nonatomic measure space and let* $F : X \to P_{dec}(L^1(T, E))$ *be a* $H$*-u.s.c. multi-valued operator. If either* $X$ *or* $L^1(T, E)$ *is separable, then for each* $\epsilon > 0$ *there is a continuous function* $f_\epsilon : X \to L^1(T, E)$ *such that* $\mathit{Graph}\, f_\epsilon \subseteq V(\mathit{Graph}\, F, \epsilon)$ *and* $f_\epsilon(X) \in P_{dec}(F(X))$.

An important result regarding a topological property of the fixed point set was established by B. Ricceri (see [10]), as follows.

**Theorem 2.4.** (see B. Ricceri [10]) *Let* $E$ *be a Banach space and let* $X$ *be a nonempty, closed, convex subset of* $E$*. Suppose* $T : X \to P_{cl,cv}(X)$ *is a multi-valued contraction. Then* $\operatorname{Fix} T$ *is an absolute retract for metric spaces.*

A decomposable version of the previous result was proved by Bressan-Cellina-Fryszkowski (see [3]).

**Theorem 2.5.** (see Bressan-Cellina-Fryszkowski [3]) *Let* $(X, d)$ *be a separable metric space,* $(T, \mathcal{A}, \mu)$ *be a complete* $\sigma$*-finite nonatomic measure space,* $E$ *be a separable Banach and let* $F : X \times L^1(T, E) \to P_{b,cl,dec}(L^1(T, E))$ *be a continuous multi-valued operator satisfying the following condition: there exists* $\alpha \in [0, 1[$ *such that* $H(F(\lambda, u), F(\lambda, v)) \leq \alpha \|u - v\|$*, for all* $u, v \in L^1(T, E)$ *and for all* $\lambda \in X$*. Then each set* $\operatorname{Fix}_\lambda F = \{u |\ u \in F(\lambda, u)\}$ *is an absolute retract. Moreover a retraction can be chosen that depends continuously on* $\lambda$*.*

**Corollary 2.6.** *Let* $F : L^1(T, E) \to P_{b,cl,dec}(L^1(T, E))$ *be a multi-valued a-contraction. Then* $\operatorname{Fix} F$ *is an absolute retract for metric spaces.*

For a multi-valued Reich-type operator with convex values, we have the following result.

**Theorem 2.7.** (see Petruşel [8]) *Let* $E$ *be a Banach space,* $X \in P_{cl,cv}(E)$ *and* $T : X \to P_{cl,cv}(X)$ *be a l.s.c. multi-valued Reich-type operator. Then* $\operatorname{Fix} T \in AR(\mathcal{M})$.

*Sketch of the Proof.* Let us remark first that $\operatorname{Fix} T \in P_{cl}(X)$. Let $K$ be a paracompact topological space, $A \in P_{cl}(K)$ and $\psi : A \to \operatorname{Fix} T$ a continuous mapping. Using Theorem 2 from B. Ricceri [10] (taking $G(t) = X$, for each $t \in K$) it follows the existence of a continuous function $\varphi_0 : K \to X$ such that $\varphi_0|_A = \psi$. We next consider $q \in ]1, (\alpha + \beta + \gamma)^{-1}[$. By induction on $n$ we can prove that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions from $K$ to $X$ with the following properties:

(i) $\varphi_n|_A = \psi$

(ii) $\varphi_n(t) \in T(\varphi_{n-1}(t))$, for all $t \in K$

(iii) $\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq [(\alpha+\beta+\gamma)q]^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for all $t \in K$.

Consider next the open covering of $K$ defined by the formula: $(\{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\})_{\lambda>0}$. Moreover, because of (iii) and the fact that $X$ is complete, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly on each of the following set $K_\lambda = \{t \in K| \ \|\varphi_1(t) - \varphi_0(t)\| < \lambda\}$ $(\lambda > 0)$. Let $\varphi : K \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}}$. Obviously $\varphi$ is continuous and $\varphi|_A = \psi$. Moreover, a simple computation ensures that : $\varphi(t) \in T(\varphi(t))$ for all $t \in K$ and this completes the proof. □

An important abstract notion is.

**Definitiom 2.8.** Let $(X, d)$ be a metric space, $F : X \to P_{cl}(X)$ be l.s.c. and $\mathcal{U} \subset \mathcal{X}$ be an arbitrary family of metric spaces. We say that $F$ has the selection property with respect to $\mathcal{U}$ when for any $Y \in \mathcal{U}$, any pair of continuous functions $f : Y \to X$ and $r : Y \to ]0, \infty[$ such that:

$$G(y) = \overline{F(f(y)) \cap B(f(y), r(y))} \neq \emptyset, \text{ for each } y \in Y,$$

and any nonempty closed set $Z \subset Y$, every continuous selection $g_0$ of $G\ |_Z$ admits a continuous extension $g$ over $Y$ fulfilling $g(y) \in G(y)$, for all $y \in Y$. When $\mathcal{U} = \mathcal{X}$ we say that $G$ has the selection property (briefly $G \in SP(X)$).

Some examples illustrating this notion are as follows.

**Example 2.9.** Let $X$ be a nonempty closed convex subset of a Banach space $E$ and $F : X \to P_{cl,cv}(X)$ be l.s.c. From Michael's Selection Theorem it follows that $F \in SP(X)$.

**Example 2.10.** Let $X$ be a nonempty closed decomposable subset of $L^1(T, E)$ and let $F : X \to P_{cl,dec}(X)$ be l.s.c. From Theorem 2.2. it follows that $F$ has the selection property with respect to the family of all separable metric spaces.

Using this abstract setting the following results was proved.

**Theorem 2.11.** (see Górniewicz-Marano [7]) *Let $\mathcal{U} \subset \mathcal{X}$ and let $X$ be a complete absolute abstract. Suppose that $F : X \to P_{cl}(X)$ is a multi-valued contraction having the selection property with respect to $\mathcal{U}$. Then* Fix $F$ *is a nonempty absolute retract.*

**Remark 2.12.** Theorem 2.11 contains as particular cases both Theorem 2.4 and Corollary 2.6.

By a similar approach to Theorem 2.7 we have the following generalization of the previous theorem.

**Theorem 2.13.** *Let $\mathcal{U} \subset \mathcal{X}$ and $X$ be a complete absolute retract. Suppose that $F : X \to P_{cl}(X)$ is a Reich type multi-function such that having the selection property with respect to $\mathcal{U}$. Then the fixed point set* $\operatorname{Fix} F$ *is a nonempty absolute retract.*

An extension of Definition 2.8. was given by Marano (see [6]), as follows.

**Definition 2.14.** Let $X, W$ be metric spaces, $F : X \to P_{cl}(W)$ be l.s.c. and $\mathcal{U} \subset \mathcal{X}$ be an arbitrary family of metric spaces. We say that $F$ has the selection property with respect to $\mathcal{U}$ when for any $Y \in \mathcal{U}$, any triple of continuous functions $f : Y \to X$, $h : Y \to Z$ and $r : Y \to ]0, \infty[$ such that: $G(y) = \overline{F(f(y)) \cap B(h(y), r(y))} \neq \emptyset$, for each $y \in Y$ and any nonempty closed set $Z \subset Y$, every continuous selection $g_0$ of $G\,|_Z$ admits a continuous extension $g$ over $Y$ fulfilling $g(y) \in G(y)$, for all $y \in Y$.

An important property of the fixed point set for the composition of two multi-functions is the following.

**Theorem 2.15.** (see Marano [6]) *Let $X, Y$ be complete absolute retracts and let $\mathcal{U} = \{[0, 1]\}$. Suppose that $F_1 : Y \to P_{cl}(X)$ and $F_2 : X \to P_{cl}(Y)$ have the following properties:*

*i) $F_1$ and $F_2$ are Lipschitz multi-functions, with Lipschitz constants $a_1$ and $a_2$ respectively.*

*ii) $F_1$ and $F_2$ have the selection property with respect $\mathcal{U}$.*

*iii) $a_1 a_2 < 1$.*

*Then the fixed point set of the composition multi-function $F = F_1 \circ F_2$ is nonempty and arcwise connected.*

Basically, the approach in the above proof is the following:

Using the fact that $F_1 : Y \to P_{cl,cv}(X)$ and $F_2 : X \to P_{cl,cv}(Y)$ are $a_1$, respectively $a_2$ Lipschitz multivalued operators and $a_1 a_2 \in ]0, 1[$, it is proved that the across cartesian product operator $T : X \times Y \multimap X \times Y$, defined by $T(x, y) := F_1(y) \times F_2(x)$, for each $(x, y) \in X \times Y$ is a multivalued contraction with closed values. Also, $T$ has the selection property with respect to $\mathcal{U}$. From Górniewicz-Marano's result (Theorem 2.11 above) we obtain that the fixed point set of $T$ is a nonempty absolute retract.

Then, it can easily see that $\operatorname{Fix}(F_1 \circ F_2)$ is the continuous image of $\operatorname{Fix} T$ via the projection $p_1 : E_1 \times E_2 \to E_1$ onto the first coordinate.

Taking into account the above, we must conclude that an important tool in nonlinear problems solved by fixed point techniques is:

**Theorem 2.16.** (see Marano [6]) *Let $X, Y$ be complete absolute retracts and let $\mathcal{U} = \{[0,1]\}$. Suppose that $F_1 : Y \to P_{cl}(X)$ and $F_2 : X \to P_{cl}(Y)$ have the following properties:*

*i) $F_1$ and $F_2$ are Lipschitz multi-functions, with Lipschitz constants $a_1$ and $a_2$ respectively.*

*ii) $F_1$ and $F_2$ have the selection property with respect $\mathcal{U}$.*

*iii) $a_1 a_2 < 1$.*

*Then the fixed point set of the across cartesian product operator $T : X \times Y \to P_{cl}(X \times Y)$, defined by $T(x, y) := F_1(y) \times F_2(x)$ is a nonempty absolute retract.*

## 3. An Application to the Game Theory

Let us first recall the notion of $n$-person game, as follows. Denote by $X_i$ the set of all strategies of the $i$ player, where $i \in \{1, 2, ..., n\}$. Then, $X := \prod X_i$ is the set of all strategy vectors. Each $x = (x_1, x_2, ..., x_n) \in X$ induces an outcome.

Players preferences are described using the preference multi-function $\widetilde{U}_i : X \multimap X$, defined by $\widetilde{U}_i(x) := \{y \in X | y \text{ is preferred to } x \}$.

We also define, the good reply multi-function.

Denote

$$x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i},$$

where $X_{-i} := \prod_{k=1, k\neq i}^{n} X_k$. and $x|y_i := (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \in X$. Then, by definition, $y_i$ is a *good reply* for the player i with respect to the strategy vector $x$ if $x|y_i \in \widetilde{U}_i(x)$.

In this setting, the good reply multi-function for the player i is $U_i : X_{-i} \multimap X_i$ defined by $U_i(x_{-i}) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x)\}$.

A game in strategic form or an abstract economy is the pair $(X_i, U_i)_{i \in \{1,2,...,n\}}$.

For example, if we consider $p_i : X \to \mathbb{R}$, for $i \in \{1, 2, ..., n\}$, the pay-off function of the $i$ player, then the good reply multi-function can be expressed by:

$$U_i(x_{-i}) := \{y_i \in X_i | F_i(x|y_i) \geq F_i(x|z_i), \text{ for each } z_i \in X_i\}.$$

By definition, $x^* \in X$ is a Nash equilibrium point for an abstract economy if $x_i^* \in U_i(x_{-i}^*)$, for $i \in \{1, 2, ..., n\}$.

Let us consider now a 2-person game (or an abstract economy with neighborhood effects) given by $(X_1, U_1), (X_2, U_2)$, where $X_1, X_2$ denote the set of strategies of the player 1, respectively player 2, and $U_1 : X_2 \multimap X_1$, $U_2 : X_1 \multimap X_2$ are the good reply multi-functions for each player.

By definition, $(x_1^*, x_2^*)$ is a Nash equilibrium point if $x_1^* \in U_1(x_2^*)$ and $x_2^* \in U_2(x_1^*)$.

The following theorem is an existence result for a Nash equilibrium point.

**Theorem 3.1.** *Let $X$ be a nonempty paracompact and convex subset of a locally convex Hausdorff topological vector space $E_1$ and $Y$ a nonempty subset of a Hausdorff topological vector space $E_2$. Let $U_1 : Y \to P_{cl,cv}(X)$ be lower semi-continuous and $U_2 : X \to P(Y)$ defined by $U_2(x) := \operatorname{conv} U(x)$, for each $x \in X$, where $U : X \to P(Y)$. If there exists a compact metrizable subset $C$ of $X$ such that $U_1(Y) \subset C$ then, there exists at least a Nash equilibrium point for the 2-person game $\{(X, U_1), (Y, U_2)\}$.*

*Proof.* Denote by $T(y) := int(U_2^{-1}(y))$, for each $y \in Y$. Then the family $T(y))_{y \in Y}$ is an open covering of the paracompact space $X$. Let $(T(y_i))_{i \in I}$ be an open locally finite covering of $X$ and $(f_i)_{i \in I}$ a continuous partition of unity subordinate to the covering $(T(y_i))_{i \in I}$. Define the continuous function $f : X \to Y$ by $f(x) := \sum f_{y_i}(x) \cdot y_i$, for each $x \in X$. Hence, if $f_{y_i}(x) \neq 0$ then $x \in \operatorname{supp} f_{y_i} \subset T(y_i) \subset U_2^{-1}(y_i)$, that is $y_i \in U_2(x)$. Since $U_2(x)$ is convex for each $x \in X$ and $f(x)$ is a convex combination of elements from $U_2(x)$, it follows that $f(x) \in U_2(x)$, for each $x \in X$. Consider now the multivalued operator $W(x) := U_1(f(x))$, for each $x \in X$. Then $W$ is l.s.c., since $f$ is continuous and $U_1$ is l.s.c. Moreover, $W(x) \in P_{cl,cv}(X)$, for each $x \in X$ and $W(X) \subset U_1(Y) \subset C$. Since $C$ is compact and metrizable, using X. Wu's Theorem (see [13] we get that there exists $x^* \in C$ such that $x^* \in W(x^*)$. It follows that $x^* \in U_1(f(x^*))$ and hence $y^* := f(x^*) \in U_2(x^*)$. The proof is complete. □

If $E_1 = E_2$ and $U_2$ has convex values, then the following existence result is the l.s.c. version of a result of Sessa [11].

**Corollary 3.2.** *Let $X$ be a nonempty paracompact and convex subset of a locally convex Hausdorff topological vector space $E$ and $Y$ a nonempty subset of a Hausdorff topological vector space $E$. Let $U_1 : Y \to P_{cl,cv}(X)$ be lower semi-continuous and $U_2 : X \to P_{cv}(Y)$. If there exists a compact metrizable subset $C$ of $X$ such that $U_1(Y) \subset C$ then, there exists at least a Nash equilibrium point for the 2-person game $\{(X, U_1), (Y, U_2)\}$.*

The following theorem is not only an existence result for the Nash equilibrium points of an 2-person game, but also produces a topological property of the Nash equilibrium point set.

**Theorem 3.3.** *Let $(X_1, U_1), (X_2, U_2)$ be a 2-person game. Suppose that:*

*(i)* $X_1, X_2$ *are nonempty, closed and convex subsets of the Banach spaces* $E_1$*, respectively* $E_2$*.*

*(ii)* $U_i$ *is an* $a_i$*-Lipschitz multi-function with nonempty, closed and convex values, for* $i \in \{1, 2\}$*.*

*(iii)* $a_1 a_2 \in ]0, 1[$.

*Then the set of all Nash equilibrium points is nonempty and arcwise connected.*

*Proof.* Let us remark that the Nash equilibrium point set is equal with the fixed point set of the multi-valued operator $T(x_1, x_2) := U_1(x_2) \times U_2(x_1)$, for each $(x_1, x_2) \in X_1 \times X_2$. □

## References


[1] J.P. Aubin, A. Cellina, *Differential Inclusions*, Springer, Berlin (1984).

[2] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.*, **90** (1988), 69-86.

[3] A. Bressan, A. Cellina, A. Fryszkowski, A class of absolute retracts in spaces of integrable functions, *Proc. A. M. S.*, **12** (1991), 413-418.

[4] A. Fryszkowski, Continuous selections for a class of non-convex multivalued maps, *Studia Math.*, **76** (1983), 163-174.

[5] L. Górniewicz, S. A. Marano, On the fixed point set of multivalued contractions, *Rend. Circ. Mat. Palermo*, **40** (1996), 139-145.

[6] S. A. Marano, Fixed points of multivalued contractions, *Rend. Circ. Mat. Palermo, Suppl.*, **48** (1997), 171-177.

[7] A. Muntean, A. Petruşel, Coincidence theorems for l. s. c. multifunctions in topological vector spaces, *Proc. Itinerant Sem. Functional Eq., Approx. and Convexity*, Srima Cluj-Napoca, 147-151 (2000).

[8] A. Petruşel, Multivalued generalized contractions, *Nonlinear Analysis T. M. A.*, **47** (2001), 649-659.

[9] A. Petruşel, G. Moţ, Convexity and decomoposability in multivalued analysis, In: *Proc. of the Generalized Convexity/Monotonicity Conference, Samos, Greece, 1999*, Lecture Notes in Economics and Mathemaical Sciences, Springer-Verlag (2001), 333-341.

[10] B. Ricceri, Une propriéré topologique de l'ensemble des points fixes d'une contraction multivoque à valeur convexes, *Atti Accad. Naz. Lincei Rend. Cl. Fis. Mat Natur.*, **81** (1987), 283-286.

[11] S. Sessa, Some remarks and applications of an extension of a lemma of Ky Fan, *C.M.U.C.*, **29** (1988), 567-575.

[12] A. A. Tolstonogov, Continuous selectors of fixed points set of multifunctions with decomposable values, *Set-Valued Analysis*, **6** (1998), 129-147.

[13] X. Wu, A new fixed point theorem and its applications, *Proc. A.M.S.*, **125** (1997), 1779-1783.