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CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}C_{12})$

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Abstract: The rank of the torsion-free part of $\mathcal{U}_1(\mathbb{Z}C_{12})$ is 1. That is, its torsion-free part is generated by a single unit. In this paper, this generator is determined by using classical ring and number theory.

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1. Introduction

Let G be a finite group and $\mathcal{U}_1(\mathbb{Z}G)$ be its integral group ring with augmentation 1. Let us recall Higman's main result for finite Abelian group [4].

Theorem 1.1. *If A is a finite Abelian group then $\mathcal{U}_1(\mathbb{Z}A) = A \times F$, where F is a finitely generated Abelian group.*

For $A = C_n =< a : a^n = 1 >$, a cyclic group of order n, rank of F can easily be obtained from the result of Dirichlet Unit Theorem [4] as follows:

$$\rho = \frac{1}{2}\varphi(n) - 1. \tag{1.1}$$

It is clear that $\rho = 1$ if and only if $n = 5, 8, 10$ or 12. For $n = 5$ and 8,

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torsion-free part of $\mathcal{U}_1(\mathbb{Z}C_n)$ is characterized by Karpilovsky [3] as follows:

$$\mathcal{U}_1(\mathbb{Z}C_5) = C_5 \text{ x } <-1+a^2+a^3> \text{ and}$$

$$\mathcal{U}_1(\mathbb{Z}C_8) = C_8 \text{ x } <-1-a-a^2+a^4+2a^5+a^6> .$$

Another description for $n = 8$ is given by Sehgal [4] using fiber product diagram. On the other hand, for $n = 7$, torsion-free part is characterized by Karpilovsky [3]. For $n = 7$ and 9, The description of $\mathcal{U}_1(\mathbb{Z}C_n)$ is given by Aleev [1]. Now, it is time to give a description of $\mathcal{U}_1(\mathbb{Z}C_n)$ for $n = 12$.

2. Main Result

At first, let us prepare some basic results before giving a characterization of $\mathcal{U}_1(\mathbb{Z}C_{12})$.

Lemma 2.1. *$\gamma = \sum \gamma_i a^i \in \mathcal{U}_1(\mathbb{Z}C_{12})$ be a torsion-free unit. Then, γ can be expressed in terms of 4 parameters.*

Proof. Let $H =< a^6 >$ and $K =< a^4 >$ be prime subgroups of C_{12}. Now, let us consider following group homomorphisms :

$$\begin{array}{c} \varphi_1 : C_{12} \to C_{12}/H \\ a^i \mapsto a^i H \end{array} \quad \text{and} \quad \begin{array}{c} \varphi_2 : C_{12} \to C_{12}/K \\ a^i \mapsto a^i K \end{array}$$

by extending these homomorphisms linearly to $\mathbb{Z}C_{12}$ over $\mathbb{Z}$, we get

$$\begin{array}{c} \bar{\varphi}_1 : ZC_{12} \longrightarrow \mathbb{Z}(C_{12}/H) \\ \sum \gamma_i a^i \mapsto \sum \gamma_i (a^i H) \end{array} \quad \text{and} \quad \begin{array}{c} \bar{\varphi}_2 : \mathbb{Z}C_{12} \longrightarrow \mathbb{Z}(C_{12}/K) \\ \sum \gamma_i a^i \mapsto \sum \gamma_i (a^i K) \end{array}$$

the natural ring homomorphisms. Since$C_{12}/H \cong C_6$ and $C_{12}/K \cong C_4$, by (1.1), $\mathcal{U}_1(\mathbb{Z}C_4)$ and $\mathcal{U}_1(\mathbb{Z}C_6)$ are trivial. Therefore, for any $\gamma \in \mathcal{U}_1(\mathbb{Z}C_{12})$ torsion-free unit, we have $\bar{\varphi}_1(\gamma) = H$ and $\bar{\varphi}_2(\gamma) = K$. Now let us consider respectively:

i) $\bar{\varphi}_1(\gamma) = H \Longrightarrow \gamma_0 + \gamma_6 = 1,\ \gamma_i + \gamma_{i+6} = 0\ (i = 1, 2, 3, 4,).$

ii) $\bar{\varphi}_2(\gamma) = K \Longrightarrow \gamma_0 + \gamma_4 + \gamma_8 = 1,\ \gamma_j + \gamma_{j+4} + \gamma_{j+8} = 0\ (j = 1, 2, 3).$

If γ_0, γ_1, γ_2 and γ_3 are chosen as free parameters, the others can be expressed by these 4 parameters as follows:

$$\begin{array}{llll} \gamma_4 = 1 - \gamma_0 + \gamma_2, & \gamma_6 = 1 - \gamma_0, & \gamma_8 = -\gamma_2, & \gamma_{10} = -1 + \gamma_0 - \gamma_2 \\ \gamma_5 = -\gamma_1 + \gamma_3, & \gamma_7 = -\gamma_1, & \gamma_9 = -\gamma_3 & \gamma_{11} = \gamma_1 - \gamma_3. \end{array} \quad \square$$

Remark 2.2. It is clear that $\tilde{\mathbb{Z}}[\sqrt{3}, i] = \{a + b\sqrt{3} + 3ci + di\sqrt{3}\}$ is a subring of $\mathbb{Z}[\sqrt{3}, i]$. For any $\gamma = a + b\sqrt{3} + 3ci + di\sqrt{3} \in \tilde{\mathbb{Z}}[\sqrt{3}, i]$, let us denote $\tilde{\gamma} = a + 3ci - (b\sqrt{3} + di\sqrt{3})$.

Proposition 2.3. *$\gamma = a + b\sqrt{3} + 3ci + di\sqrt{3} \in \tilde{\mathbb{Z}}[\sqrt{3}, i]$ is a torsion-free unit if and only if $c = d = 0$ and $a^2 - 3b^2 = 1$.*

Proof. Let $\gamma = a + b\sqrt{3} + 3ci + di\sqrt{3} \in \tilde{\mathbb{Z}}[\sqrt{3}, i]$be a torsion-free unit and let us consider the following multiplicative group homomorphism.

$$f : U(\tilde{\mathbb{Z}}[\sqrt{3}, i]) \to U(\mathbb{Z}[i]),$$
$$\gamma \mapsto \gamma\tilde{\gamma}.$$

Due to $f(\gamma) = (a^2 - 3b^2 - 9c^2 + 3d^2) + 6(ac - bd)i$ the following equations can be obtained easily.

$$f(\gamma) = 1 \Rightarrow (a^2 - 3b^2 - 9c^2 + 3d^2) = 1 \text{ and } ac - bd = 0. \tag{2.2}$$

For $k \in \mathbb{Z}$, we write $ac = bd = k$. Now, let us assume that k $\neq 0$, then we write,

$$c = \frac{k}{a} \text{ and } d = \frac{k}{b}. \tag{2.3}$$

By substituting these equations in (2.3) in the first equation in (2.2), then we get

$$a^2 - 3b^2 - \frac{9k^2}{a^2} + \frac{3k^2}{b^2} = 1 \Rightarrow (a^2 - 3b^2) = \frac{a^2b^2}{a^2b^2 + 3k^2}$$
$$\Rightarrow 0 < \frac{a^2b^2}{a^2b^2 + 3k^2} < 1, (0 < a^2, b^2, k^2)$$
$$\Rightarrow (a^2 - 3b^2) \notin Z,$$

a contradiction with the assumption. Therefore, $k = 0$. Hence,

$$\left.\begin{array}{c} ac = 0 \quad \text{and} \quad bd = 0 \\ a^2 - 3b^2 - 9c^2 + 3d^2 = 1 \end{array}\right\} \Rightarrow c = d = 0 \text{ and } a^2 - 3b^2 = 1.$$

The converse is clear. □

Let us get characterize $\mathcal{U}_1(\mathbb{Z}C_{12})$ after two basic facts about torsion-free units.

Theorem 2.4. *$\mathcal{U}_1(\mathbb{Z}C_{12})$=$C_{12}\times < 3 + 2a + a^2 - a^4 - 2a^5 - 2a^6 - 2a^7 - a^8 + a^{10} + 2a^{11} >$.*

Proof. Let $\varepsilon = e^{2\pi i/12}$ be a 12-th primitive root of unity and consider the following isomorphism:

$$\Psi : C_{12} \to < \varepsilon >,$$
$$a^i \mapsto \varepsilon^i.$$

By extending this isomorphism linearly to $\mathbb{Z}C_{12}$ over $\mathbb{Z}$, we get the following embedding

$$\bar{\Psi} : \mathbb{Z}C_{12} \longrightarrow \mathbb{Z}[\varepsilon],$$
$$\sum \gamma_i a^i \mapsto \sum \gamma_i \varepsilon^i.$$

The image of $\gamma \in \mathcal{U}_1(\mathbb{Z}C_{12})$, a torsion-free unit,

$$\bar{\Psi}(\gamma) = \sum_{i=0}^{11} \gamma_i \varepsilon^i = \sum_{i=0}^{5} (\gamma_i - \gamma_{i+6})\varepsilon^i \quad (\varepsilon^6 = -1),$$

$$\bar{\Psi}(\gamma) = (2\gamma_0 - 1) + 2\gamma_1\varepsilon + 2\gamma_2\varepsilon^2 + 2\gamma_3\varepsilon^3 + 2(1 - \gamma_0 + \gamma_2)\varepsilon^4 \quad (\text{ Lemma 2.1})$$
$$+2(-\gamma_1 + \gamma_3)\varepsilon^5,$$

$$\bar{\Psi}(\gamma) = (3\gamma_0 - 2) + (2\gamma_1 - \gamma_3)\sqrt{3} + 3\gamma_3 i + (1 - \gamma_0 + 2\gamma_2)i\sqrt{3} \quad (\varepsilon = \frac{\sqrt{3}+i}{2}),$$

and hence $\bar{\Psi}(\gamma) \in \mathbb{Z}[\sqrt{3}, i]$. By Proposition 2.3, we have

$$\gamma_3 = 0, \gamma_0 = 1 + 2\gamma_2 \text{ and } (3\gamma_0 - 2)^2 - 3(2\gamma_1)^2 = 1. \tag{2.4}$$

By Pell-equation [2] we write,

$$\bar{\Psi}(\gamma) = (3\gamma_0 - 2) + (2\gamma_1)\sqrt{3} \in U(\mathbb{Z}[\sqrt{3}]) = \{\pm 1\}x < 2 + \sqrt{3} > .$$

Because of torsion-freeness of γ, there exist a smallest $k \in \mathbb{N}$ such that

$$\bar{\Psi}(\gamma) = (2 + \sqrt{3})^k . \tag{2.5}$$

The equality (2.5) holds for k=2 but not smaller k. So, we obtain

$$\gamma_0 = 3, \gamma_1 = 2, \gamma_2 = 1 \text{ and } \gamma_3 = 0.$$

By Lemma 2.1 and Theorem 1.1, we can get the generator of torsion-free part of $\mathcal{U}_1(\mathbb{Z}C_{12})$ as follows:

$$3 + 2a + a^2 - a^4 - 2a^5 - 2a^6 - 2a^7 - a^8 + a^{10} + 2a^{11}. \qquad \square$$

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