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FACTORIZATION OF WIENER-HOPF PLUS HANKEL OPERATORS WITH APW FOURIER SYMBOLS

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Abstract: Wiener-Hopf plus Hankel operators with Fourier symbols in the Wiener subalgebra of almost periodic functions are considered as operators acting between L^2 Lebesgue spaces. Conditions for left, right, or both-sided invertibility of such operators are obtained upon certain characteristics of their Fourier symbols, and new factorization notions. The introduced factorizations also allow the exposition of dependencies between the invertibility of Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same Fourier symbol.

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1. Introduction

Nowadays there is a growing interest in the study of regularity properties [3, 4, 11] of the so-called Wiener-Hopf plus Hankel operators, mainly because of their appearance in different kinds of applications. This is the case of *wave diffraction theory* where for some diffraction problems such operators occur in a natural manner [5, 6].

In the present paper we are going to consider Wiener-Hopf plus Hankel

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operators with Fourier symbols in a subalgebra of the almost periodic functions, and in the framework of the $L^2(\mathbb{R})$ Banach space of complex-valued Lebesgue measurable functions φ on $\mathbb{R}$, for which $|\varphi|^2$ is integrable. More precisely, we will consider operators of the form

$$WH_\phi \;=\; W_\phi + H_\phi \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1}$$

with W_ϕ and H_ϕ being, respectively, Wiener-Hopf and Hankel operators defined by

$$W_\phi \;=\; r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{2}$$
$$H_\phi \;=\; r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{3}$$

where $L^2_+(\mathbb{R})$ denotes the subspace of $L^2(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+ = (0, +\infty)$, r_+ represents the operator of restriction from $L^2_+(\mathbb{R})$ into $L^2(\mathbb{R}_+)$, $\mathcal{F}$ denotes the Fourier transformation, J is the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x),\ x \in \mathbb{R}$, and ϕ belongs to the so-called APW algebra.

For defining the APW functions, let us first consider the algebra of the almost periodic functions, usually denoted by AP, i.e. the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$), where $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$. Since every function in AP may be represented by a series, but not every function in AP may be represented by an absolutely convergent series, APW is precisely the subclass of all functions $\varphi \in AP$ which can be written in the form of an absolutely convergent series:

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x} \quad (x \in \mathbb{R}), \qquad \lambda_j \in \mathbb{R}, \qquad \sum_j |\varphi_j| < \infty.$$

Let us fix the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. By Bohr's Theorem, for each $\phi \in \mathcal{G}AP$ there exists a real number $k(\phi)$ and a function $\psi \in AP$ such that

$$\phi(x) = e^{ik(\phi)x} e^{\psi(x)}\,,$$

for all $x \in \mathbb{R}$. Since $k(\phi)$ is uniquely determined, $k(\phi)$ is usually called the *mean motion* of ϕ.

Let $A : X \to Y$ be a bounded linear operator acting between Banach spaces. If $\operatorname{Im} A$ is closed, the cokernel of A is defined as $\operatorname{Coker} A = Y/\operatorname{Im} A$. Then A is said to be *properly d-normal* if $\dim \operatorname{Coker} A$ is finite and $\dim \operatorname{Ker} A$ is infinite, *properly n-normal* if $\dim \operatorname{Ker} A$ is finite and $\dim \operatorname{Coker} A$ is infinite, and *Fredholm* if both $\dim \operatorname{Ker} A$ and $\dim \operatorname{Coker} A$ are finite.

For Wiener-Hopf operators with Fourier symbols in $\mathcal{G}AP$, there is a semi-Fredholm and invertibility criterion due to Gohberg, Feldman, Coburn and Douglas (cf. [8] and [10], or [2, Theorem 2.28]) based on the sign of the mean motion of the Fourier symbol of the operator. That criterion says that if the mean motion of the symbol is negative, then the Wiener-Hopf operator is properly d-normal and right-invertible; if the mean motion of the symbol is positive, the Wiener-Hopf operator is properly n-normal and left-invertible and in the case where the mean motion of the symbol is zero, the Wiener-Hopf operator is invertible. Such kind of criterion was the starting motivation for the present work. Thus, the main purpose of this paper is to establish a corresponding invertibility criterion for Wiener-Hopf plus Hankel operators with APW Fourier symbols.

2. Composition Identities for Wiener-Hopf Plus Hankel Operators upon the Structure of their Symbols

According to (1), (2) and (3), we have

$$WH_\phi = r_+(\mathcal{F}^{-1}\phi \cdot \mathcal{F} + \mathcal{F}^{-1}\phi \cdot \mathcal{F}J) = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J),$$

where $I_{L^2_+(\mathbb{R})}$ denotes the identity operator in $L^2_+(\mathbb{R})$.

Furthermore, since $I_{L^2_+(\mathbb{R})} + J = \ell^e r_+$, where $\ell^e : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes the even extension operator, we may write the Wiener-Hopf plus Hankel operator as

$$WH_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}\ell^e r_+. \tag{4}$$

From the Wiener-Hopf and Hankel operator theory, are known the following relations:

$$W_{\phi\varphi} = W_\phi\, \ell_0\, W_\varphi + H_\phi\, \ell_0\, H_{\widetilde{\varphi}},$$
$$H_{\phi\varphi} = W_\phi\, \ell_0\, H_\varphi + H_\phi\, \ell_0\, W_{\widetilde{\varphi}},$$

where $\ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is the zero extension operator. Additionally, from the last two identities, it follows that

$$WH_{\phi\varphi} = W_\phi\, \ell_0\, WH_\varphi + H_\phi\, \ell_0\, WH_{\widetilde{\varphi}}$$

and

$$WH_{\phi\varphi} = WH_\phi\, \ell_0\, WH_\varphi + H_\phi\, \ell_0\, WH_{\widetilde{\varphi}-\varphi}. \tag{5}$$

Let AP^- (AP^+) denote the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ, $\lambda \leq 0$ ($\lambda \geq 0$) and let APW^- (APW^+) be the set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset (-\infty, 0]$ ($\Omega(\psi) \subset [0, +\infty)$, respectively). Here $\Omega(\psi) = \{\lambda \in \mathbb{R} : M(\psi e_{-\lambda}) \neq 0\}$ is the *Bohr-Fourier spectrum* of ψ, where $M(\sigma)$ represents the *mean value* of σ (see, e.g., [2]). Naturally, $APW^- \subset AP^-$ and $APW^+ \subset AP^+$.

Let $H^\infty(\mathbb{C}_-)$ denote the set of all bounded and analytic functions in $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and $H^\infty_-(\mathbb{R})$ be the set of all functions in $L^\infty(\mathbb{R})$ that are non-tangential limits of elements in $H^\infty(\mathbb{C}_-)$. Due to (5), if we consider $\phi \in H^\infty_-(\mathbb{R})$ or φ being an even function, then we obtain a multiplicative relation

$$WH_{\phi\varphi} = WH_\phi \, \ell_0 \, WH_\varphi. \tag{6}$$

Note that if the symbol of a Wiener-Hopf operator admits a factorization of the form $\varphi_- \, \psi \, \varphi_+$, where $\varphi_\pm \in H^\infty_\pm(\mathbb{R})$ and $\psi \in L^\infty(\mathbb{R})$, it is possible to apply to the Wiener-Hopf operator the multiplicative property $W_{\varphi_- \, \psi \, \varphi_+} = W_{\varphi_-} \, \ell_0 \, W_\psi \, \ell_0 \, W_{\varphi_+}$ (see e.g. [2, Proposition 2.17]). With a convenient change, it is possible to construct for Wiener-Hopf plus Hankel operators a corresponding result as that one obtained for Wiener-Hopf operators. In the present case we may apply the multiplicative property on the left if the left factor belongs to $H^\infty_-(\mathbb{R})$ and on the right if the right factor is an even function, like it is asserted in the following proposition.

Proposition 1. *If $\varphi \in H^\infty_-(\mathbb{R})$ and $\psi, \phi \in L^\infty(\mathbb{R})$ such that $\phi = \widetilde{\phi}$, then*

$$\begin{aligned} WH_{\varphi\,\psi\,\phi} = WH_\varphi \, \ell_0 \, WH_{\psi\phi} &= WH_\varphi \, \ell_0 \, WH_\psi \, \ell_0 \, WH_\phi \\ &= W_\varphi \, \ell_0 \, WH_\psi \, \ell_0 \, WH_\phi. \end{aligned}$$

Proof. Since $\varphi \in H^\infty_-(\mathbb{R})$, we may apply the already presented multiplicative relation for Wiener-Hopf plus Hankel operators, see (6). Thus

$$WH_{\varphi\,\psi\phi} = WH_\varphi \, \ell_0 \, WH_{\psi\phi}. \tag{7}$$

In addition, since $\phi = \widetilde{\phi}$, it also follows from (6) that

$$WH_{\psi\phi} = WH_\psi \, \ell_0 \, WH_\phi. \tag{8}$$

From (7) and (8), we have that

$$WH_{\varphi\,\psi\,\phi} = WH_\varphi \, \ell_0 \, WH_\psi \, \ell_0 \, WH_\phi. \tag{9}$$

Since $\varphi \in H^\infty_-(\mathbb{R})$, we have $H_\varphi = 0$ due to the structure of the Hankel operators. Therefore $WH_\varphi = W_\varphi$ and it follows from (9) that $W_{\varphi\,\psi\,\phi} = W_\varphi \, \ell_0 \, WH_\psi \, \ell_0 \, WH_\phi$. □

Proposition 2. *If* $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ *and* $\widetilde{\phi_e} = \phi_e$*, then* WH_{ϕ_e} *is invertible and its inverse is the operator* $\ell_0\, WH_{\phi_e^{-1}}\, \ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R})$*.*

Proof. On one hand, we have

$$WH_{\phi_e \cdot \phi_e^{-1}}\, \ell_0 = WH_1\, \ell_0 = W_1\, \ell_0 = I_{L^2(\mathbb{R}_+)}, \tag{10}$$

where $I_{L^2(\mathbb{R}_+)}$ represents the identity operator in $L^2(\mathbb{R}_+)$. On the other hand, since $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ and $\widetilde{\phi_e} = \phi_e$, then $\widetilde{\phi_e^{-1}} = \phi_e^{-1}$ and therefore we may apply the multiplicative relation for Wiener-Hopf plus Hankel operators. So we have

$$WH_{\phi_e \cdot \phi_e^{-1}} = WH_{\phi_e}\, \ell_0\, WH_{\phi_e^{-1}}. \tag{11}$$

Thus, combining (10) and (11), we get that

$$WH_{\phi_e}\, \ell_0\, WH_{\phi_e^{-1}}\, \ell_0 = I_{L^2(\mathbb{R}_+)}. \tag{12}$$

In the same way, we obtain that

$$\ell_0\, WH_{\phi_e^{-1}}\, \ell_0\, WH_{\phi_e} = I_{L^2_+(\mathbb{R})}. \tag{13}$$

Therefore, (12)–(13) show that WH_{ϕ_e} is invertible and its inverse is $\ell_0\, WH_{\phi_e^{-1}} \ell_0$. □

3. *APW* Asymmetric Factorization

The following definition is motivated by the role of the *APW* factorization in the theory of Wiener-Hopf operators with *APW* symbols [2], and by the recent works on Toeplitz plus Hankel operators [1, 9] and convolution type operators with symmetry [6, 7].

Definition 3. We will say that a function $\phi \in \mathcal{G}APW$ admits an *APW asymmetric factorization* if it can be represented in the form

$$\phi = \phi_-\, e_\lambda\, \phi_e\,,$$

where $\lambda \in \mathbb{R}$, $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, $\phi_- \in \mathcal{G}APW^-$, $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ and $\widetilde{\phi_e} = \phi_e$.

The particular case of an *APW* asymmetric factorization with $\lambda = 0$ will be referred to as a *canonical APW asymmetric factorization.*

Such as the weak asymmetric factorization refereed in [1], the *APW* asymmetric factorization here introduced is also unique up to a constant.

Proposition 4. *Let $\phi \in \mathcal{G}APW$. Suppose that ϕ admits two APW asymmetric factorizations:*

$$\phi = \phi_-^{(1)}\; e_{\lambda_1}\; \phi_e^{(1)}\;,$$
$$\phi = \phi_-^{(2)}\; e_{\lambda_2}\; \phi_e^{(2)}\;.$$

Then $\lambda_1 = \lambda_2$, $\phi_-^{(1)} = \gamma\phi_-^{(2)}$ and $\phi_e^{(1)} = \gamma^{-1}\phi_e^{(2)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. The equality $\phi_-^{(1)}\; e_{\lambda_1}\; \phi_e^{(1)} = \phi_-^{(2)}\; e_{\lambda_2}\; \phi_e^{(2)}$, implies that

$$(\phi_-^{(2)})^{-1}\; \phi_-^{(1)}\; e_{\lambda_1} \;= e_{\lambda_2}\; \phi_e^{(2)}\; (\phi_e^{(1)})^{-1}. \tag{14}$$

Assume, without loss of generality, that $\lambda_1 \leq \lambda_2$. Then $\lambda = \lambda_1 - \lambda_2 \leq 0$. From (14) it follows that

$$(\phi_-^{(2)})^{-1}\; \phi_-^{(1)}\; e_{\lambda} \;= \phi_e^{(2)}\; (\phi_e^{(1)})^{-1}. \tag{15}$$

Since the right-hand side of (15) is an even function, $(\phi_-^{(2)})^{-1}\; \phi_-^{(1)}\; e_{\lambda}$ is also an even function. Put

$$\varphi = (\phi_-^{(2)})^{-1}\; \phi_-^{(1)}. \tag{16}$$

Thus $\varphi(x)\; e_\lambda(x) = \widetilde{\varphi}(x)\; \widetilde{e_\lambda}(x)$, i.e. $\varphi(x)\; e_\lambda(x) = \widetilde{\varphi}(x)\; e_{-\lambda}(x)$, or equivalently

$$\varphi(x)\; e_{2\lambda}(x) = \widetilde{\varphi}(x). \tag{17}$$

On one hand, since $\varphi \in \mathcal{G}APW^-$, we may apply the well-known characterization of $\mathcal{G}APW^-$ which assures the existence of a $\psi \in APW^-$ such that $\varphi = e^\psi$ (cf. e.g. [2, Lemma 3.4]). On the other hand, because $\widetilde{\varphi} \in \mathcal{G}APW^+$, by a corresponding characterization of $\mathcal{G}APW^+$, there exists a $\eta \in APW^+$ such that $\widetilde{\varphi} = e^\eta$. From (17), it follows that

$$e^{\psi(x)+i2\lambda x} = e^{\eta(x)},$$

which implies that $\lambda = 0$ and $\psi \in APW^- \cap APW^+$, i.e., $\lambda_1 = \lambda_2$ and ψ is a constant function. From (16), we get $\phi_-^{(1)} = \gamma\phi_-^{(2)}$ with $\gamma \in \mathbb{C}\setminus\{0\}$. By (15), we obtain $\phi_e^{(1)} = \gamma^{-1}\phi_e^{(2)}$. □

The following result shows that every invertible function in APW possesses an APW asymmetric factorization, and reveals through this the first indicator of the importance of such a factorization.

Theorem 5. *If $\phi \in \mathcal{G}APW$, then ϕ admits an APW asymmetric factorization.*

Proof. Suppose $\phi \in \mathcal{G}APW$. From a reformulation of Bohr's Theorem to the class $\mathcal{G}APW$ (see [2, Theorem 8.11]), there exists a $\varphi \in APW$ such that

$$\phi(x) = e^{ik(\phi)x} e^{\varphi(x)}$$

(and recalling that $k(\phi) \in \mathbb{R}$). Since $\varphi \in APW$, φ can be represented in the form of an absolutely convergent series

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x} \ , \quad \lambda_j \in \mathbb{R}.$$

Once again, since $\varphi \in APW$, we may write

$$\varphi(x) = \sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} + \sum_{\lambda_j \geq 0} \varphi_j e^{i\lambda_j x}, \tag{18}$$

where $\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} \in APW^-$ and $\sum_{\lambda_j \geq 0} \varphi_j e^{i\lambda_j x} \in APW^+$.

From (18), it follows that

$$\begin{aligned}
\varphi(x) &= \Big(\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\lambda_j \geq 0} \varphi_j e^{-i\lambda_j x} \Big) + \Big(\sum_{\lambda_j \geq 0} \varphi_j e^{i\lambda_j x} + \sum_{\lambda_j \geq 0} \varphi_j e^{-i\lambda_j x} \Big) \\
&= \Big(\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\alpha_j \leq 0} \varphi_j e^{i\alpha_j x} \Big) + \sum_{\lambda_j \geq 0} \varphi_j (e^{i\lambda_j x} + e^{-i\lambda_j x}),
\end{aligned}$$

with $\alpha_j = -\lambda_j$, for all j such that $\lambda_j \geq 0$.

Let $\varphi_-(x) = \sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\alpha_j \leq 0} \varphi_j e^{i\alpha_j x}$ and $\varphi_e(x) = \sum_{\lambda_j \geq 0} \varphi_j (e^{i\lambda_j x} + e^{-i\lambda_j x})$. Then we have $\varphi = \varphi_- + \varphi_e$, where $\varphi_- \in APW^-$ and $\varphi_e \in L^\infty(\mathbb{R})$ is an even function. Since $\phi(x) = e^{ik(\phi)x} e^{\varphi(x)}$, it results that

$$\begin{aligned}
\phi(x) &= e^{\varphi_-(x)} e^{ik(\phi)x} e^{\varphi_e(x)} \\
&= \phi_-(x) e^{ik(\phi)x} \phi_e(x),
\end{aligned}$$

where $\phi_- = e^{\varphi_-} \in \mathcal{G}APW^-$ and $\phi_e = e^{\varphi_e}$ is an (invertible) even function. □

4. Invertibility Criterion

Consider now two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces. The operators T and S are said to be *equivalent* if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E\, S\, F. \tag{19}$$

It follows from (19) that if two operators are equivalent, then they belong to the same regularity class [3, 4, 11]. More precisely, one of these operators is invertible, one-sided invertible, Fredholm, properly n-normal, properly d-normal, one-sided regularizable, generalized invertible or normally solvable, if and only if the other operator enjoys the same property.

Theorem 6. *Let $\phi \in \mathcal{G}APW$.*

(a) *If $k(\phi) < 0$, then WH_ϕ is properly d-normal and right-invertible.*

(b) *If $k(\phi) > 0$, then WH_ϕ is properly n-normal and left-invertible.*

(c) *If $k(\phi) = 0$, then WH_ϕ is invertible.*

Proof. From Theorem 5 it follows that ϕ admits an APW asymmetric factorization,

$$\phi = \phi_-\, e_{k(\phi)}\, \phi_e.$$

In the case where $k(\phi) < 0$, we have that $e_{k(\phi)} \in AP^-$. Since $AP^- = AP \cap H^\infty_-(\mathbb{R})$, it holds that $e_{k(\phi)} \in H^\infty_-(\mathbb{R})$ and hence

$$WH_\phi = W_{\phi_-}\, \ell_0\, W_{e_{k(\phi)}}\, \ell_0\, WH_{\phi_e}, \tag{20}$$

due to Proposition 1 and also taking into account that, because $e_{k(\phi)} \in H^\infty_-(\mathbb{R})$, $WH_{e_{k(\phi)}} = W_{e_{k(\phi)}}$. Since $\phi_- \in \mathcal{G}APW^-$, by the characterization of $\mathcal{G}APW^-$, there exists a $\psi \in APW^-$ such that $\phi_- = e^\psi$. Thus, the mean motion of ϕ_- is zero and by the theorem due to Gohberg, Feldman, Coburn and Douglas mentioned before, W_{ϕ_-} is invertible. From Proposition 2, we know that WH_{ϕ_e} is invertible. Therefore, since $\ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R})$ is also an invertible operator, (20) shows that WH_ϕ is equivalent to $W_{e_{k(\phi)}}$. Once again, by the theorem due to Gohberg, Feldman, Coburn and Douglas, since the mean motion of $W_{e_{k(\phi)}}$ is $k(\phi)$ and $k(\phi) < 0$, we have that the operator $W_{e_{k(\phi)}}$ is properly d-normal and right-invertible. Consequently, due to the equivalence relation (20), the operator WH_ϕ is also properly d-normal and right-invertible. This completes the proof of part (a).

Part (b) can be derived from part (a) by passage to adjoint operators.

Finally, let us now suppose that $k(\phi) = 0$. Then $\phi = \phi_- \, \phi_e$ and $WH_\phi = W_{\phi_-} \, \ell_0 \, WH_{\phi_e}$. Since W_{ϕ_-} and WH_{ϕ_e} are invertible, then WH_ϕ is also invertible. □

Theorem 7. *Let $\phi \in \mathcal{G}APW$ and*

$$T = \ell_0 r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell^e r_+ \mathcal{F}^{-1} e_{-k(\phi)} \cdot \mathcal{F} \ell^e r_+ \mathcal{F}^{-1} \phi_-^{-1} \cdot \mathcal{F} \ell : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}),$$

where ϕ_e^{-1}, $e_{-k(\phi)}$, and ϕ_-^{-1} are the inverses of the corresponding factors of an APW asymmetric factorization of ϕ, $\phi = \phi_- \, e_{k(\phi)} \, \phi_e$, and the operator $\ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes an arbitrary extension operator (i.e. T is independent of the choice of the extension ℓ). Then the operator T is a reflexive generalized inverse of WH_ϕ and, moreover, it is:

(a) the right-inverse of WH_ϕ, if $k(\phi) < 0$,

(b) the left-inverse of WH_ϕ, if $k(\phi) > 0$,

(c) the inverse of WH_ϕ, if $k(\phi) = 0$.

Proof. Since $\phi \in \mathcal{G}APW$, due to Theorem 5, we have that ϕ admits an APW asymmetric factorization, $\phi = \phi_- \, e_{k(\phi)} \, \phi_e$. Consequently, from (4), it follows that

$$WH_\phi = r_+ \, A_- \, E \, A_e \, \ell^e r_+,$$

where $A_- = \mathcal{F}^{-1} \phi_- \cdot \mathcal{F}$, $E = \mathcal{F}^{-1} e_{k(\phi)} \cdot \mathcal{F}$ and $A_e = \mathcal{F}^{-1} \phi_e \cdot \mathcal{F}$.

(i) If $k(\phi) \le 0$, consider

$$\begin{aligned} WH_\phi \, T &= r_+ \, A_- \, E \, A_e \, \ell^e r_+ \;\; \ell_0 r_+ A_e^{-1} \ell^e r_+ \, E^{-1} \ell^e r_+ \; A_-^{-1} \ell \\ &= r_+ \, A_- \, E \, A_e \, \ell^e r_+ A_e^{-1} \ell^e r_+ \, E^{-1} \ell^e r_+ \, A_-^{-1} \ell, \end{aligned} \tag{21}$$

where the term $\ell_0 r_+$ was omitted due to the fact that $r_+ \ell_0 r_+ = r_+$. Since A_e^{-1} preserves the even property of its symbol, we may also drop the first $\ell^e r_+$ term in (21), and obtain

$$WH_\phi \, T = r_+ \, A_- \, E \, \ell^e r_+ \, E^{-1} \ell^e r_+ \, A_-^{-1} \ell. \tag{22}$$

Additionally, since in the present case (due to $k(\phi) \le 0$) E^{-1} is a *plus type factor* [7, 11], we have $\ell^e r_+ \, E^{-1} \ell^e r_+ = E^{-1} \ell^e r_+$; also because A_- is a *minus type factor* it follows

$$WH_\phi \, T = r_+ \, A_- \ell^e r_+ \, A_-^{-1} \ell = r_+ \, \ell = I_{L^2(\mathbb{R}_+)}. \tag{23}$$

(ii) If $k(\phi) \geq 0$, we will now analyze the composition

$$T\,WH_\phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ \, E^{-1} \ell^e r_+ \; A_-^{-1} \ell \;\, r_+ \, A_- \, E \, A_e \, \ell^e r_+ \, . \tag{24}$$

In the present case E^{-1} is a *minus type factor* and for this reason $\ell^e r_+ E^{-1} \ell^e r_+ = \ell^e r_+ \, E^{-1}$. The same reasoning applies to the factor A_-^{-1}, and therefore the equality (24) takes the form

$$T\,WH_\phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ \; A_e \, \ell^e r_+ = \ell_0 r_+ \, \ell^e r_+ = \ell_0 r_+ = I_{L_+^2(\mathbb{R})} \, , \tag{25}$$

where we took into account that $\ell^e r_+ A_e \ell^e r_+ = A_e \ell^e r_+$.

(iii) Intersecting the last two cases, (i) and (ii), it follows that for $k(\phi) = 0$, the operator T is the (both-sided) inverse of WH_ϕ (cf. (23) and (25)).

Finally, please observe that all the above three situations are stronger than

$$WH_\phi \, T \, WH_\phi = WH_\phi \, , \qquad T \, WH_\phi \, T = T \, ,$$

i.e. T is a reflexive generalized inverse of WH_ϕ. □

5. *APW* Antisymmetric Factorization and its Consequences

Let us recall that ϕ is said to admit a *right APW factorization* [2] if $\phi = \varphi_- e_\lambda \varphi_+$, where $\varphi_- \in \mathcal{G}APW^-$, $\varphi_+ \in \mathcal{G}APW^+$, and $\lambda \in \mathbb{R}$. In addition, if $\lambda = 0$ this factorization is called a *canonical right APW factorization*.

In the present section the *APW* asymmetric factorization is related to a special case of *right APW factorization*, which we will call *APW* antisymmetric factorization. In this new kind of factorization a strong dependence between the left and the right factor occurs.

Definition 8. A function $\phi \in \mathcal{G}APW$ admits an *APW antisymmetric factorization* if it is possible to write

$$\phi = \phi_- \; e_{2\lambda} \; \widetilde{\phi_-^{-1}},$$

where $\lambda \in \mathbb{R}$, $e_{2\lambda}(x) = e^{2i\lambda x}$, $x \in \mathbb{R}$, and $\phi_- \in \mathcal{G}APW^-$.

Proposition 9. *Let $\phi \in \mathcal{G}APW$ and put $\Phi = \phi\widetilde{\phi^{-1}}$.*

(a) *If ϕ admits an APW asymmetric factorization, $\phi = \phi_- \; e_\lambda \; \phi_e$, then Φ admits an APW antisymmetric factorization with the same factor ϕ_- and the same index λ.*

(b) *If Φ admits an APW antisymmetric factorization, $\Phi = \psi_-\, e_{2\lambda}\, \widetilde{\psi_-^{-1}}$, then ϕ admits an APW asymmetric factorization with the same minus factor ψ_-, the same index λ and the even factor $\phi_e = e_{-\lambda}\psi_-^{-1}\phi$.*

Proof. (a) From the *APW* asymmetric factorization of ϕ, $\phi = \phi_-\, e_\lambda\, \phi_e$, we have $\widetilde{\phi^{-1}} = \phi_e^{-1}\, e_\lambda\, \widetilde{\phi_-^{-1}}$, with $\phi_- \in \mathcal{G}APW^-$. Hence $\Phi = \phi\widetilde{\phi^{-1}} = \phi_-\, e_{2\lambda}\, \widetilde{\phi_-^{-1}}$.

(b) It follows from the definition of the factor ϕ_e that $\phi = \psi_-\, e_\lambda\, \phi_e$. Thus it remains to prove that ϕ_e is an even function. Once again by the definition of ϕ_e, we obtain

$$\widetilde{\phi_e} = e_\lambda\widetilde{\psi_-^{-1}}\widetilde{\phi} = e_\lambda e_{-2\lambda}\psi_-^{-1}\phi = e_{-\lambda}\psi_-^{-1}\phi = \phi_e,$$

since $\widetilde{\psi_-^{-1}}\widetilde{\phi} = e_{-2\lambda}\psi_-^{-1}\phi$ (due to the *APW* antisymmetric factorization of Φ). Therefore ϕ_e is an even function. □

Theorem 10. *Let $\phi \in \mathcal{G}APW$. If ϕ admits a right APW factorization, $\phi = \varphi_- e_\lambda \varphi_+$, then ϕ admits an APW asymmetric factorization, $\phi = \phi_- e_\lambda \phi_e$, with $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.*

Proof. Suppose that ϕ admits a right *APW* factorization, i.e, $\phi = \varphi_- e_\lambda \varphi_+$, where $\varphi_- \in \mathcal{G}APW^-$, $\varphi_+ \in \mathcal{G}APW^+$. Considering $\Phi = \phi\widetilde{\phi^{-1}}$, we have

$$\Phi = \varphi_- e_\lambda \varphi_+ \widetilde{\varphi_+^{-1}} e_\lambda \widetilde{\varphi_-^{-1}} = \varphi_- \widetilde{\varphi_+^{-1}} e_{2\lambda} \varphi_+ \widetilde{\varphi_-^{-1}}. \tag{26}$$

Since $\varphi_- \in \mathcal{G}APW^-$ and $\varphi_+ \in \mathcal{G}APW^+$, then $\widetilde{\varphi_+^{-1}} \in \mathcal{G}APW^-$, $\widetilde{\varphi_-^{-1}} \in \mathcal{G}APW^+$ and therefore $\varphi_-\widetilde{\varphi_+^{-1}} \in \mathcal{G}APW^-$ and $\varphi_+\widetilde{\varphi_-^{-1}} \in \mathcal{G}APW^+$. Putting $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, it follows from (26) that $\Phi = \phi_- e_{2\lambda}\widetilde{\phi_-^{-1}}$. Since $\phi_- \in \mathcal{G}APW^-$, it results that Φ admits a *APW* antisymmetric factorization. By Proposition 9, that implies that ϕ admits a *APW* asymmetric factorization, $\phi = \phi_- e_\lambda \phi_e$, with $\phi_e = e_{-\lambda}\phi_-^{-1}\phi$. Rewriting ϕ_- and ϕ_e at the cost of the factors of the right *APW* factorization, φ_- and φ_+, we have $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$ and $\phi_e = \widetilde{\varphi_+}\varphi_+$. □

Corollary 11. *Let $\phi \in \mathcal{G}APW$. If ϕ admits a canonical right APW factorization, $\phi = \varphi_-\varphi_+$, then ϕ admits a canonical APW asymmetric factorization, $\phi = \phi_-\phi_e$, with $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.*

Proof. The result is a direct consequence of Theorem 10, if we take there $\lambda = 0$. □

Theorem 12. *Let $\phi \in \mathcal{G}APW$. If W_ϕ is invertible, then WH_ϕ is invertible.*

Proof. According to [2, Corollary 9.8], if $\phi \in \mathcal{G}APW$, then W_ϕ is invertible if and only if ϕ admits a canonical right APW factorization. Thus, as W_ϕ is invertible, ϕ admits a canonical right APW factorization. Suppose that $\phi = \varphi_- \varphi_+$ is a canonical right APW factorization of ϕ. By Corollary 11, ϕ admits a canonical APW asymmetric factorization, $\phi = \phi_- \phi_e$, where

$$\phi_- = \varphi_- \, \widetilde{\varphi_+^{-1}} \, ,$$
$$\phi_e = \widetilde{\varphi_+} \, \varphi_+ \, .$$

Because $\phi = \phi_- \phi_e$, we obtain $WH_\phi = WH_{\phi_-} \, \ell_0 \, WH_{\phi_e}$. As $\phi_- \in \mathcal{G}APW^-$, we have $k(\phi_-) = 0$ and, from Theorem 6, it follows that WH_{ϕ_-} is invertible. Since WH_{ϕ_e} is also invertible (cf. Proposition 2), it results that WH_ϕ is invertible. $\square$

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