International Journal of Pure and Applied Mathematics
Volume 16 No. 1 2004, 125-135

ON δ-PREIRRESOLUTE MULTIFUNCTIONS

Erdal Ekici
Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus
Canakkale, 17020, TURKEY
e-mail: eekici@comu.edu.tr

Abstract: In this paper, upper and lower δ-preirresolute multifunctions are introduced and studied. Some characterizations and several properties concerning upper and lower δ-preirresolute multifunctions are obtained.

AMS Subject Classification: 54C60, 54C08, 54C10
Key Words: upper (lower) δ-preirresolute, multifunction, δ-preopen sets

1. Introduction and Preliminaries

In 1993, Raychaudhuri and Mukherjee [8] introduced a weak form of open sets called δ-preopen sets. Recently, some stronger and weaker forms of continuous functions and multifunctions have researched and studied and some kinds of continuous functions and multifunctions have studied in terms of δ-preopen sets by several authors. For example Raychaudhuri and Mukherjee [8], Ekici [4], Park, Lee, and Son [7]. The purpose of the present paper is to define upper (lower) δ-preirresolute multifunctions and to obtain several characterizations of upper (lower) δ-preirresolute multifunctions and several properties of such multifunctions.

Received: August 11, 2004

Throughout the paper, spaces X and Y always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1, 2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$. For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$ [9]. Other basic concepts and terminology about multifunctions are in [3] and [5]. If $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{cl}\,(A)$ and $\mathrm{int}(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A of a space X is said to be regular open (respectively regular closed) if $A = \mathrm{int}\,(\mathrm{cl}\,(A))$ (respectively $A = \mathrm{cl}\,(\mathrm{int}\,(A))$) [10]. The δ-interior [11] of a subset A of X is the union of all regular open sets of X contained in A is denoted by $\delta\text{-int}\,(A)$. A subset A is called δ-open [11] if $A = \delta\text{-int}\,(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. A set A of (X, τ) is called δ-closed [11] if $A = \delta\text{-cl}\,(A)$, where $\delta\text{-cl}\,(A) = \{x \in X : A \cap \mathrm{int}\,(\mathrm{cl}\,(U)) \neq \emptyset,\ U \in \tau$ and $x \in U\}$.

A subset A of a space X is said to be δ-preopen [8] if $A \subset \mathrm{int}\,(\delta\text{-cl}\,(A))$. The family of all δ-preopen sets of X containing a point $x \in X$ is denoted by $\delta PO(X, x)$. The complement of a δ-preopen set is said to be δ-preclosed. The intersection of all δ-preclosed sets of X containing A is called the δ-preclosure [8] of A and is denoted by $\delta\text{-pcl}\,(A)$. The union of all δ-preopen sets of X contained A is called δ-preinterior of A and is denoted by $\delta\text{-pint}\,(A)$ [8].

The family of all δ-preopen (resp. δ-preclosed, δ-open) sets of X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$, $\delta O(X)$).

2. δ-Preirresolute Multifunctions

In this section, the notion of δ-preirresolute multifunctions is introduced and characterizations and basic properties of δ-preirresolute multifunctions are studied.

Definition 1. A multifunction $F : X \to Y$ is said to be:

1. Lower δ-preirresolute at a point $x \in X$ if for each δ-preopen set V of Y such that $x \in F^-(V)$, there exists a $U \in \delta PO(X, x)$ such that $U \subset F^-(V)$,
2. Upper δ-preirresolute at a point $x \in X$ if for each δ-preopen set V of Y such that $x \in F^+(V)$, there exists a $U \in \delta PO(X, x)$ such that $U \subset F^+(V)$.
3. Lower (upper) δ-preirresolute if F has this property at each point of X.

Proposition 2. *For any subset A of X, $\delta\text{-pcl}\,(A) = A \cup \text{cl}\,(\delta\text{-int}\,(A))$ and $\delta\text{-pint}\,(A) = A \cap \text{int}\,(\delta\text{-cl}\,(A))$, see [8].*

The following theorem give some characterizations of lower δ-preirresolute multifunction.

Theorem 3. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is lower δ-preirresolute multifunction,

(2) for each $x \in X$ and for each δ-preopen set V with $F(x) \cap V \neq \emptyset$, there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \cap V \neq \emptyset$,

(3) $F^-(V) \in \delta PO(X)$ for any δ-preopen set $V \subset Y$,

(4) $F^+(K) \in \delta PC(X)$ for any δ-preclosed set $K \subset Y$,

(5) $\delta\text{-pcl}\,(F^+(B)) \subset F^+(\delta\text{-pcl}\,(B))$ for every subset B of Y,

(6) $F^-(\delta\text{-pint}\,(B)) \subset \delta\text{-pint}\,(F^-(B)$ for every subset B of Y,

(7) $\text{cl}\,(\delta\text{-int}\,(F^+(B))) \subset F^+(\delta\text{-pcl}\,(B))$ for every subset B of Y,

(8) $F(\text{cl}\,(\delta\text{-int}\,(A))) \subset \delta\text{-pcl}\,(F(A))$ for every subset A of X,

(9) $F(\delta\text{-pcl}\,(A)) \subset \delta\text{-pcl}\,(F(A))$ for every subset A of X.

Proof. (1)$\Leftrightarrow$(2). This is obvious.

(1)$\Rightarrow$(3): Let $x \in X$ and G be a δ-preopen set of Y such that $x \in F^-(G)$. By (1), there exists a $U_x \in \delta PO(X, x)$ such that $U_x \subset F^-(G)$. Therefore, we have $F^-(G) = \bigcup_{x \in F^-(G)} U_x$ and hence $F^-(G) \in \delta PO(X)$.

(3)$\Rightarrow$(1): Let $V \in \delta PO(Y)$ and $x \in F^-(V)$. By (3), $F^-(V) \in \delta PO(X)$. Take $U = F^-(V)$. We obtain $U \subset F^-(V)$.

(3)$\Leftrightarrow$(4): Let K be any δ-preclosed set of Y. Then, $Y \backslash K$ is an δ-preopen set of Y. By (3), $F^-(Y \backslash K) \in \delta PO(X)$. Since $F^-(Y \backslash K) = X \backslash F^+(K)$, we obtain that $F^+(K)$ is δ-preclosed in X.

The converse is similar.

(4)⇒(5): For any subset B of Y, δ-pcl(B) is δ-preclosed in Y and then $F^+(\delta\text{-pcl}(B))$ is δ-preclosed in X. Hence $\delta\text{-pcl}(F^+(B)) \subset F^+(\delta\text{-pcl}(B))$.

(5)⇒(4): Let K be any δ-preclosed set in Y. Then $\delta\text{-pcl}(F^+(K)) \subset F^+(\delta\text{-pcl}(K)) = F^+(K)$ and hence $F^+(K)$ is a δ-preclosed set in X.

(3)⇒(6): For any subset B of Y, δ-pint(B) is δ-preopen in Y and then $F^-(\delta\text{-pint}(B))$ is δ-preopen in X. Hence $F^-(\delta\text{-pint}(B)) \subset \delta\text{-pint}(F^-(B))$.

(6)⇒(3): Let V be any δ-preopen set of Y. Then $F^-(V) = F^-(\delta\text{-pint}(V)) \subset \delta\text{-pint}(F^-(V))$ and hence $F^-(V) \in \delta PO(X)$.

(4)⇒(7): Let B be any subset of Y. Since δ-pcl(B) is δ-preclosed, $F^+(\delta\text{-pcl}(B))$ is δ-preclosed in X and $F^+(B) \subset F^+(\delta\text{-pcl}(B))$. Therefore, we obtain $\delta\text{-pcl}(F^+(B)) \subset F^+(\delta\text{-pcl}(B))$ and hence

$$\mathrm{cl}\,(\delta - \mathrm{int}\,(F^+(B)) \subset \mathrm{cl}\,(\delta - \mathrm{int}\,(F^+(\delta - \mathrm{pcl}\,(B))) \subset F^+(\delta - \mathrm{pcl}\,(B)).$$

(7)⇒(8): Let A be any subset of X. By (7), we have $\mathrm{cl}\,(\delta\text{-int}\,(A)) \subset \mathrm{cl}\,(\delta\text{-int}\,(F^+(F(A))) \subset F^+(\delta\text{-pcl}\,(F(A)))$. Therefore we obtain $F(\mathrm{cl}\,(\delta\text{-int}\,(A))) \subset \delta\text{-pcl}\,(F(A))$.

(8)⇒(9): Let A be any subset of X. By the previous proposition and by (8), $F(\delta\text{-pcl}\,(A)) = F(A \cup \mathrm{cl}\,(\delta\text{-int}\,(A))) \subset \delta\text{-pcl}\,(F(A))$.

(9)⇒(5): Let B be any subset of Y. Then we have

$$F(\delta - \mathrm{pcl}\,(F^+(B))) \subset \delta - \mathrm{pcl}\,(F(F^+(B)))$$

and hence $\delta\text{-pcl}\,(F^+(B)) \subset F^+(\delta\text{-pcl}\,(F(F^+(B))) \subset F^+(\delta\text{-pcl}\,(B))$. □

The following theorem give some characterizations of upper δ-preirresolute multifunction.

Theorem 4. *The following are equivalent for a multifunction $F : X \to Y$:*
(1) F is upper δ-preirresolute,
(2) for each $x \in X$ and for each δ-preopen set V such that $F(x) \subset V$, there exists a $U \in \delta PO(X, x)$ such that if $y \in U$, then $F(y) \subset V$,
(3) $F^+(V) \in \delta PO(X)$ for any δ-preopen set $V \subset Y$,
(4) $F^-(K) \in \delta PC(X)$ for any δ-preclosed set $K \subset Y$,
(5) $\delta\text{-pcl}\,(F^-(B)) \subset F^-(\delta\text{-pcl}\,(B))$ for any subset B of Y,
(6) $F^+(\delta\text{-pint}\,(N)) \subset \delta\text{-pint}\,(F^+(N))$ for any subset N of Y,
(7) $\mathrm{cl}\,(\delta\text{-}int(F^-(B))) \subset F^-(\delta\text{-pcl}\,(B))$ for any subset B of Y.

Proof. It can be obtained similarly as the previous theorem. □

Theorem 5. *Let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If is upper (lower) δ-preirresolute and G is upper (lower) δ-preirresolute, then $G \circ F : X \to Z$ is a upper (lower) δ-preirresolute multifunction.*

Proof. Let $V \subset Z$ be any δ-preopen set. From the definition of $G \circ F$, we have $(G \circ F)^{+}(V) = F^{+}(G^{+}(V))$ (resp. $(G \circ F)^{-}(V) = F^{-}(G^{-}(V))$). Since G is upper (lower) δ-preirresolute multifunction, it follows that $G^{+}(V)$ (resp. $G^{-}(V)$) is a δ-preopen set. Since F is upper (lower) δ-preirresolute multifunction, it follows that $F^{+}(G^{+}(V))$ (resp. $F^{-}(G^{-}(V))$) is a δ-preopen set. It shows that $G \circ F$ is a upper (resp. lower) δ-preirresolute multifunction. □

Lemma 6. *Let A be a subset of a space (X, τ). Then $A \in \delta PO(X)$ if and only if $A \cap U \in \delta PO(X)$ for each regular open (δ-open) set U of X, see [8].*

Lemma 7. *For a multifunction $F : X \to Y$, (1) $G_F^{+}(A \times B) = A \cap F^{+}(B)$, (2) $G_F^{-}(A \times B) = A \cap F^{-}(B)$ for any subsets $A \subset X$ and $B \subset Y$, see [6].*

Lemma 8. *Let A and B be subsets of spaces (X, τ) and (Y, σ), respectively. If $A \in \delta PO(X)$ and $B \in \delta PO(Y)$, then $A \times B \in \delta PO(X \times Y)$, see [8].*

Theorem 9. *Let $F : X \to Y$ be a multifunction. If the graph multifunction of F is upper δ-preirresolute, F is upper δ-preirresolute.*

Proof. Suppose that $G_F : X \to X \times Y$ is upper δ-preirresolute. Let $x \in X$ and V be any δ-preopen set of Y containing $F(x)$. Since $X \times V$ is δ-preopen in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \delta PO(X, x)$ such that $G_F(U) \subset X \times V$. By Lemma 7, we have $U \subset G_F^{+}(X \times V) = F^{+}(V)$ and $F(U) \subset V$. This shows that F is upper δ-preirresolute. □

Theorem 10. *Let $F : X \to Y$ be a multifunction. $F : X \to Y$ is lower δ-preirresolute if $G_F : X \to X \times Y$ is lower δ-preirresolute.*

Proof. Suppose that G_F is lower δ-preirresolute. Let $x \in X$ and V be any δ-preopen set of Y such that $x \in F^{-}(V)$. Then $X \times V$ is δ-preopen in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower δ-preirresolute, there exists a δ-preopen set U containing x such that $U \subset G_F^{-}(X \times V)$. By Lemma 7, we have $U \subset F^{-}(V)$. This shows that F is lower δ-preirresolute. □

Lemma 11. *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta PO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta PO(X_0)$, see [8].*

Lemma 12. *Let $A \subset X_0 \subset X$. If $X_0 \in \delta O(X)$ and $A \in \delta PO(X_0)$, then $A \in \delta PO(X)$, see [8].*

Theorem 13. *Let $F : X \to Y$ be a multifunction and let U be a δ-open set in X. If F is a lower (upper) δ-preirresolute, then the restriction multifunction $F|_U : U \to Y$ is a lower (resp. upper) δ-preirresolute.*

Proof. Suppose that V is a δ-preopen set in Y. Let $x \in U$ and let $x \in (F|_U)^-(V)$. Since F is lower δ-preirresolute multifunction, it follows that there exists a δ-preopen set G such that $x \in G \subset F^-(V)$. By Lemma 11, we obtain that $x \in G \cap U \in \delta PO(U)$ and $G \cap U \subset (F|_U)^-(V)$. Thus, we show that the restriction multifunction $F|_U$ is a lower δ-preirresolute.

The proof of the upper δ-preirresoluteness of $F|_U$ is similar. □

Theorem 14. *Let $\{U_i : i \in I\}$ be a δ-open cover of a space X. Then a multifunction $F : X \to Y$ is upper δ-preirresolute (resp. lower δ-preirresolute) if and only if the restriction $F|_{U_i} : U_i \to Y$ is upper δ-preirresolute (resp. lower δ-preirresolute) for each $i \in I$.*

Proof. Let $i \in I$ and V be any δ-preopen set of Y. Since F is upper δ-preirresolute, $F^+(V)$ is δ-preopen in X. By Lemma 11, $(F|_{U_i})^+(V) = F^+(V) \cap U_i$ is δ-preopen in U_i and hence $F|_{U_i}$ is upper δ-preirresolute.

Conversely, let V be any δ-preopen set of Y. Since $F|_{U_i}$ is upper δ-preirresolute for each $i \in I$, $(F|_{U_i})^+(V) = F^+(V) \cap U_i$ is δ-preopen in U_i. By Lemma 12, $(F|_{U_i})^+(V)$ is δ-preopen in X for each $i \in I$. We obtain that $F^+(V) = \bigcup_{i \in I} (F|_{U_i})^+(V)$ is δ-preopen in X. Hence F is upper δ-preirresolute.

The proof for F lower δ-preirresolute is similar. □

Definition 15. The δ-prefrontier of a subset A of a space X, denoted by δ-pFr(A), is defined by δ-pFr$(A) = \delta$-pcl$(A) \cap \delta$-pcl$(X \backslash A) = \delta$-pcl$(A) \backslash \delta$-pint$(A)$ [7].

Theorem 16. *The set all points of X at which a multifunction $F : X \to Y$ is not upper δ-preirresolute (lower δ-preirresolute) is identical with the union of the δ-prefrontier of the upper (lower) inverse images of δ-preopen sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper δ-preirresolute. Then there exists a δ-preopen set V of Y containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in \delta PO(X, x)$. Therefore, we have $x \in \delta$-pcl$(X \backslash F^+(V)) = X \backslash \delta$-pint$(F^+(V))$ and x$\in F^+(V)$. Thus, we obtain $x \in \delta$-pFr$(F^+(V))$.

Conversely, suppose that V is a δ-preopen set of Y containing $F(x)$ such that $x \in \delta$-pFr$(F^+(V))$. If F is upper δ-preirresolute at x, then there exists

$U \in \delta PO(X,x)$ such that $U \subset F^+(V)$; hence $x \in \delta\text{-pint}\,(F^+(V))$. This is a contradiction and hence F is not upper δ-preirresolute at x.

The case for lower δ-preirresolute is similarly shown. □

Theorem 17. *If $F: X \to Y$ is upper δ-preirresolute multifunction such that $F(x)$ is compact for each $x \in X$ and Y is Hausdorff space, then the multigraph $G(F)$ of F is δ-preclosed in $X \times Y$.*

Proof. $(x,y) \notin G(F)$. That is $y \notin F(x)$. Since Y is Hausdorff, for each $k \in F(x)$, there exist disjoint open sets $V(k)$ and $U(k)$ of Y such that $k \in U(k)$ and $y \in V(k)$. Then $\{U(k) : k \in F(x)\}$ is open cover of $F(x)$ and since $F(x)$ is compact, there exists a finite number of points, say, k_1, k_2, k_3,..., k_n in $F(x)$ such that $F(x) \subset \bigcup\{U(k_i) : i = 1,2,3,...,n\}$. Put $U = \bigcup\{U(k_i) : i = 1,2,3,...,n\}$ and $V = \bigcap\{V(k_i) : i = 1,2,3,...,n\}$. Then U and V are open in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. Since F is upper δ-precontinuous multifunction, there exists $H \in \delta PO(X,x)$ such that $F(H) \subset U$. Since V is open, by Lemma 8, it follows that $H \times V \in \delta PO(X \times Y)$ and $(x,y) \in H \times V \subset (X \times Y)\backslash G(F)$. We obtain that $(X \times Y)\backslash G_F = \bigcup\limits_{(x,y)\in(X\times Y)\backslash G(F)} H \times V$is δ-preopen in $X \times Y$ and hence $G(F)$ is δ-preclosed in $X \times Y$. □

A multifunction $F: X \to Y$ is said to be punctually connected if, for each $x \in X$, $F(x)$ is connected.

Definition 18. A space X is called δ-preconnected provided that X is not the union of two disjoint nonempty δ-preopen sets, see [4].

Theorem 19. *Let F be a multifunction from a δ-preconnected topological space X onto a topological space Y such that F is punctually connected. If F is upper δ-preirresolute multifunction, then Y is a connected space.*

Proof. Let $F: X \to Y$ be a upper δ-preirresolute multifunction from a δ-preconnected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is upper δ-preirresolute multifunction, $F^+(H)$ and $F^+(K)$ are δ-preopen subsets of X. Since $F^+(H)$, $F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the δ-preconnectedness of X. Thus, it is obtained that Y is a connected space. □

Definition 20. A space X is said to be δ-pre-compact if every δ-preopen cover of X has a finite subcover [4].

Theorem 21. *Let $F : X \to Y$ be an upper δ-preirresolute surective multifunction such that $F(x)$ is δ-pre-compact for each $x \in X$. If X is a δ-pre-compact space, then Y is δ-pre-compact.*

Proof. Let $\{V_i : i \in I\}$ be a δ-preopen cover of Y. Since $F(x)$ is δ-pre-compact for each $x \in X$, there exists a finite subset $I(x)$ of I such that $F(x) \subset \bigcup\{V_i : i \in I(x)\}$. Put $V(x) = \bigcup\{V_i : i \in I(x)\}$. Since F is upper δ-preirresolute, there exists a preopen set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then the family $\{U(x) : x \in X\}$ is a δ-preopen cover of X and since X is δ-pre-compact, there exists a finite number of points, say, x_1, x_2, x_3, ...,x_n in X such that $X = \bigcup\{U(x_j) : j = 1, 2, 3, ..., n\}$. Thus $Y = F(X) = F(\bigcup_{j=1}^{n} U(x_j)) = \bigcup_{j=1}^{n} F(U(x_j)) \subset \bigcup_{j=1}^{n} V(x_j) = \bigcup_{j=1}^{n} \bigcup_{i \in I(x_j)} V_i$. This shows that Y is δ-pre-compact. □

Definition 22. A space X is said to be δ-pre-Lindelof if every δ-preopen cover of X has a countable subcover [4].

Theorem 23. *Let $F : X \to Y$ be an upper δ-preirresolute surjective multifunction such that $F(x)$ is δ-pre-Lindelof for each $x \in X$. If X is a δ-pre-Lindelof space, then Y is δ-pre-Lindelof.*

Proof. The proof is similar to that of the previous theorem. □

Definition 24. A space X is said to be δ-pre-T_2 if for each pair of distinct points x and y in X, there exist disjoint δ-preopen sets U and V in X such that $x \in U$ and $y \in V$ [4].

Definition 25. A multifunction $F : X \to Y$ is said to be punctually δ-preclosed if, for each $x \in X$, $F(x)$ is δ-preclosed.

Definition 26. A space X is said to be weakly δ-pre-normal if for each pair of disjoint δ-preclosed subsets F and K of X, there exist disjoint δ-preopen sets G and H such that $F \subset G$ and $K \subset H$.

Theorem 27. *Let $F : X \to Y$ be an upper δ-preirresolute multifunction and punctually δ-preclosed from a topological space X to a weakly δ-pre-normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a δ-pre-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a weakly δ-pre-normal space, it follows that there exists disjoints δ-preopen sets U and V containing $F(x)$ and $F(y)$ respectively. Thus

$F^+(U)$ and $F^+(V)$ are disjoint δ-preopen sets containing x and y respectively. Thus, it is obtained that X is δ-pre-T_2. □

A net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 28. Let (X, τ) be a topological space and let (x_α) be a net in X. It is said that the net (x_α) δ-preconverges to x if for each δ-preopen set G containing x in X, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.

Theorem 29. *Let $F : X \to Y$ be a multifunction. If F is lower (upper) δ-preirresolute multifunction, then for each $x \in X$ and for each net (x_α) which δ-preconverges to x in X and for each δ-preopen set $V \subset Y$ such that $x \in F^-(V)$ (resp. $x \in F^+(V)$), the net (x_α) is eventually in $F^-(V)$ (resp. $F^+(V)$).*

Proof. Let (x_α) be a net which δ-preconverges to x in X and let V be any δ-preopen set in Y such that $x \in F^-(V)$. Since F is lower δ-preirresolute multifuction, it follows that there exists a δ-preopen set U in X containing x such that $U \subset F^-(V)$. Since (x_α) δ-preconverges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^-(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(V)$.

The other proof is similar. □

Theorem 30. *Let $F_1 : X \to Y$, $F_2 : X \to Z$ be multifunctions and $F_1 \times F_2 : X \to Y \times Z$ be a multifunction defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) δ-preirresolute multifunction, then F_1 and F_2 are upper (resp. lower) δ-preirresolute multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be δ-preopen sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper δ-preirresolute multifunction, it follows that there exists a δ-preopen set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper δ-preirresolute multifunctions.

The other proof is similar. □

A multifunction $F : X \to Y$ is called upper (lower) completely δ-preirresolute if $F^+(V)$ (resp. $F^-(V)$) is regular open for each δ-preopen set V of Y.

Theorem 31. *Let F be upper δ-preirresolute punctually δ-preclosed and G be upper completely δ-preirresolute punctually δ-preclosed multifunctions from a space X to a weakly δ-pre-normal space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is δ-preclosed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually δ-preclosed multifunctions and Y is a weakly δ-pre-normal space, it follows that there exists disjoint δ-preopen sets U and V containing $F(x)$ and $G(x)$ respectively. Since F and G are upper δ-preirresolute and upper completely δ-preirresolute, respectively then the sets $F^+(U)$ and $G^+(V)$ are δ-preopen and regular open, respectively such that contain x. Let $H = F^+(U) \cap G^+(V)$. Then H is a δ-preopen set containing x and $H \cap K = \emptyset$. Hence, K is δ-preclosed in X. □

Theorem 32. *If Y is weakly δ-pre-normal space and $F_i : X_i \to Y$ is upper δ-preirresolute multifunction such that F_i is punctually δ-preclosed for $i = 1$, 2, then a set $K = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is δ-preclosed set in $X_1 \times X_2$.*

Proof. Take $(x_1, x_2) \in (X_1 \times X_2) \backslash K$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is weakly δ-pre-normal and F_i is punctually δ-preclosed for $i = 1$, 2, there exist disjoint δ-preopen sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1$, 2. Since F_i is upper δ-preirresolute, $F_i^+(V_i)$ is δ-preopen for $i = 1$, 2. Put $U = F_1^+(V_1) \times F_2^+(V_2)$, then U is δ-preopen and $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash K$. This shows that $(X_1 \times X_2) \backslash K$ is δ-preopen and hence K is δ-preclosed in $X_1 \times X_2$. □

Theorem 33. *Let X and X_α be topological spaces where $\alpha \in J$ and J is finite. Suppose that $F : X \to \prod_{\alpha \in J} X_\alpha$ is a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ is the projection for each $\alpha \in J$. If F is upper (lower) δ-preirresolute multifunction, then $P_\alpha \circ F$ is upper (resp. lower) δ-preirresolute multifunction for each $\alpha \in J$.*

Proof. Let $\alpha_0 \in J$. Take any δ-preopen set V_{α_0} in X_{α_0}. We obtain $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$. Since F is upper δ-preirresolute multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a δ-preopen set, $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ is δ-preopen in (X, τ). Hence, $P_{\alpha_0} \circ F$ is upper δ-preirresolute multifunction.

The other proof is similar. □

References

[1] T. Banzaru, Multifunctions and M-product spaces, *Bull. Stiin. Teh. Inst. Politeh Timisoara Ser. Mat. Fiz Mer. Teor. Apl.*, **17**, No. 31 (1972), 17-23, In Romanian.

[2] C. Berge, *Espaces Topologiques, Fonctions Multivoques*, Dunod, Paris (1959).

[3] C. Berge, *Topological Spaces*, Oliver and Boyd Ltd. (1963).

[4] E. Ekici, (δ-pre, s)-continuous functions, *Bulletin of the Malaysian Mathematical Society* (2004), Accepted.

[5] E. Klein, A. Thompson, *Theory of Correspondences*, A Wiley-Interscience Publications, John Wiley and Sons (1984).

[6] T. Noiri, V. Popa, Almost weakly continuous multifunctions, *Demonstratio Math.*, **26** (1993), 363-380.

[7] J.H. Park, B.Y. Lee, M.J. Son, On upper and lower δ-precontinuous multifunctions, *Chaos, Solitions and Fractals*, **19** (2004), 1231-1237.

[8] S. Raychaudhuri, N. Mukherjee, On δ-almost continuity and δ-preopen sets, *Bull. Inst. Math. Acad. Sinica*, **21**, No. 4 (1993), 357-366.

[9] R.E. Smithson, Almost and weak continuity for multifunctions, *Bull. Calcutta Math. Soc.*, **70** (1978), 383-390.

[10] M.H. Stone, Applications of the theory of Boolean rings to general topology, *TAMS*, **41** (1937), 375-381.

[11] N.V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103-118.