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SEMI-GLOBAL STABILIZATION AND OUTPUT REGULATION FOR DISCRETE-TIME SINGULAR SYSTEMS WITH ACTUATOR SATURATION

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Abstract: This paper studies the problems of semi-global stabilization and output regulation for linear discrete-time singular systems subject to actuator saturation. The singular system is first transformed into a reduced-order normal system by a standard coordinate transformation. The solvability conditions of the semi-global stabilization and the output regulation for the reduced-order normal system are verified. Furthermore, it is shown that the controller solving the stabilization (output regulation) problem for the reduced-order normal system also solves the stabilization (output regulation) problem of the original singular system.

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1. Introduction

Singular systems arise in many engineering systems (such as power systems, chemical processing), social systems, biological systems and so on. Control of singular systems has been extensively studied in the past decades (see, for example, [3], [4], [7]). singular systems are also called descriptor systems, implicit systems, differential/algebraic systems, and generalized state space systems.

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On the other hand, in control systems design, actuator saturation is inevitable. It can severely degrade the closed-loop performance and sometimes even drive the system to instability. For their practical and theoretical importance, the analysis and synthesis of systems with actuator saturation have received much attention (see [1], [6], [9], [10], [11] and references therein). However, as far as we know, the controller design of singular systems subject to actuator saturation is seldom investigated. The problem of eigenstructure assignment for linear continuous-time systems was discussed by Tarbouriech et al [12] and an approach for discrete singular constrained regulation was established by Georgiou et al [5]. In these two papers, the controllers are designed such that the saturations do not occur. Local stabilization for linear continuous-time singular systems with input saturation was considered by Tarbouriech et al [13] using the generalized Riccati and Lyapunov equation approaches. The semi-global stabilization and the output regulation problems for linear continuous-time singular systems subject to actuator saturation were addressed by Lan et al [8].

In this paper, based on the results by Lan et al [8], we will further study semi-global stabilization and output regulation for linear discrete-time singular systems subject to saturation nonlinearities. Our main results can be viewed as an extension of those for the continuous-time case by Lan et al [8]. The proofs for both semi-global stabilization and output regulation can be divided into three steps. First, the singular system is converted into a reduced-order normal system via a coordinate transformation. Second, we verify that the transformed normal system satisfies all the conditions of the semi-global stabilization (output regulation) and therefore design the corresponding controller. Finally, we prove that the designed controller also solves the semi-global stabilization (output regulation) problem for the original singular system.

The paper is organized as follows. Section 2 formulates the problem and gives some preliminaries. The semi-global stabilization and output regulation problems via output feedback are considered in Section 3 and Section 4, respectively. A numerical example illustrating our design procedure and its effectiveness is given in Section 5. Finally, in Section 6, concluding remarks end the paper.

Notations. The following notations are used throughout this paper. $\Re$ is the set of real numbers and $\Re^n$, $\Re^{m\times n}$ are sets of real vectors with dimension n and real matrices with dimension $m \times n$, respectively. The notation $X > Y$, where X and Y are symmetric matrices, means that the matrix $X-Y$ is positive definite. I and 0 denote the identity matrix and zero matrix of compatible dimensions. $\Lambda(E, A) = \{z|z \in \mathcal{C}, \det(zE - A) = 0\}$. $U^+ = \{z|z \in \mathcal{C}, |z| < 1\}$

and $\bar{U}^+ = \{z | z \in \mathcal{C}, |z| \le 1\}$. $\|\cdot\|$ denotes the standard Euclidean 2 norm. For a n-dimensional vector x, $|x|_\infty = \max_i \{|x_i|\}$. $\|v\|_\infty = \sup_{t\ge 0} |v|_\infty$ and for any integer $L \ge 0$, $\|v\|_{\infty,L} = \sup_{t\ge L} |v|_\infty$.

2. Preliminaries and Problems Statement

Consider the following linear discrete-time singular system subject to actuator saturation

$$Ex(k+1) = Ax(k) + B\sigma(u(k))\,, \tag{1}$$

$$y(k) = Cx(k)\,, \tag{2}$$

where $x(k) \in \Re^n$ is the state, $u(k) \in \Re^m$ the control input, $y(k) \in \Re^p$ the measured output. E, A, B, C are constant matrices with appropriate dimensions. $\mathrm{rank}(E) = r < n$. $\sigma(\cdot)$ is the standard saturation function, i.e., $\sigma(u) = \left[\ \sigma(u_1), \sigma(u_2), \cdots, \sigma(u_m)\ \right]^T$, where $\sigma(u_i) = \mathrm{sign}(u_i)\min\{1, |u_i|\}$.

First, let us introduce some definitions about singular system

$$Ex(k+1) = Ax(k). \tag{3}$$

Definition 1. (see [4]) The singular system (3) is said to be regular if $\det(zE - A)$ is not identically zero.

Definition 2. (see [4]) The singular system (3) is said to be causal if $\deg(\det(zE - A)) = \mathrm{rank}(E)$.

Definition 3. (see [4]) The singular system (3) is said to be stable if all of its finite poles are within the unit circle, i.e., $\Lambda(E, A) \subset U^+$.

In the sequel, we will use the following terms about linear discrete-time singular systems. The pair (E, A, B) is said to be stabilizable if there exists a matrix K such that $(E, A+BK)$ is stable. (E, A, C) is detectable if there exists a matrix L such that $(E, A + LC)$ is stable. (E, A, B) is said to be strongly stabilizable if there exists a matrix K such that $(E, A + BK)$ is stable and causal. (E, A, C) is said to be strongly detectable if there exists a matrix L such that $(E, A + LC)$ is stable and causal. (E, A) is said to be semistable if $\Lambda(E, A) \subset \bar{U}^+$. (E, A) is said to be strongly stable if $\Lambda(E, A) \subset U^+$ and $\deg(\det(zE - A)) = \mathrm{rank}(E)$.

Here we introduce a dynamic output feedback controller of the form

$$\begin{aligned} z(k+1) &= A_z z(k) + B_z y(k), \\ u(k) &= C_z z(k) + D_z y(k), \end{aligned} \tag{4}$$

where $z \in \Re^{n_z}$, A_z, B_z, C_z, D_z are constant matrices to be solved.

In this paper, we consider two interesting problems of discrete-time singular system (1) and (2) subject to actuator saturation. More specifically, we consider the discrete-time counterpart of the results by Lan et al [8]. Our aim is to solve two problems, namely, the semi-global stabilization problem and the output regulation problem with actuator saturation.

Problem 1. (Semi-Global Stabilization Problem) Given any compact sets $X_0 \subset \Re^n$ containing the origin and $Z_0 \subset \Re^{n_z}$ containing the origin, find an output feedback controller (4) such that for all $(x(0), z(0)) \in X_0 \times Z_0$, the solution of the closed-loop system

$$\begin{array}{rcl} Ex(k+1) & = & Ax(k) + B\sigma(C_z z(k) + D_z Cx(k)), \\ z(k+1) & = & A_z z(k) + B_z Cx(k), \end{array} \tag{5}$$

exists for all $k = 0, 1, \cdots$, and there exist a scalar $\alpha > 0$ and a scalar $0 < \beta < 1$ which satisfy

$$\left\| \begin{array}{c} x(k) \\ z(k) \end{array} \right\| \leq \alpha\beta^k \left\| \begin{array}{c} x(0) \\ z(0) \end{array} \right\|.$$

Another interesting problem considered in this paper is the output regulation problem for discrete-time singular system subject to actuator saturation together with an exosystem that generates disturbance and reference signals described as follows

$$\begin{array}{rcl} Ex(k+1) & = & Ax(k) + B\sigma(u(k)) + Pw(k), \\ w(k+1) & = & Sw(k), \\ y(k) & = & Cx(k) + Qw(k), \end{array} \tag{6}$$

where $w \in \Re^q$ is the exogenous signal, P, S, Q are constant matrices. Under the output feedback controller (4), the closed-loop system becomes

$$\begin{array}{rcl} Ex(k+1) & = & Ax(k) + B\sigma(C_z z(k) + D_z Cx(k)) + Pw(k), \\ w(k+1) & = & Sw(k), \\ z(k+1) & = & A_z z(k) + B_z y(k), \\ y(k) & = & Cx(k) + Qw(k). \end{array} \tag{7}$$

Problem 2. (Semi-Global Output Regulation Problem) Given any compact sets $X_0 \subset \Re^n$ containing the origin and $Z_0 \subset \Re^{n_z}$ containing the origin, find an output feedback controller (4) such that the closed-loop system (7) satisfies:

1) (Internal Stability) When $w = 0$, the equilibrium point $(x, z) = (0, 0)$ of the closed-loop system (7) is stable with $X_0 \times Z_0$ contained in its basin of attraction.

2) (Output Regulation) For all $(x(0), z(0), w(0)) \in X_0 \times Z_0 \times W_0$ where $W_0 \in \Re^q$ is a compact set containing the origin of $\Re^q$, the solution of the closed-loop system satisfies

$$\lim_{k\to+\infty} y(k) = 0.$$

In order to obtain our main results, the following lemmas are needed.

Lemma 1. (see [9]) *For normal linear systems, i.e., $E = I$, the discrete-time semi-global stabilization problem is solvable if (A, B) is stabilizable, (A, C) is detectable and A has all eigenvalues inside or on the unit circle. Moreover, the family of output feedback laws take the following form*

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B\sigma(u) - L(y(k) - C\hat{x}(k)), \\ u(k) &= -(B^T P(\varepsilon)B + I)^{-1} B^T P(\varepsilon) A\hat{x}(k), \end{aligned} \tag{8}$$

where L is chosen such that $A + LC$ is asymptotically stable, and $P(\varepsilon) > 0$ is the solution to the discrete-time algebraic Riccati equation (DARE)

$$P = A^T PA - A^T PB(B^T PB + I)^{-1} B^T PA + \varepsilon I, \varepsilon \in (0, \varepsilon^*], \tag{9}$$

for some $\varepsilon^* \in (0, 1]$; $\|u\|_\infty = \| - (B^T P(\varepsilon)B + I)^{-1} B^T P(\varepsilon) A\hat{x}(k)\|_\infty \leq 1$, for all $\varepsilon \in (0, \varepsilon^*]$.

Lemma 2. (see [9]) *For normal linear systems, i.e., $E = I$, the semi-global output regulation problem is solvable for all $w(0) \in W_0 \subset \Re^q$ if:*

1) (A, B) is stabilizable, and A has all eigenvalues inside or on the unit circle. Moreover, the pair $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix}\right)$ is detectable.

2) The eigenvalues of S are on or outside of the unit circle.

3) There exist matrices Π and Γ such that

(a) They solve the following linear matrix equations

$$\begin{cases} \Pi S = A\Pi + B\Gamma + P, \\ C\Pi + Q = 0. \end{cases} \tag{10}$$

(b) There exist $\delta > 0$ and $L \geq 0$ such that $\|\Gamma w\|_{\infty,L} \leq 1 - \delta$ for all w with $w(0) \in W_0$.

Moreover, an output feedback controller can be constructed as

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{w}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \sigma(u) - \begin{bmatrix} L_A \\ L_S \end{bmatrix} \left(y - \begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{w}(k) \end{bmatrix} \right),$$

$$u(k) = F(\varepsilon)\hat{x}(k) + (-F(\varepsilon)\Pi + \Gamma)\hat{w}(k), \tag{11}$$

where $F(\varepsilon) = -(B^T P(\varepsilon) B + I)^{-1} B^T P(\varepsilon) A$, $P(\varepsilon) > 0$ *is the unique solution to the DARE (9). The matrices* L_A *and* L_S *are chosen such that the following matrix is asymptotically stable*

$$\begin{bmatrix} A + L_A C & P + L_A Q \\ L_S C & S + L_S Q \end{bmatrix}.$$

Before presenting our main results, we make some assumptions.

A1) (E, A, B) is strongly stabilizable, and (E, A) is semistable and causal;

A2) (E, A, C) is strongly detectable;

A3) $\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix} \right)$ is strongly detectable;

A4) The eigenvalues of S are on or outside of the unit circle;

A5) There exist matrices Π and Γ such that:

(a) They solve the following linear matrix equations

$$\begin{cases} E\Pi S = A\Pi + B\Gamma + P, \\ C\Pi + Q = 0. \end{cases} \tag{12}$$

(b) There exist $\delta > 0$ and $L \geq 0$ such that $\|\Gamma w\|_{\infty,L} \leq 1 - \delta$ for all w with $w(0) \in W_0$.

3. Semi-Global Stabilization via Normal Output Feedback

In this section, we will present a result on semi-global stabilization for singular systems with actuator saturation.

Theorem 1. *Under the assumptions* **A1)** *and* **A2)**, *the semi-global stabilization problem is solvable.*

Proof. Similar to the proof of that for the continuous-time case in [8], we divide the proof into three steps.

Step 1. Use a standard coordinate transformation. By matrix theory, there exist two nonsingular matrices $M_1, N_1 \in \Re^{n \times n}$ such that

$$\bar{E} = M_1 E N_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \bar{A} = M_1 A N_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \tag{13}$$

Now partition $M_1 B$, $C N_1$ according to $M_1 E N_1$, and it follows that

$$\begin{aligned} &\bar{B} = M_1 B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \bar{C} = C N_1 = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \\ &\bar{x}(k) = N_1^{-1} x(k) = \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix}. \end{aligned} \tag{14}$$

By (1), (2), (13) and (14), we have

$$\bar{x}_1(k+1) = \bar{A}_{11}\bar{x}_1(k) + \bar{A}_{12}\bar{x}_2(k) + \bar{B}_1\sigma(u), \tag{15}$$
$$0 = \bar{A}_{21}\bar{x}_1(k) + \bar{A}_{22}\bar{x}_2(k) + \bar{B}_2\sigma(u), \tag{16}$$
$$y(k) = \bar{C}_1\bar{x}_1(k) + \bar{C}_2\bar{x}_2(k). \tag{17}$$

It can be easily verified that under the assumptions **A1)** and **A2)**, we know that $(\bar{E}, \bar{A}, \bar{B})$ is strongly stabilizable, $(\bar{E}, \bar{A})$ is semistable and causal, and $(\bar{E}, \bar{A}, \bar{C})$ is strongly detectable. We can also prove that $\bar{A}_{22}$ is nonsingular since

$$\deg(\det(z\bar{E} - \bar{A})) = \deg(\det \begin{bmatrix} zI_r - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix}) = \mathrm{rank}(\bar{E}) = r.$$

Thus, (16) can be rewritten as

$$\bar{x}_2(k) = -\bar{A}_{22}^{-1}(\bar{A}_{21}\bar{x}_1(k) + \bar{B}_2\sigma(u)). \tag{18}$$

Substituting (18) into (15) and (17), it follows

$$\begin{aligned} \bar{x}_1(k+1) &= A_r\bar{x}_1(k) + B_r\sigma(u), \\ y(k) &= C_r\bar{x}_1(k) + D_r\sigma(u), \end{aligned} \tag{19}$$

where $A_r = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$, $B_r = \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2$, $C_r = \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21}$, $D_r = -\bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2$.

Step 2. Now, we will prove that under assumptions **A1)** and **A2)**, (A_r, B_r) is stabilizable, (A_r, C_r) is detectable and all the eigenvalues of A_r are located in or on the unit circle. Note that

$$\begin{aligned} \det(zE - A) &= \det(M_1^{-1}(zM_1EN_1 - M_1AN_1)N_1^{-1}) \\ &= \det(M_1^{-1})\det(N_1^{-1})\det\left(\begin{bmatrix} zI_r - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix}\right) \\ &= \det(M_1^{-1})\det(N_1^{-1})\det(-\bar{A}_{22})\det(zI_r - A_r). \end{aligned}$$

Since (E, A) is semistable, we have $\Lambda(I, A_r) \subset \bar{U}^+$.

Since $(\bar{E}, \bar{A}, \bar{B})$ is strongly stabilizable, there exists a matrix K such that $(\bar{E}, \bar{A} + \bar{B}K)$ is stable and causal. Let $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, we have

$$\begin{aligned} &\det(z\bar{E} - (\bar{A} + \bar{B}K)) \\ &= \det\left\{\begin{bmatrix} zI_r - (A_{11} + \bar{B}_1K_1) & -(A_{12} + \bar{B}_1K_2) \\ -(A_{21} + \bar{B}_2K_1) & -(A_{22} + \bar{B}_2K_2) \end{bmatrix}\right\} = \det(-\tilde{A}_{22}) \\ &\times \det\{zI_r - (\bar{A}_{11} + \bar{B}_1K_1) + (\bar{A}_{12} + \bar{B}_1K_2)\tilde{A}_{22}^{-1}(\bar{A}_{21} + \bar{B}_2K_1)\}. \end{aligned} \tag{20}$$

The fact that matrix $\tilde{A}_{22} = \bar{A}_{22} + \bar{B}_2 K_2$ is invertible is guaranteed since $(\bar{E}, \bar{A} + \bar{B}K)$ is causal.

Using the matrix inverse formula, we get

$$(\bar{A}_{11} + B_1 K_1) - (\bar{A}_{12} + B_1 K_2)(\bar{A}_{22} + \bar{B}_2 K_2)^{-1}(\bar{A}_{21} + \bar{B}_2 K_1) = A_r + B_r \bar{K},$$

where $\bar{K} = K_1 - (K_2 \bar{A}_{22}^{-1} \bar{B}_2 + I)^{-1} K_2 \bar{A}_{22}^{-1} (\bar{A}_{21} + \bar{B}_2 K_1)$.

Then we have

$$\det(z\bar{E} - (\bar{A} + \bar{B}K)) = \det(-\bar{A}_{22} - \bar{B}_2 K_2) \times \det(zI_r - (A_r + B_r \bar{K})).$$

The fact that (A_r, B_r) is stabilizable follows directly since $(\bar{E}, \bar{A}, \bar{B})$ is stabilizable. The proof that (A_r, C_r) is detectable is similar to that of the stabilizability of (A_r, B_r), and is thus omitted.

By Lemma 1, the semi-global stabilization problem of the normal system (19) is solvable and the corresponding output feedback controller can be designed as

$$\begin{aligned} \hat{x}(k+1) &= A_r \hat{x}(k) + B_r \sigma(u) - L(y(k) - C_r \hat{x}(k) - D_r \sigma(u)), \\ u(k) &= -(B_r^T P(\varepsilon) B_r + I)^{-1} B_r^T P(\varepsilon) A_r \hat{x}(k), \end{aligned} \tag{21}$$

where L is chosen such that $A_r + LC_r$ is asymptotically stable, and $P(\varepsilon)$ solves the following DARE

$$P(\varepsilon) = A_r^T P(\varepsilon) A_r - A_r^T P(\varepsilon) B_r (B_r^T P(\varepsilon) B_r + I)^{-1} B_r^T P(\varepsilon) A_r + \varepsilon I, \quad 0 < \varepsilon \leq 1.$$

Step 3. Given any compact set $X_0 \times Z_0 \subset \Re^n \times \Re^r$, by (1) and (21), it follows that

$$\begin{aligned} &Ex(k+1) = Ax(k) + B\sigma(u), \\ &\hat{x}(k+1) = A_r \hat{x}(k) + B_r \sigma(u) - L(y(k) - C_r \hat{x}(k) - D_r \sigma(u)), \\ &u(k) = -(B_r^T P(\varepsilon) B_r + I)^{-1} B_r^T P(\varepsilon) A_r \hat{x}(k). \end{aligned} \tag{22}$$

Let $\bar{X}_0 = \bar{X}_{10} \times \bar{X}_{20} \subset \Re^r \times \Re^{n-r}$ be compact and be sufficiently large such that $x(k) \in X_0$ implies $\bar{x} = N_1^{-1} x(k) \in \bar{X}_0$. Then, by Lemma 1, for $\bar{X}_{10} \times Z_0 \subset \Re^r \times \Re^r$, there exists an $\varepsilon^* \in (0, 1]$ such that for each $\varepsilon \in (0, \varepsilon^*]$, $\|u\|_\infty = \| - (B_r^T P(\varepsilon) B_r + I)^{-1} B_r^T P(\varepsilon) A_r \hat{x}(k)\|_\infty \leq 1$, and the equilibrium point at the origin of the closed-loop system composed of (19) and (21) is stable with $\bar{X}_{10} \times Z_0$ contained in its basin of attraction. There exist $\alpha_1 > 0$ and $0 < \beta < 1$ such that for all $\bar{x}_1(0) \in \bar{X}_{10}$ and $z(0) \in Z_0$, it follows that

$$\left\| \begin{array}{c} \bar{x}_1(k) \\ \hat{x}(k) \end{array} \right\| \leq \alpha_1 \beta^k \left\| \begin{array}{c} \bar{x}_1(0) \\ \hat{x}(0) \end{array} \right\|.$$

By (18) and the fact that the actuator never saturates under the control law, we have

$$\begin{aligned}\|\bar{x}_2(k)\| &\le \|\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1(k)\| + \|\bar{A}_{22}^{-1}\bar{B}_2(B_r^T P(\varepsilon)B_r + I)^{-1}B_r^T P(\varepsilon)A_r\hat{x}(k)\| \\ &\le \|\bar{A}_{22}^{-1}\bar{A}_{21}\|\|\bar{x}_1(k)\| + \|\bar{A}_{22}^{-1}\bar{B}_2(B_r^T P(\varepsilon)B_r + I)^{-1}B_r^T P(\varepsilon)A_r\|\|\hat{x}(k)\| \\ &\le \left(\|\bar{A}_{22}^{-1}\bar{A}_{21}\| + \|\bar{A}_{22}^{-1}\bar{B}_2(B_r^T P(\varepsilon)B_r + I)^{-1}B_r^T P(\varepsilon)A_r\|\right)\alpha_1\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \hat{x}(0)\end{matrix}\right\| \\ &\le \alpha_2\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \hat{x}(0)\end{matrix}\right\|, \end{aligned} \tag{23}$$

where $\alpha_2 = \left(\|\bar{A}_{22}^{-1}\bar{A}_{21}\| + \|\bar{A}_{22}^{-1}\bar{B}_2(B_r^T P(\varepsilon)B_r + I)^{-1}B_r^T P(\varepsilon)A_r\|\right)\alpha_1$.
Then

$$\begin{aligned}\left\|\begin{matrix}\bar{x}_1(k)\\ \bar{x}_2(k)\\ \hat{x}(k)\end{matrix}\right\| &\le \left\|\begin{matrix}\bar{x}_1(k)\\ \hat{x}(k)\end{matrix}\right\| + \|\bar{x}_2(k)\| \le \alpha_1\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \hat{x}(0)\end{matrix}\right\| + \alpha_2\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \hat{x}(0)\end{matrix}\right\| \\ &= (\alpha_1+\alpha_2)\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \hat{x}(0)\end{matrix}\right\| \le (\alpha_1+\alpha_2)\beta^k \left\|\begin{matrix}\bar{x}_1(0)\\ \bar{x}_2(0)\\ \hat{x}(0)\end{matrix}\right\|. \end{aligned} \tag{24}$$

Thus, we have

$$\begin{aligned}\left\|\begin{matrix}x(k)\\ \hat{x}(k)\end{matrix}\right\| &\le \left\|\begin{matrix}N_1 & 0\\ 0 & I\end{matrix}\right\| \left\|\begin{matrix}\bar{x}(k)\\ \hat{x}(k)\end{matrix}\right\| \le \left\|\begin{matrix}N_1 & 0\\ 0 & I\end{matrix}\right\| (\alpha_1+\alpha_2)\beta^k \left\|\begin{matrix}\bar{x}(0)\\ \hat{x}(0)\end{matrix}\right\| \\ &\le \left\|\begin{matrix}N_1 & 0\\ 0 & I\end{matrix}\right\| \left\|\begin{matrix}N_1^{-1} & 0\\ 0 & I\end{matrix}\right\| (\alpha_1+\alpha_2)\beta^k \left\|\begin{matrix}x(0)\\ \hat{x}(0)\end{matrix}\right\|, \end{aligned} \tag{25}$$

for any $(x(0), \hat{x}(0)) \in X_0 \times Z_0$, which shows that the output feedback controller (21) also solves the semi-global stabilization problem for the singular system (1) and (2). This completes the proof. □

4. Semi-Global Output Regulation via Normal Output Feedback

In this section, we will deal with the problem of semi-global output regulation for discrete-time singular system (1) and (2) subject to actuator saturation using the output feedback controller (4).

Theorem 2. *For linear discrete-time singular system (1) and (2) with the output feedback controller (4), if the assumptions* **A1)**, **A3)**, **A4)** *and* **A5)** *are satisfied, the problem of semi-global output regulation via normal output feedback subject to actuator saturation is solvable for all $w(0) \in W_0 \subset \Re^q$.*

Proof. Here we adopt the same strategy and procedures as that in Theorem 1 to solve the semi-global output regulation problem.

Step 1. By the same coordinate transformation in (13), let

$$\bar{P} = M_1 P = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix}$$

and (6) becomes

$$\bar{x}_1(k+1) = \bar{A}_{11}\bar{x}_1(k) + \bar{A}_{12}\bar{x}_2(k) + \bar{B}_1\sigma(u) + \bar{P}_1 w(k), \tag{26}$$

$$0 = \bar{A}_{21}\bar{x}_1(k) + \bar{A}_{22}\bar{x}_2(k) + \bar{B}_2\sigma(u) + \bar{P}_2 w(k), \tag{27}$$

$$y(k) = \bar{C}_1\bar{x}_1(k) + \bar{C}_2\bar{x}_2(k) + Qw(k). \tag{28}$$

Since $\bar{A}_{22}$ is nonsingular, if we substitute (27) into (26) and (28) to eliminate $\bar{x}_2(k)$, we have

$$\begin{array}{rcl} \bar{x}_1(k+1) & = & A_r\bar{x}_1(k) + B_r\sigma(u) + P_r w(k), \\ y(k) & = & C_r\bar{x}_1(k) + D_r\sigma(u) + Q_r w(k), \end{array} \tag{29}$$

where A_r, B_r, C_r, D_r are defined as above, and

$$P_r = \bar{P}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{P}_2, \quad Q_r = Q - \bar{C}_2\bar{A}_{22}^{-1}\bar{P}_2.$$

Step 2. In order to prove the semi-global output regulation problem of the normal system (29), we just need to verify that all the conditions stated in Lemma 2 are satisfied.

In the proof of Theorem 1, we can see that (A_r, B_r) is stabilizable and all the eigenvalues of A_r are located within the unit circle.

The condition that $\left(\begin{bmatrix} A_r & P_r \\ 0 & S \end{bmatrix}, \begin{bmatrix} C_r & Q_r \end{bmatrix}\right)$ is detectable can be proved as follows.

Let us introduce two nonsingular matrices

$$M_2 = \begin{bmatrix} I_r & -\bar{A}_{12}\bar{A}_{22}^{-1} & 0 \\ 0 & I_{n-r} & 0 \\ 0 & 0 & I_q \end{bmatrix}, \quad N_2 = \begin{bmatrix} I_r & 0 & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I_{n-r} & -\bar{A}_{22}^{-1}\bar{P}_2 \\ 0 & 0 & I_q \end{bmatrix}.$$

It is easy to verify that

$$M_2 \begin{bmatrix} \bar{E} & 0 \\ 0 & I_q \end{bmatrix} N_2 = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_q \end{bmatrix},$$

$$M_2 \begin{bmatrix} \bar{A} & \bar{P} \\ 0 & S \end{bmatrix} N_2 = \begin{bmatrix} A_r & 0 & P_r \\ 0 & \bar{A}_{22} & 0 \\ 0 & 0 & S \end{bmatrix},$$

$$\begin{bmatrix} \bar{C} & Q \end{bmatrix} N_2 = \begin{bmatrix} C_r & \bar{C}_2 & Q_r \end{bmatrix}.$$

The strong detectability of

$$\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_r & P_r & 0 \\ 0 & S & 0 \\ 0 & 0 & \bar{A}_{22} \end{bmatrix}, \begin{bmatrix} C_r & Q_r & \bar{C}_2 \end{bmatrix} \right)$$

guarantees that there exist matrices G_1, G_2 and G_3 such that all the eigenvalues of

$$\Pi = \det \left\{ \begin{bmatrix} \lambda I_r & 0 & 0 \\ 0 & \lambda I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_r & P_r & 0 \\ 0 & S & 0 \\ 0 & 0 & \bar{A}_{22} \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \begin{bmatrix} C_r & Q_r & \bar{C}_2 \end{bmatrix} \right\}$$

are located in the unit circle and $-A_{22} + G_3\bar{C}_2$ is nonsingular. Thus we have

$$\Pi = \det(-A_{22} + G_3\bar{C}_2) \times \det \left\{ \begin{bmatrix} \lambda I_r & 0 \\ 0 & \lambda I_q \end{bmatrix} - \begin{bmatrix} A_r & P_r \\ 0 & S \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \hat{A}_{22} \begin{bmatrix} C_r & Q_r \end{bmatrix} \right\},$$

which implies that

$$\left(\begin{bmatrix} A_r & P_r \\ 0 & S \end{bmatrix}, \begin{bmatrix} C_r & Q_r \end{bmatrix} \right)$$

is detectable. Here $\hat{A}_{22} = I + \bar{C}_2(A_{22} - G_3\bar{C}_2)^{-1}G_3$.

Note that assumption **A4)** is equivalent to the condition 2) in Lemma 2. Now we just need to verify 3) in Lemma 2. By (12), we have

$$\bar{\Pi}_1 S = \bar{A}_{11}\bar{\Pi}_1 + \bar{A}_{12}\bar{\Pi}_2 + \bar{B}_1\Gamma + \bar{P}_1, \tag{30}$$

$$0 = \bar{A}_{21}\bar{\Pi}_1 + \bar{A}_{22}\bar{\Pi}_2 + \bar{B}_2\Gamma + \bar{P}_2, \tag{31}$$

$$0 = \bar{C}_1\bar{\Pi}_1 + \bar{C}_2\bar{\Pi}_2 + Q, \tag{32}$$

where $\bar{\Pi} = \begin{bmatrix} \bar{\Pi}_1 \\ \bar{\Pi}_2 \end{bmatrix} = N_1^{-1}\Pi$.

Since $\bar{A}_{22}$ is nonsingular, substitute (31) into (30) and (32) to eliminate $\bar{\Pi}_2$, it follows that

$$\begin{cases} \bar{\Pi}_1 S = A_r\bar{\Pi}_1 + B_r\Gamma + P_r, \\ C_r\bar{\Pi}_1 + D_r\Gamma + Q_r = 0. \end{cases} \tag{33}$$

Note that $\bar{\Pi}_1$ and Γ in (33) are the solutions of the regulator equations associated with system (29), so condition $3(a)$ in Lemma 2 is satisfied. Furthermore, assumption **A5)** (b) guarantees $3(b)$ in Lemma 2 with the same (Γ, W_0, δ, L) described in (33) and **A5)** (b).

We know the following output feedback control law solves the problem of semi-global output regulation

$$\begin{aligned} \begin{bmatrix} \hat{x}_1(k+1) \\ \hat{w}(k+1) \end{bmatrix} &= \begin{bmatrix} A_r & P_r \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{w}(k) \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} \sigma(u) \\ &\quad - \begin{bmatrix} L_A \\ L_S \end{bmatrix} \begin{bmatrix} C_r & Q_r \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) - \hat{x}_1(k) \\ w(k) - \hat{w}(k) \end{bmatrix}, \\ u(k) &= F_1(\varepsilon)\hat{x}_1(k) + (-F_1(\varepsilon)\bar{\Pi}_1 + \Gamma)\hat{w}(k), \end{aligned} \tag{34}$$

where $F_1(\varepsilon) = -(B_r^T P(\varepsilon) B_r + I)^{-1} B_r^T P(\varepsilon) A_r$ and $P(\varepsilon) > 0$ is the unique solution to the DARE

$$P = A_r^T P A_r - A_r^T P B_r (B_r^T P B_r + I)^{-1} B_r^T P A_r + \varepsilon I, \quad 0 < \varepsilon \leq 1\,.$$

The matrices L_A and L_S are chosen such that the following matrix is asymptotically stable

$$\begin{bmatrix} A_r + L_A C_r & P_r + L_A Q_r \\ L_S C_r & S + L_S Q_r \end{bmatrix}.$$

Step 3. Consider the closed-loop system composed of (6) and (34)

$$\begin{aligned} Ex(k+1) = Ax(k) + B\sigma(F_1(\varepsilon)\hat{x}_1(k) \\ + (-F_1(\varepsilon)\bar{\Pi}_1 + \Gamma)\hat{w}(k)) + Pw(k), \end{aligned} \tag{35}$$

$$w(k+1) = Sw(k), \tag{36}$$

$$\begin{aligned} \begin{bmatrix} \hat{x}_1(k+1) \\ \hat{w}(k+1) \end{bmatrix} &= \begin{bmatrix} A_r & P_r \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{w}(k) \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} \sigma(u) \\ &\quad - \begin{bmatrix} L_A \\ L_S \end{bmatrix} \begin{bmatrix} C_r & Q_r \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) - \hat{x}_1(k) \\ w(k) - \hat{w}(k) \end{bmatrix}, \end{aligned} \tag{37}$$

$$y(k) = Cx(k) + Qw(k)\,. \tag{38}$$

By the same coordinate transformation as (13), we have

$$\bar{x}_1(k+1) = A_r\bar{x}_1(k) + B_r\sigma(u) + P_r w(k)\,, \tag{39}$$

$$\bar{x}_2(k) = -\bar{A}_{22}^{-1}(\bar{A}_{21}\bar{x}_1(k) + \bar{B}_2\sigma(u) + \bar{P}_2 w(k))\,, \tag{40}$$

$$y(k) = C_r\bar{x}_1(k) + D_r\sigma(u) + Q_r w(k)\,. \tag{41}$$

Obviously, (37), (39) and (41) are the right expression for the normal system (29) and output controller (34).

For any compact set $X_0 \times Z_0 \subset \Re^n \times \Re^{r+q}$, let $\bar{X}_0 = \bar{X}_{10} \times \bar{X}_{20} \subset \Re^r \times \Re^{n-r}$ be compact and be sufficiently large such that $x(k) \in X_0$ implies $\bar{x}(k) = N_1^{-1}x(k) \in \bar{X}_0$. Then, by Lemma 2, for $\bar{X}_{10} \times Z_0 \subset \Re^r \times \Re^r$, there exists an $\varepsilon^* \in (0,1]$ such that for each $\varepsilon \in (0,\varepsilon^*]$, the equilibrium point at the origin of the closed-loop system composed of (37) and (39) is stable with $\bar{X}_{10} \times Z_0$ contained in its basin of attraction. By the result of [9], we can see that $\bar{x}_1$ and $\bar{x}_2$ approach 0 as $L \to +\infty$ when $w = 0$, which proves that the system composed of (35) and (37) with $w = 0$ is also exponentially stable with $X_0 \times Z_0$ contained in its basin of attraction. Furthermore, we can also obtain that

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty}(Cx(k) + Qw(k)) = 0\,.$$

This completes the proof. □

5. Example

Consider the linear discrete-time singular system (1) and (2) with

$$E = \begin{bmatrix} 4 & 0 & 0 & 4 \\ -8 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} -6 & 2 & 0 & -4 \\ 7 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.1 & 0 \end{bmatrix}.$$

We ca easily see that assumptions **A1) - A4)** are satisfied and

$$\Pi = \begin{bmatrix} 1.5250 & 0.2500 \\ 1.4375 & 0.3750 \\ 1.6250 & 1.2500 \\ -1.6250 & -0.2500 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -0.8125 & -0.1250 \end{bmatrix}$$

solve the regulation equation (12). When $W_0 = \{w \in \Re^2 : \|w\| \leq 1\}$, we know that $\|\Gamma w(k)\|_\infty \leq 0.9375$ for all $w(0) \in W_0$. Therefore, assumption **A5)** is also verified. Now we can say that the problem of output regulation for linear discrete-time singular systems with actuator saturation is solvable.

6. Conclusion

We have discussed the stabilization and output regulation problems for linear discrete-time singular systems subject to actuator saturation. The existence conditions and the corresponding output feedback controllers for the semi-global stabilization and output regulation have been given. The main results we obtained can be viewed as an extension of the stabilization and output regulation of the normal linear discrete-time systems. It is a topic for further research to consider the control problems for singular systems with unstable poles in the presence of actuator saturation.

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